The general solution of Eq. (2-59) is[11]

$$\mathcal{V}(\mathbf{r}) = \int \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\, dV' \tag{2-60}$$

Physically, Eq. (2-60) expresses the fact that the potential at $\mathbf{r}$ is the sum of individual potential contributions

$$d\mathcal{V} = \frac{\rho(\mathbf{r}')\, dV'}{|\mathbf{r}-\mathbf{r}'|}$$

produced by charges $\rho(\mathbf{r}')\, dV'$. For a point charge e at $\mathbf{r}_0$,

$$\rho(\mathbf{r}') = e\, \delta(\mathbf{r}'-\mathbf{r}_0) \tag{2-61}$$

where $\delta(\mathbf{r}'-\mathbf{r}_0)$ is the Dirac delta function, with the property

$$\int f(\mathbf{r}')\, \delta(\mathbf{r}'-\mathbf{r}_0)\, dV' = f(\mathbf{r}_0) \tag{2-62}$$

with $f(\mathbf{r}')$ any function. Inserting Eq. (2-61) into (2-60) gives the Green's function for the potential arising from a point charge,

$$\mathcal{V}(\mathbf{r}) = \frac{e}{|\mathbf{r}-\mathbf{r}_0|} \tag{2-63}$$

and from Eq. (2-59),

$$\nabla^2 \frac{1}{|\mathbf{r}-\mathbf{r}_0|} = -4\pi\, \delta(\mathbf{r}-\mathbf{r}_0) \tag{2-64}$$

Thus in the Green's function method the potential at any point can be determined by the integration of Eq. (2-60) over a continuous distribution of charge, each charge element $\rho(\mathbf{r}')\, dV'$ contributing to the potential like a point charge situated at $\mathbf{r}'$.

Application to Elasticity

In the elasticity case the elastic displacement $\mathbf{u}$ is analogous to the electrostatic potential $\mathcal{V}$, and the body forces $\mathbf{f}$ to the charge density ρ. Our aim is to develop an expression in the form of Poisson's equation (2-59), so that the displacements can be determined from an expression like Eq. (2-60). The procedure is somewhat more complicated for elasticity than in the preceding cases, because vector quantities are involved instead of scalars.

For the case of isotropy, the substitution of Eq. (2-47) into (2-8) yields the result

$$(\lambda+\mu)\frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j}+\mu\frac{\partial^2 u_i}{\partial x_j^{\,2}}+f_i=0$$

or, in vector notation,

$$(\lambda+\mu)\nabla(\nabla\cdot\mathbf{u})+\mu\nabla^2\mathbf{u}+\mathbf{f}=0 \tag{2-65}$$

Let us specialize to only one component of the body force and then generalize the result. Suppose that a point force $f_1\delta(\mathbf{r})$ is acting at the origin. Equation (2-65) then becomes

$$\begin{aligned}(\lambda+\mu)\frac{\partial}{\partial x_1}(\nabla\cdot\mathbf{u})+\mu\nabla^2 u_1+f_1\delta(\mathbf{r})&=0\\ (\lambda+u)\frac{\partial}{\partial x_2}(\nabla\cdot\mathbf{u})+\mu\nabla^2 u_2&=0\\ (\lambda+\mu)\frac{\partial}{\partial x_3}(\nabla\cdot\mathbf{u})+\mu\nabla^2 u_3&=0\end{aligned} \tag{2-66}$$

The results of potential theory[12] show that a vector $\mathbf{u}$ can be represented in terms of a scalar potential ϕ and a vector potential $\mathbf{A}$,

$$\mathbf{u}=\nabla\phi+\operatorname{curl}\mathbf{A} \tag{2-67}$$

In this representation, Eq. (2-66) becomes, after the sequence of some derivatives is interchanged,

$$\begin{aligned}(\lambda+2\mu)\nabla^2\frac{\partial\phi}{\partial x_1}+\mu\nabla^2\left(\frac{\partial A_3}{\partial x_2}-\frac{\partial A_2}{\partial x_3}\right)+f_1\delta(\mathbf{r})&=0\\ (\lambda+2\mu)\nabla^2\frac{\partial\phi}{\partial x_2}+\mu\nabla^2\left(\frac{\partial A_1}{\partial x_3}-\frac{\partial A_3}{\partial x_1}\right)&=0\\ (\lambda+2\mu)\nabla^2\frac{\partial\phi}{\partial x_3}+\mu\nabla^2\left(\frac{\partial A_2}{\partial x_1}-\frac{\partial A_1}{\partial x_2}\right)&=0\end{aligned} \tag{2-68}$$

[12] See I. S. Sokolnikoff and R. M. Redheffer, "Mathematics of Physics and Modern Engineering," McGraw-Hill, New York, 1966, p. 408.

