4

The Theory of Curved Dislocations

4-1. INTRODUCTION

This chapter is concerned with an extension to generally curved dislocations of the ideas developed for straight dislocations. The development mainly involves the mathematical formalism required to derive four key equations in dislocation theory: the Burgers formula for the displacements produced by an infinitesimal element of dislocation line; the two Peach-Koehler formulas for the stress produced by such an element and for the force on it produced by an external stress; and the Blin formula for the interaction energy between two such elements. These formulas are all very important in providing tools to handle interactions between complex arrays of dislocations as discussed in the subsequent two chapters. Because of their importance, we present their derivation in sufficient detail that it can be followed through step by step.

4-2. CONSERVATIVE AND NONCONSERVATIVE MOTION

Let us extend the concepts of Chap. 3 to generally curved dislocations. Consider a closed loop C, bounding some surface A (Fig. 4-1). Ascribe a sense ξ to C. The positive normal $\mathbf{n}$ to an element dA is defined by the requirement that if C were made to shrink continuously in A until it just bounded dA, it would encircle $\mathbf{n}$ in the positive sense, by the right-hand rule. Also $d\mathbf{A}=\mathbf{n}\,dA$.

C becomes a dislocation line of Burgers vector $\mathbf{b}$ if, over the surface A, one removes (or inserts) material

$$\delta V=\mathbf{b}\cdot d\mathbf{A} \tag{4-1}$$

displaces the surface on the negative side of the cut by $\mathbf{b}$ relative to the positive side, and then pastes the surfaces together. The surface A is perfect again, and a pure line defect, the dislocation line C, results. Any surface A bounded by C could be used for the operation. For example, the identical dislocation could

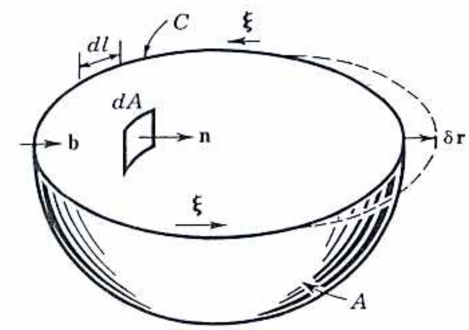

FIGURE 4-1. Closed dislocation loop C bounding surface A.

be produced by cutting and displacing any of the three surfaces shown in Fig. 4-2.

If the dislocation moves by $\delta\mathbf{r}$, matter

$$\delta V = \oint_C \mathbf{b} \cdot (\delta\mathbf{r} \times d\mathbf{l}) \qquad d\mathbf{l} = \boldsymbol{\xi}\, dl \tag{4-2}$$

must be removed, according to Eq. (4-1). $\mathbf{r}$ is variable along C. Surface elements $\delta\mathbf{A} = \delta\mathbf{r} \times d\mathbf{l}$ are added to A by the motion.

If everywhere along C the quantity $\mathbf{b} \cdot (\delta\mathbf{r} \times d\mathbf{l})$ is zero, the motion is one of pure slip. This occurs if $\delta\mathbf{r}$ is perpendicular to $\mathbf{b} \times d\mathbf{l}$, since $\mathbf{b} \cdot (\delta\mathbf{r} \times d\mathbf{l}) = -(\mathbf{b} \times d\mathbf{l}) \cdot \delta\mathbf{r}$. *The dislocation can move conservatively, by pure slip, on the cylindrical surface containing C and* $\mathbf{b}$.

Equation (4-2) can be interpreted geometrically as follows:

Project the dislocation onto a screen normal to $\mathbf{b}$ *(Fig. 4-3). The total mass transport to the dislocation during dislocation motion is given as the magnitude of the Burgers vector* $\mathbf{b}$ *times the change in projected enclosed area, counted negative if the projection of C encircles* $\mathbf{b}$ *in the positive sense.*

Exercise 4-1. The above condition for conservative motion, i.e., that $\delta\mathbf{r}$ be perpendicular to $\mathbf{b} \times d\mathbf{l}$, is sufficient but not necessary. A segment for which

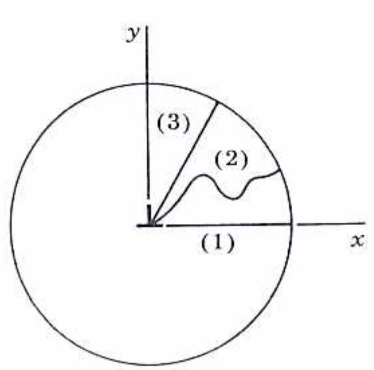

FIGURE 4-2. Three possible surface cuts, 1, 2, and 3, for producing a pure edge dislocation in a cylinder. Projection is along the z axis.

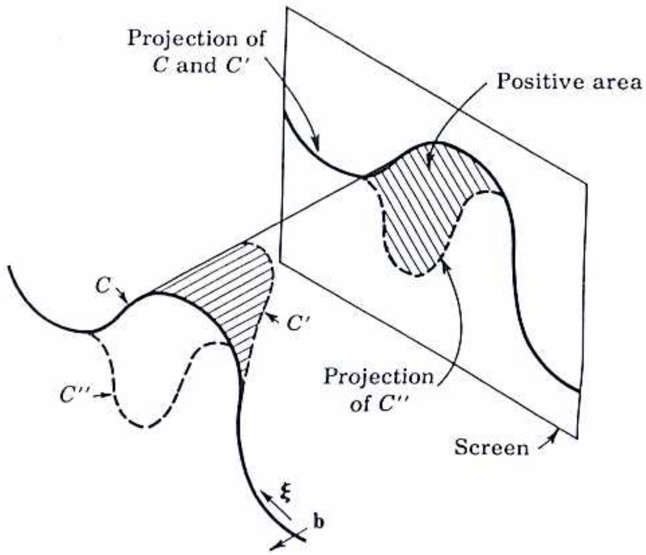

FIGURE 4-3. Projection on a screen normal to **b** of a dislocation line C which has undergone glide to position C' and climb to position C''.

$\mathbf{b}\times d\mathbf{l}=0$ ($d\mathbf{l}$ being a pure screw segment) is not restricted to glide on the cylindrical surface. Discuss how the statements of the preceding section should be qualified to take this special case into account. How will such conservative motion out of the cylindrical surface appear on the screen?

4-3. DISPLACEMENTS CAUSED BY CURVED DISLOCATIONS

Application of Green's Function Method

The derivation of the displacements associated with a dislocation loop of arbitrary shape involves the application of the theory, and Theorem 2-1, presented in Sec. 2-7. Consider a material of infinite extent, and suppose that a closed dislocation loop C of Burgers vector **b** is created (Fig. 4-4). The creation of the dislocation produces some displacement $\mathbf{u}(\mathbf{r})$ at **r**. Imagine for the moment that a point force acts at **r**. If a point force **F** acts at **r** while the dislocation is created, it does work

$$W=\mathbf{F}\cdot\mathbf{u}(\mathbf{r})=F_m u_m(\mathbf{r}) \tag{4-3}$$

where u_m and F_m are the components of **u** and **F**, respectively. If the displacements relieve the point force, they decrease the energy of the mechanism producing the point force by an amount W, the interaction energy. Therefore, a positive W represents a decrease in the system energy. *Since, by Theorem* 2-1, *there is no cross term in the elastic energy between the stress field of the*

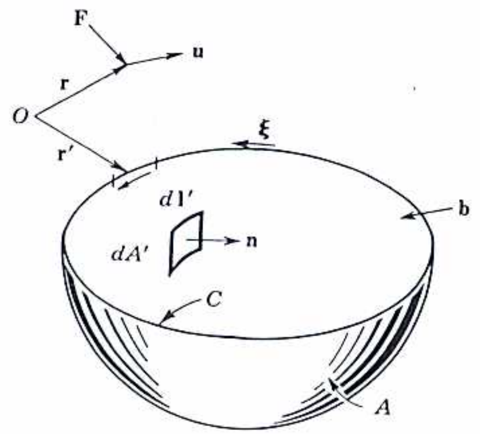

FIGURE 4-4. A point force **F** acting within an elastic continuum containing a closed dislocation loop.

dislocation and the stress field produced by the force **F**, *and since the dislocation line is not a sink of interaction energy, the entire energy W is spent as work done on the surface A*,[1]

$$W = -\int_A dA_j b_i F_m \sigma_{ij_m}(\mathbf{r}' - \mathbf{r}) \tag{4-4}$$

where $F_m \sigma_{ij_m}(\mathbf{r}'-\mathbf{r})$ is the stress σ_{ij} at $\mathbf{r}'$ caused by the components F_m of a point force **F** at **r**. The b_i are the components of **b**, dA_j are the components of $d\mathbf{A}$, etc.

In terms of the displacement functions $u_{mk}(\mathbf{r}'-\mathbf{r})$ introduced in Eq. (2-70), the so-called Green's functions of elasticity, Eq. (4-4), can be written as

$$W = -\int_A dA_j b_i c_{ijkl} \frac{\partial}{\partial x_l'} F_m u_{mk}(\mathbf{r}' - \mathbf{r}) \tag{4-5}$$

with summation over i, j, k, and m understood. Equating Eqs. (4-3) and (4-5) with only one nonvanishing component of **F**, $F_{m'} = 0$ when $m' \neq m$, one obtains the displacement field u_m, caused by the dislocation, by canceling F_m,

$$u_m(\mathbf{r}) = -\int_A dA_j b_i c_{ijkl} \frac{\partial}{\partial x_l'} u_{mk}(\mathbf{r}' - \mathbf{r}) \tag{4-6}$$

In other words, we use the point force **F** as a *test probe* to determine, via Eq. (2-71), the displacement field of the dislocation. The surface work is written in terms of the surface displacement b_i and the stress in A, produced by the point

[1]That is, less energy is introduced at the cut surface, and since there is no cross term, less by exactly the amount W if the point force has such a sign that it contributes to the energy of formation of the loop. The total energy contributed by the external mechanisms producing the surface displacements and the point force equals the elastic energy of the self-stress field of the loop.

force, at the point $\mathbf{r}'$. By the elasticity equations (2-3) and (2-15), this stress can in turn be written in terms of the displacements at $\mathbf{r}'$ produced by the point force. Thus, component by component in the point force, one can equate the work done by the point force at $\mathbf{r}$ and the surface work, cancel F_m, and deduce the displacement field of the dislocation.

Mura[2] has demonstrated that the Burgers integral expression for $u_m(\mathbf{r})$ can be transformed to a line integral for the gradient of $u_m(\mathbf{r})$. Because they are relative coordinates (Fig. 4-4) $dx_s' = -dx_s$, so the derivative of Eq. (4-6) can be written, with $\mathbf{R}=\mathbf{r}'-\mathbf{r}$, as

$$\frac{\partial u_m(\mathbf{r})}{\partial x_s} = b_i c_{ijkl}\int_A dA_j \frac{\partial^2 u_{mk}(\mathbf{R})}{\partial x_s' \partial x_l'} \tag{4-7}$$

This quantity enters the strains [Eq. (2-3)], which must be continuous across the cut A.

In accord with Eq. (2-8), we obtain

$$c_{ijkl}\frac{\partial^2 u_{mk}(\mathbf{R})}{\partial x_j' \partial x_l'} = 0 \tag{4-8}$$

when $R \neq 0$. This condition is similar to the requirement for mechanical equilibrium in the strain field of a point force. Since A can always be chosen so that $R \neq 0$ on the cut surface, Eqs. (4-7) and (4-8) can be combined to give

$$\frac{\partial u_m(\mathbf{r})}{\partial x_s} = b_i c_{ijkl}\int_A \left[dA_j \frac{\partial^2 u_{mk}}{\partial x_s' \partial x_l'} - dA_s \frac{\partial^2 u_{mk}}{\partial x_j' \partial x_l'} \right] \tag{4-9}$$

Stokes' theorem[3] has the form

$$\int_A \left(\frac{\partial \phi}{\partial x_j} dA_i - \frac{\partial \phi}{\partial x_i} dA_j \right) = \epsilon_{ijk} \oint_C \phi \, dx_k \tag{4-10}$$

where, in terms of the orthogonal unit vectors, the Einstein permutation operator [Eq. (3-94)] is given by

$$\epsilon_{ijk} = \mathbf{e}_i \cdot (\mathbf{e}_j \times \mathbf{e}_k) \tag{4-11}$$

[2] T. Mura, *Phil. Mag.*, **8**; 843 (1963).

[3] Stokes' theorem is widely known in the form

$$\int_A \operatorname{curl} \mathbf{M} \cdot d\mathbf{A} = \oint_C \mathbf{M} \cdot d\mathbf{l}$$

See I. S. Sokolnikoff and E. S. Sokolnikoff, "Higher Mathematics for Engineers and Physicists," McGraw-Hill, New York, 1941, p. 421. With $\mathbf{M} = \phi \mathbf{e}_k$ and $d\mathbf{l} = dx_i \mathbf{e}_i$, Eq. (4-10) follows directly.

Application of Stokes' theorem to Eq. (4-9) yields the line integral

$$\frac{\partial u_m(\mathbf{r})}{\partial x_s} = \epsilon_{jsn} b_i c_{ijkl} \oint_C \frac{\partial}{\partial x_l'} u_{mk}(\mathbf{R})\, dx_n' \tag{4-12}$$

which is Mura's formula.

Equations (4-6) and (4-12) are generally valid for an anisotropic material. If the Green's function $u_{mk}(\mathbf{r}'-\mathbf{r})$ is known, the displacement and strain fields caused by a dislocation of any shape can in principle be calculated. In the further development we shall assume elastic isotropy. Much of what follows is based on a review article by R. de Wit.[4]

The Burgers Displacement Equation

With the use of Eq. (2-47), Eq. (4-6) becomes, in a more extended form,

$$u_m(\mathbf{r}) = -\lambda \int_A dA_j b_j \frac{\partial u_{mk}}{\partial x_k'} - \mu \int_A dA_j b_i \frac{\partial u_{mi}}{\partial x_j'} - \mu \int_A dA_j b_i \frac{\partial u_{mj}}{\partial x_i'} \tag{4-13}$$

In order to derive the first term on the right-hand side of this equation, a change of dummy indices $dA_j b_j = dA_i b_i$ is required. Next, introducing Eq. (2-70) for the Green's functions into Eq. (4-13), one obtains

$$u_m(\mathbf{r}) = -\frac{1}{8\pi\mu} \int_A \Bigg[\left(\lambda b_j \frac{\partial}{\partial x_m'} \nabla'^2 R\, dA_j + \mu b_m \frac{\partial}{\partial x_j'} \nabla'^2 R\, dA_j + \mu b_i \frac{\partial}{\partial x_i'} \nabla'^2 R\, dA_m \right)$$
$$- \frac{\lambda+\mu}{\lambda+2\mu} \left(\lambda b_j \frac{\partial^3 R}{\partial x_m' \partial^2 x_k'} dA_j \right.$$
$$\left. + 2\mu b_i \frac{\partial^3 R}{\partial x_m' \partial x_i' \partial x_j'} dA_j \right) \Bigg] \tag{4-14}$$

Here

$$R = |\mathbf{r}' - \mathbf{r}| \tag{4-15}$$

and

$$\mathbf{R} = \mathbf{r}' - \mathbf{r} \tag{4-16}$$

[4] R. de Wit, *Solid State Phys.*, **10**: 249 (1960).

Finally, since $\partial^2/\partial x_i'^2 = \nabla'^2$, etc., one can rewrite Eq. (4-14) in a more symmetrical form suitable for the introduction of vector notation,

$$u_m(\mathbf{r}) = -\frac{1}{8\pi}\int_A b_m \frac{\partial}{\partial x_j'}\nabla'^2 R\, dA_j - \frac{1}{8\pi}\int_A \left(b_i \frac{\partial}{\partial x_i'}\nabla'^2 R\, dA_m - b_i \frac{\partial}{\partial x_m'}\nabla'^2 R\, dA_i \right)$$

$$+\frac{1}{4\pi}\frac{\lambda+\mu}{\lambda+2\mu}\int_A \left(b_i \frac{\partial}{\partial x_i'}\frac{\partial^2 R}{\partial x_m' \, dx_j'} dA_j - b_i \frac{\partial}{\partial x_j'}\frac{\partial^2 R}{\partial x_m' \partial x_j'} dA_i \right) \tag{4-17}$$

Again some changes in dummy indices are required. Equations (4-10) and (4-17) combine to give

$$u_m(\mathbf{r}) = \frac{-1}{8\pi}\int_A b_m \frac{\partial}{\partial x_j'}\nabla'^2 R\, dA_j - \frac{1}{8\pi}\oint_C b_i \epsilon_{mik}\nabla'^2 R\, dx_k'$$

$$-\frac{1}{8\pi(1-\nu)}\oint_C b_i \epsilon_{ijk}\frac{\partial^2 R}{\partial x_m' \partial x_j'} dx_k' \tag{4-18}$$

where use is made of Eq. (2-50) in the form $(\lambda+\mu)/(\lambda+2\mu) = 1/2(1-\nu)$. Since

$$\nabla'^2 R = \frac{2}{R}$$

and

$$\text{grad}'\frac{1}{R} = -\frac{\mathbf{R}}{R^3} \tag{4-19}$$

the vector form of Eq. (4-18) is

$$\mathbf{u}(\mathbf{r}) = -\frac{\mathbf{b}}{4\pi}\Omega - \frac{1}{4\pi}\oint_C \frac{\mathbf{b}\times d\mathbf{l}'}{R} + \frac{1}{8\pi(1-\nu)}\,\text{grad}\oint_C \frac{(\mathbf{b}\times\mathbf{R})\cdot d\mathbf{l}'}{R} \tag{4-20}$$

Here

$$\Omega = -\int_A \frac{\mathbf{R}\cdot d\mathbf{A}}{R^3} \tag{4-21}$$

is the solid angle through which the positive side of A is seen from $\mathbf{r}$. The change in sign in the last term from Eq. (4-18) to Eq. (4-20) comes about because the variable with respect to which differentiation is performed has been changed; grad $= -$grad$'$. Equation (4-20) was first derived by Burgers[5] (with a difference in sign because of his different definition of $\mathbf{b}$).

[5] J. M. Burgers, *Proc. Kon. Ned. Akad. Wetenschap.*, **42**: 293, 378 (1939).

The first term in Eq. (4-20) gives a discontinuity $\Delta \mathbf{u}=\mathbf{b}$ over the surface A, consistent with the operation of producing the dislocation by cutting and displacing A. The other two terms are continuous except at the dislocation line.

With the use of Eq. (4-20), the displacement produced at a point $\mathbf{r}$ by an arbitrarily curved dislocation, or array of dislocations, can be determined by integration over the dislocation line.

4-4. SELF-STRESS OF A CURVED DISLOCATION

The stresses for the isotropic case could be derived starting with Eq. (4-12). Instead, the stresses are derived following deWit's derivation,[6] developed before Mura's formula[7] was known. The stresses are obtained by the differentiation of Eq. (4-18) and insertion of the result into Eq. (2-46). In the formula for the displacements, there is a discontinuity over the surface A. In the formula for the stresses, this discontinuity must disappear for continuity and equilibrium to be maintained. Hence, unlike the case for displacements, the stresses can be expressed in terms of line integrals alone; the dislocation is defined by an operation that leaves the material perfect and continuous except at the dislocation core.

The only term that is not expressed as a line integral in Eq. (4-20) is the one involving the solid angle Ω. Consider $\partial\Omega/\partial x_j$. Evidently, $\delta x_j(\partial\Omega/\partial x_j)$ can be interpreted as the change in solid angle as seen from $\mathbf{r}$ if C is displaced by $\delta\mathbf{r}'=-\delta x_j \mathbf{e}_j$ (Fig. 4-5). Thus Eq. (4-21) indicates that

$$\frac{\partial\Omega}{\partial x_j}=\oint_C \frac{\mathbf{R}\cdot(\mathbf{e}_j\times d\mathbf{l})}{R^3} \tag{4-22}$$

or, written out in the notation of Eq. (4-18),

$$\frac{\partial\Omega}{\partial x_j}=-\frac{1}{2}\oint_C \epsilon_{ijk}\frac{\partial}{\partial x_i'}\nabla'^2 R\, dx_k' \tag{4-23}$$

The stresses are obtained from Eqs. (2-7) and (2-47) in the form

$$\sigma_{\alpha\beta}=\left[\lambda\delta_{\alpha\beta}\delta_{ml}+\mu\left(\delta_{\alpha l}\delta_{\beta m}+\delta_{\alpha m}\delta_{\beta l}\right)\right]\frac{\partial u_m}{\partial x_l} \tag{4-24}$$

A simple but tedious rearrangement of Eqs. (4-18), (4-20), and (4-24), involving

[6] R. deWit, *Solid State Phys.*, **10**: 249 (1960).

[7] T. Mura, *Phil. Mag.*, **8**: 843 (1963).

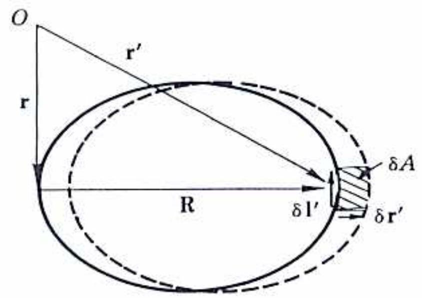

FIGURE 4-5. A dislocation loop.

changes of dummy indices, leads to the expression

$$\begin{aligned}\sigma_{\alpha\beta} = {} & \frac{\mu}{8\pi}\oint_C \big(\overset{(1)}{\delta_{\alpha l}\delta_{\beta m}\epsilon_{ilk}} + \overset{(2)}{\delta_{\alpha i}\delta_{\beta l}\epsilon_{lmk}} + \overset{(3)}{\delta_{\alpha m}\delta_{\beta l}\epsilon_{ilk}} + \overset{(4)}{\delta_{\alpha l}\delta_{\beta i}\epsilon_{lmk}}\big) b_m \frac{\partial}{\partial x'_i}\nabla'^2 R\, dx'_k \\ & + \frac{\mu(\lambda+\mu)}{2\pi(\lambda+2\mu)}\oint_C b_i\epsilon_{ijk}\frac{\partial^3 R}{\partial x'_\alpha\partial x'_\beta\partial x'_j}dx'_k \\ & + \frac{\mu\lambda\delta_{\alpha\beta}}{4\pi(\lambda+2\mu)}\oint_C b_m\epsilon_{imk}\frac{\partial}{\partial x'_i}\nabla'^2 R\, dx'_k \end{aligned} \tag{4-25}$$

This relation can be simplified by the following procedure of de Wit.[8] By use of the relation[9]

$$\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl} \tag{4-26}$$

one can transform the first term in Eq. (4-25) to leave a more symmetrical form

[8] R. de Wit, *Solid State Phys.*, **10**: 249 (1960).

[9] Relation (4-26) is proved as follows: Equation (4-11) can be rewritten as

$$\epsilon_{ijk} = \mathbf{e}_i\cdot(\mathbf{e}_j\times\mathbf{e}_k) = \mathbf{e}_k\cdot(\mathbf{e}_i\times\mathbf{e}_j)$$

Taking the dot product of $\mathbf{e}_k$ and the above equation, one finds

$$\mathbf{e}_i\times\mathbf{e}_j = \epsilon_{ijk}\mathbf{e}_k \qquad \text{so that} \qquad \mathbf{e}_l\times\mathbf{e}_m = \epsilon_{klm}\mathbf{e}_k$$

Multiplying the two last equations, one obtains

$$\begin{aligned}\epsilon_{ijk}\epsilon_{klm} &= (\mathbf{e}_i\times\mathbf{e}_j)\cdot(\mathbf{e}_l\times\mathbf{e}_m) = \mathbf{e}_i\cdot[\mathbf{e}_j\times(\mathbf{e}_l\times\mathbf{e}_m)] \\ &= \mathbf{e}_i\cdot[(\mathbf{e}_j\cdot\mathbf{e}_m)\mathbf{e}_l - (\mathbf{e}_j\cdot\mathbf{e}_l)\mathbf{e}_m] \\ &= (\mathbf{e}_i\cdot\mathbf{e}_l)(\mathbf{e}_j\cdot\mathbf{e}_m) - (\mathbf{e}_i\cdot\mathbf{e}_m)(\mathbf{e}_j\cdot\mathbf{e}_l)\end{aligned}$$

which proves Eq. (4-26).

for $\sigma_{\alpha\beta}$. With the aid of Eq. (4-26), the four subterms (1), (2), (3), and (4) in the first term of Eq. (4-25) can be expanded as follows:

$$\overset{(1)}{\delta_{\alpha l}\delta_{\beta m}\epsilon_{ilk}} = \overset{(1)'}{\epsilon_{\beta lp}\epsilon_{pm\alpha}\epsilon_{ilk}} + \delta_{\alpha\beta}\delta_{lm}\epsilon_{ilk}$$

$$\overset{(2)}{\delta_{\alpha i}\delta_{\beta l}\epsilon_{lmk}} = \overset{(2)'}{\epsilon_{\beta ip}\epsilon_{pl\alpha}\epsilon_{lmk}} + \delta_{\alpha\beta}\delta_{il}\epsilon_{lmk}$$

$$\overset{(3)}{\delta_{\alpha m}\delta_{\beta l}\epsilon_{ilk}} = \overset{(3)'}{\epsilon_{\beta mp}\epsilon_{pl\alpha}\epsilon_{ilk}} + \delta_{\alpha\beta}\delta_{lm}\epsilon_{ilk}$$

$$\overset{(4)}{\delta_{\alpha l}\delta_{\beta i}\epsilon_{lmk}} = \overset{(4)'}{\epsilon_{\beta lp}\epsilon_{pi\alpha}\epsilon_{lmk}} + \delta_{\alpha\beta}\delta_{il}\epsilon_{lmk} \tag{4-27}$$

A second application of Eq. (4-26) to the factor $\epsilon_{\beta lp}\epsilon_{ilk}$ in (1)′ gives

$$\epsilon_{\beta lp}\epsilon_{ilk} = \delta_{pk}\delta_{\beta i} - \delta_{pi}\delta_{\beta k} \tag{4-28}$$

and a term

$$\delta_{\beta i}\delta_{pk}\epsilon_{pm\alpha} = \delta_{\beta i}\epsilon_{km\alpha}$$

is obtained, which is evidently the negative of term (4),

$$\delta_{\alpha l}\delta_{\beta i}\epsilon_{lmk} = \delta_{\beta i}\epsilon_{\alpha mk} = -\delta_{\beta i}\epsilon_{km\alpha}$$

In this fashion, (1)′ is decomposed to yield a term $-(4)$, (2)′ yields a term $-(3)$, (3)′ yields a term $-(2)$, and (4)′ yields a term $-(1)$. The sum of terms $(1)+(2)+(3)+(4)$ is

$$(1)+(2)+(3)+(4) = -\epsilon_{im\alpha}\delta_{\beta k} - \epsilon_{im\beta}\delta_{\alpha k} + 2\delta_{\alpha\beta}\epsilon_{imk} \tag{4-29}$$

After the substitution of Eq. (4-29), and of ν for λ, Eq. (4-25) takes the neater form

$$\sigma_{\alpha\beta} = -\frac{\mu}{8\pi}\oint_C b_m\epsilon_{im\alpha}\frac{\partial}{\partial x_i'}\nabla'^2 R\, dx_\beta' - \frac{\mu}{8\pi}\oint_C b_m\epsilon_{im\beta}\frac{\partial}{\partial x_i'}\nabla'^2 R\, dx_\alpha'$$

$$-\frac{\mu}{4\pi(1-\nu)}\oint_C b_m\epsilon_{imk}\left(\frac{\partial^3 R}{\partial x_i'\,\partial x_\alpha'\,\partial x_\beta'} - \delta_{\alpha\beta}\frac{\partial}{\partial x_i'}\nabla'^2 R\right)dx_k' \tag{4-30}$$

This equation was first derived by Peach and Koehler[10] (with a different sign because of their different definition of **b**). Equation (4-30) enables one to determine the stress field of an arbitrarily curved dislocation by line integration. Specific examples of such a procedure are given in Chap. 5. A more contracted form of Eq. (4-30) is given in dyadic notation

$$\sigma = \frac{\mu}{4\pi}\oint_C (\mathbf{b}\times\nabla')\frac{1}{R}\otimes d\mathbf{l}' + \frac{\mu}{4\pi}\oint_C d\mathbf{l}'\otimes(\mathbf{b}\times\nabla')\frac{1}{R} - \frac{\mu}{4\pi(1-\nu)}\oint_C \nabla'\cdot(\mathbf{b}\times d\mathbf{l}')(\nabla\otimes\nabla - \mathbf{I}\nabla^2)R \tag{4-31}$$

where **I** is the unit dyadic or idemfactor, given by $\mathbf{e}_1\otimes\mathbf{e}_1+\mathbf{e}_2\otimes\mathbf{e}_2+\mathbf{e}_3\otimes\mathbf{e}_3$.

Exercise 4-2. Derive the stress σ_{xx} of a straight pure edge dislocation in an infinite medium, $\mathbf{b}=(b_x,0,0)$, from Eq. (4-30). The result is given in Eq. (3-43). Notice that the derivatives in (4-30) are expressed in the $\mathbf{r}'$ coordinates of Fig. 4-4, and that $\mathbf{R}=\mathbf{r}'-\mathbf{r}$, $R=[(x'-x)^2+(y-y)^2+(z'-z)^2]^{1/2}$. *Hint*: Even though $x'=y'=0$ in this problem, in terms involving $\partial/\partial x'$ and $\partial/\partial y'$, the derivatives must be taken before setting $x'=y'=0$. Equation (4-30) formally applies to a closed loop of dislocation, yet here line integrals are taken only over the z' axis. Imagine that the dislocation is a portion of a closed loop with the portions other than that along z' infinitely far removed, and rationalize the fact that these portions do not contribute to the integral.

4-5. ENERGY OF INTERACTION BETWEEN TWO DISLOCATION LOOPS

If loop 1 is created while loop 2 is present, the stresses originating from loop 2 do work $-W_{12}$, where W_{12} is the interaction energy between the two loops (Fig. 4-6). Since, by Eq. (4-3), the work done on the surface of loop 1 is $W=-W_{12}$, and since, by Theorem 2-1, no energy flows in or out of loop 2, the work done on loop 1 represents a decrease in the strain energy of the total system. Therefore, if W_{12} is negative, the energy of the system decreases if loop 1 is created in the presence of loop 2, and an attractive force exists between the loops.

The interaction energy is, from Eq. (4-4),

$$W_{12} = \int_{A_1} dA_{1_\beta} b_{1_\alpha} \sigma_{2_{\alpha\beta}} \tag{4-32}$$

[10] M. O. Peach and J. S. Koehler, *Phys. Rev.*, **80**: 436 (1950).

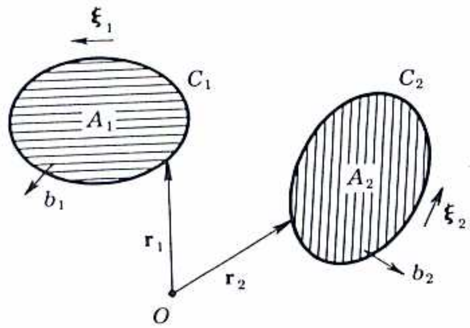

FIGURE 4-6. Two dislocation loops within the same elastic continuum.

If one inserts Eq. (4-30) into this formula, recalling that $\partial/\partial x_{1_i} = -(\partial/\partial x_{2_i})$, one finds

$$W_{12} = \frac{\mu}{8\pi}\int_{A_1}\oint_{C_2} dA_{1_\beta}\, dx_{2_\beta}\, b_{1_\alpha} b_{2_m} \epsilon_{im\alpha} \frac{\partial}{\partial x_{1_i}} \nabla^2 R \tag{4-33a}$$

$$+ \frac{\mu}{8\pi}\int_{A_1}\oint_{C_2} dA_{1_\beta}\, dx_{2_\alpha}\, b_{1_\alpha} b_{2_m} \epsilon_{im\beta} \frac{\partial}{\partial x_{1_i}} \nabla^2 R \tag{4-33b}$$

$$+ \frac{\mu}{4\pi(1-\nu)}\int_{A_1}\oint_{C_2} dA_{1_\beta}\, dx_{2_k}\, b_{1_\alpha} b_{2_m} \epsilon_{imk} \frac{\partial^3 R}{\partial x_{1_i}\partial x_{1_\alpha}\partial x_{1_\beta}} \tag{4-33c}$$

$$- \frac{\mu}{4\pi(1-\nu)}\int_{A_1}\oint_{C_2} dA_{1_\beta}\, dx_{2_k}\, b_{1_\alpha} b_{2_m} \epsilon_{imk} \delta_{\alpha\beta} \frac{\partial}{\partial x_{1_i}} \nabla^2 R \tag{4-33d}$$

By Stokes' theorem, Eq. (4-10), term (4-33a) becomes

$$(4\text{-}33a) = \frac{\mu}{8\pi}\int_{A_1}\oint_{C_2} dA_{1_i}\, dx_{2_\beta}\, b_{1_\alpha} b_{2_m} \epsilon_{im\alpha} \frac{\partial}{\partial x_{1_\beta}} \nabla^2 R$$

$$+ \frac{\mu}{8\pi}\oint_{C_1}\oint_{C_2} \epsilon_{\beta il}\, dx_{1_l}\, dx_{2_\beta}\, b_{1_\alpha} b_{2_m} \epsilon_{im\alpha} \nabla^2 R \tag{4-34}$$

The first term in (a) vanishes because

$$\oint_{C_2} dx_{2_\beta} \frac{\partial}{\partial x_{1_\beta}} \nabla^2 R = -\oint_{C_2} dx_{2_\beta} \frac{\partial}{\partial x_{2_\beta}} \nabla^2 R = -\oint_{C_2} d(\nabla^2 R) = 0$$

Thus (a) is given by the second term in Eq. (4-34), which is

$$(4\text{-}33a) = -\frac{\mu}{4\pi}\oint_{C_1}\oint_{C_2} \frac{(\mathbf{b}_1 \times \mathbf{b}_2)\cdot(d\mathbf{l}_1 \times d\mathbf{l}_2)}{R} \tag{4-35}$$

More simply for (b),

$$(4\text{-}33b)=\frac{\mu}{8\pi}\oint_{C_2}\mathbf{b}_1\cdot d\mathbf{l}_2\int_{A_1}(\text{grad}_1\nabla^2R\times\mathbf{b}_2)\cdot d\mathbf{A}_1$$

$$=\frac{\mu}{8\pi}\oint_{C_2}\mathbf{b}_1\cdot d\mathbf{l}_2\int_{A_1}\text{curl}_1\nabla^2R\,\mathbf{b}_2\cdot d\mathbf{A}_1$$

$$=\frac{\mu}{4\pi}\oint_{C_1}\oint_{C_2}\frac{(\mathbf{b}_1\cdot d\mathbf{l}_2)(\mathbf{b}_2\cdot d\mathbf{l}_1)}{R} \tag{4-36}$$

The subscript 1 on grad and curl is inserted as a reminder that the derivatives are to be taken in terms of the $\mathbf{r}_1$ coordinates in Fig. (4-6).

For term (c), Stokes' theorem, in the form of Eq. (4-10), yields[11]

$$(4\text{-}33c)=\frac{\mu}{4\pi(1-\nu)}\int_{A_1}\oint_{C_2}dA_{1_\alpha}dx_{2_k}b_{1_\alpha}b_{2_m}\epsilon_{imk}\frac{\partial^3R}{\partial x_{1_i}\partial^2x_{1_\beta}}$$

$$+\frac{\mu}{4\pi(1-\nu)}\oint_{C_1}\oint_{C_2}dx_{1_l}dx_{2_k}b_{1_\alpha}b_{2_m}\epsilon_{imk}\epsilon_{\beta\alpha l}\frac{\partial^2R}{\partial x_i\partial x_\beta} \tag{4-37}$$

The first term in (c) exactly cancels term (d). Thus

$$(4\text{-}33c)+(4\text{-}33d)=\frac{\mu}{4\pi(1-\nu)}\oint_{C_1}\oint_{C_2}(\mathbf{b}_1\times d\mathbf{l}_1)\cdot\mathbf{T}\cdot(\mathbf{b}_2\times d\mathbf{l}_2) \tag{4-38}$$

where $\mathbf{T}$ is a tensor with components

$$T_{ij}=\frac{\partial^2R}{\partial x_i\partial x_j} \tag{4-39}$$

Collection of terms and use of the relation

$$(\mathbf{b}\times\mathbf{b}_2)\cdot(d\mathbf{l}_1\times d\mathbf{l}_2)=(\mathbf{b}_1\cdot d\mathbf{l}_1)(\mathbf{b}_2\cdot d\mathbf{l}_2)-(\mathbf{b}_2\cdot d\mathbf{l}_1)(\mathbf{b}_1\cdot d\mathbf{l}_2)$$

to eliminate (4-36) now yields

$$W_{12}=-\frac{\mu}{2\pi}\oint_{C_1}\oint_{C_2}\frac{(\mathbf{b}_1\times\mathbf{b}_2)\cdot(d\mathbf{l}_1\times d\mathbf{l}_2)}{R}+\frac{\mu}{4\pi}\oint_{C_1}\oint_{C_2}\frac{(\mathbf{b}_1\cdot d\mathbf{l}_1)(\mathbf{b}_2\cdot d\mathbf{l}_2)}{R}$$

$$+\frac{\mu}{4\pi(1-\nu)}\oint_{C_1}\oint_{C_2}(\mathbf{b}_1\times d\mathbf{l}_1)\cdot\mathbf{T}\cdot(\mathbf{b}_2\times d\mathbf{l}_2) \tag{4-40}$$

[11] We simply write $\partial^2R/\partial x_i\partial x_\beta$, since $\partial^2R/\partial x_{1_i}\partial x_{1_\beta}=\partial^2R/\partial x_{2_i}\partial x_{2_\beta}$. In symmetrical terms, where the subscript does not matter, it is left out.

The above formula was first obtained by Blin.[12] An alternative form[13] for the last term in Eq. (4-40) is

$$\frac{\mu}{4\pi(1-\nu)}\oint_{C_1}\oint_{C_2}\left\{-(\mathbf{b}_1\cdot\mathbf{T}\cdot\mathbf{b}_2)(d\mathbf{l}_1\cdot d\mathbf{l}_2)+\frac{2}{R}(\mathbf{b}_1\times d\mathbf{l}_1)\cdot(\mathbf{b}_2\times d\mathbf{l}_2)\right\}$$

The integrands are not identical but differ by terms which give no net contribution after integration over *complete* loops C_1 and C_2. They would differ for integrals over segments; Chap. 6. The form derived by Blin is always used in this book. Equation (4-40) has a much more extensive application than simply to two loops of dislocation; in Chap. 6, Eq. (4-40) is extended to yield the interaction energy between two arbitrarily positioned *segments* of dislocation line.

4-6. FORCE PRODUCED BY AN EXTERNAL STRESS ACTING ON A DISLOCATION LOOP

Let $\boldsymbol{\sigma}$ denote the stress tensor in the medium, excluding the self-stress of the dislocation loop under consideration. As the loop is created, the stress does work

$$W=\int_A -\mathbf{b}\cdot(\boldsymbol{\sigma}\cdot d\mathbf{A}) \tag{4-41}$$

If every line element $d\mathbf{l}$ of the loop is displaced by some distance, the area A changes by increments $\delta\mathbf{r}\times d\mathbf{l}$ and the stress $\boldsymbol{\sigma}$ does addition work

$$\delta W=\oint_C d\mathbf{F}\cdot\delta\mathbf{r}=-\oint_C \mathbf{b}\cdot[\boldsymbol{\sigma}\cdot(\delta\mathbf{r}\times d\mathbf{l})]$$

$$=-\oint_C(\mathbf{b}\cdot\boldsymbol{\sigma})\cdot(\delta\mathbf{r}\times d\mathbf{l})=\oint_C[(\mathbf{b}\cdot\boldsymbol{\sigma})\times d\mathbf{l}]\cdot\delta\mathbf{r} \tag{4-42}$$

Thus, since $\delta\mathbf{r}$ is arbitrarily variable along C,

$$d\mathbf{F}=(\mathbf{b}\cdot\boldsymbol{\sigma})\times d\mathbf{l} \tag{4-43}$$

Each element $d\mathbf{l}$ is acted upon by a force $d\mathbf{F}$, as given in Eq. (4-43), in agreement with Eq. (3-90). Equation (4-43) also was derived first by Peach and Koehler.[14] Together with Eq. (4-30), Eq. (4-43) can be used to determine the interaction force between dislocation segments, as demonstrated in Chap. 5.

[12] J. Blin, *Acta. Met.*, **3**: 199 (1955).

[13] R. Fuentes-Samaniego, research in progress.

[14] M. O. Peach and J. S. Koehler, *Phys. Rev.*, **80**: 436 (1950).

4-7. SELF-ENERGY OF A DISLOCATION LOOP

In the preceding section we considered the work done by stresses other than the self-stresses when a loop is created. In forming the dislocation, work must also be done against the self-stress of the loop. Each element $d\mathbf{l}$ of the loop is acted upon by a force caused by the stress originating from all other parts of the loop, and the work done against all these forces is the work done to supply the self-energy. Only when the total force on each element $d\mathbf{l}$ of the loop is zero is the loop in equilibrium, and only then does one find an extremum in the total energy of the system.

Formally, the self-energy W_s is obtained if in Eq. (4-40) one inserts $C_1 = C_2 = C$ and $\mathbf{b}_1 = \mathbf{b}_2 = \mathbf{b}$ and then divides by 2:

$$W_s = \frac{\mu}{8\pi} \oint_{C_1 = C} \oint_{C_2 = C} \frac{(\mathbf{b} \cdot d\mathbf{l}_1)(\mathbf{b} \cdot d\mathbf{l}_2)}{R}$$

$$+ \frac{\mu}{8\pi(1-\nu)} \oint_{C_1 = C} \oint_{C_2 = C} (\mathbf{b} \times d\mathbf{l}_1) \cdot \mathbf{T} \cdot (\mathbf{b} \times d\mathbf{l}_2) \qquad (4\text{-}44)$$

$$T_{ij} = \frac{\partial^2 R}{\partial x_i \, \partial x_j}$$

The factor of 2 can be justified by the following reasoning, which is generally useful in the consideration of dislocation interactions. Imagine that the loop is created in infinitesimal increments of the Burgers vector $\mathbf{b}$; for each increment the self-energy increases by the interaction energy of a loop of fractional Burgers vector $\mathbf{b}f$ and a loop of vector $\mathbf{b}\,df$. The sum of all interactions leads to an average factor

$$\int_0^1 f\,df = \tfrac{1}{2}$$

with which Eq. (4-40) must be multiplied.

Alternatively, one can regard the self-energy as the interaction energy between all segments of the loop. Since the integrations in Eq. (4-44) count the interactions between two given elements twice, one again concludes that the result must be divided by 2.

As is expected from the discussion of the core region of straight dislocations, Eq. (4-44) diverges as the separation between elements, R, approaches zero. Cutoff procedures must be introduced to avoid the divergence, as discussed in detail in Chap. 6. This problem is characteristic of all types of interaction between two adjacent segments of the same dislocation line.

PROBLEMS

4-1. Consider the originally pure screw dislocation lying along ABC in Fig. 4-7. If AB is moved conservatively through positions $A'B'$, $A''B''$, etc., while BC is held fixed, a screw with a loop results. Is the loop a vacancy or an interstitial loop?

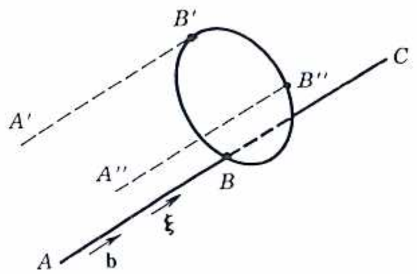

FIGURE 4-7. A screw dislocation along ABC. AB is moved conservatively in a clockwise manner viewed along $\boldsymbol{\xi}$ to form a closed loop $BB'B''B$.

4-2. The formation of the loop in Prob. 4-1 requires material transport. Where is the source or sink for this matter? *Hint*: Consider that the screw emerges normal to a free surface at A and study the configuration at A as the loop is formed.

4-3. Derive the stress field for a pure screw dislocation in an infinite medium from Eq. (4-30).

4-4. Demonstrate that the displacements $\mathbf{u}(\mathbf{r})$, given by Eq. (4-20), change discontinuously by $\Delta\mathbf{u} = \pm\mathbf{b}$ if the point $\mathbf{r}$ is intersected by the surface.

4-5. Show that the displacements given by Eq. (4-20) are dependent only on the configuration of the dislocation *line* forming the loop, and not on the shape of the surface A, provided the surface A is not intersected in the manner of Prob. 4-4.

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