It is easily verified that $\nabla^2|\mathbf{r}|=2/|\mathbf{r}|$, so that Eq. (2-64) is equivalent to

$$\nabla^2\nabla^2|\mathbf{r}|=-8\pi\,\delta(\mathbf{r}) \tag{2-69}$$

Now, if Eqs. (2-68) and (2-69) can be shown to be equivalent, then the Green's function analog will be established. A solution which gives this equivalency, as can be verified by direct substitution, is

$$\phi=\frac{f_1}{8\pi(\lambda+2\mu)}\frac{\partial r}{\partial x_1} \qquad A_2=-\frac{f_1}{8\pi\mu}\frac{\partial r}{\partial x_3}$$

$$A_1=0 \qquad A_3=\frac{f_1}{8\pi\mu}\frac{\partial r}{\partial x_2}$$

Substituting these definitions into (2-67) and generalizing the result to a point force $f_j\delta(\mathbf{r})$, one finds that the ith component of the displacement $u_{ij}(\mathbf{r})$ caused by a unit point force $f_j=1$ applied in the jth direction at the origin is

$$u_{ij}(\mathbf{r})=\frac{1}{8\pi\mu}\left(\delta_{ij}\nabla^2 r-\frac{\lambda+\mu}{\lambda+2\mu}\frac{\partial^2 r}{\partial x_i\,\partial x_j}\right) \tag{2-70}$$

Also, $u_{ij}(\mathbf{r})=u_{ji}(\mathbf{r})$, by symmetry. $u_{ij}(\mathbf{r})$ is called the *tensor Green's function for the elastic displacements*. A continuous distribution of forces $f_j(\mathbf{r})$ in an elastic medium causes displacements

$$u_i(\mathbf{r})=\int u_{ij}(\mathbf{r}-\mathbf{r}')f_j(\mathbf{r}')\,dV' \tag{2-71}$$

Equation (2-71) is analogous to Eq. (2-60), and the analogy with the electrostatic case is complete.

Equation (2-70) gives the response of an infinite body to a point force. In a finite body, boundary conditions at the surface must be satisfied. For example, no forces can act on a free surface,

$$\sigma_{ij}n_j=0 \tag{2-72}$$

where the n_j are the components of $\mathbf{n}$, the local surface normal. The displacements in a finite body subjected to a point force can be described as a superposition of the displacements (2-70) and displacements caused by "image" stresses applied on the external surface of the body in order to satisfy boundary conditions. The image displacements are continuous throughout the entire body. At a point sufficiently close to that at which the point force is applied, Eq. (2-70) gives the dominant part of the nonuniform displacement giving rise to stress.

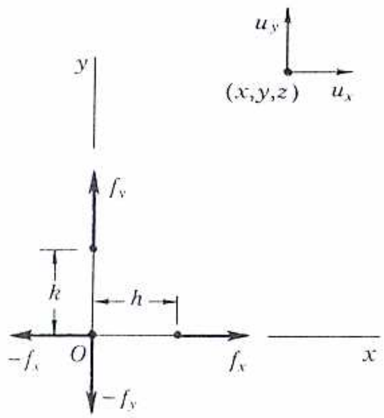

FIGURE 2-5. Displacements u_x and u_y associated with point forces f_x and f_y.

Point Source of Expansion

As an example of the application of Eq. (2-70), consider the displacement field of a point source of expansion. Three perpendicular double forces produce a stress field typical of such a source. In Fig. 2-5, let h, k, and l tend to zero, while $f_x h = f_y k = f_z l = M$ is kept constant. The force $-f_x$ at the origin produces a displacement $-f_x u_{11}(\mathbf{r})$, given by Eq. (2-70). The opposite force f_x at h produces a displacement $f_x u_{11}(\mathbf{r}) - f_x(\partial u_{11}/\partial x)h$. Thus the force pair produces a displacement $-(\partial u_{11}/\partial x) f_x h$. Proceeding similarly for the other force couples, one finds that the total displacement in the x direction is, from Eq. (2-70),

$$u_x = -M\frac{\partial}{\partial x}u_{11} - M\frac{\partial}{\partial y}u_{12} - M\frac{\partial}{\partial z}u_{13} = -\frac{M}{8\pi(\lambda+2\mu)}\frac{\partial}{\partial x}\nabla^2 r$$

$$= -\frac{M}{4\pi(\lambda+2\mu)}\frac{\partial}{\partial x}\frac{1}{r} = \frac{M}{4\pi(\lambda+2\mu)}\frac{x}{r^3} \tag{2-73}$$

u_y and u_z follow from symmetry. Inspection of Eq. (2-73) reveals that u_x, u_y, and u_z can be derived from a purely radial displacement

$$u_r = \frac{M}{4\pi(\lambda+2\mu)}\frac{1}{r^2} \tag{2-74}$$

This displacement is representative of a point of expansion with a strength

$$\delta v = 4\pi r^2 u_r = \frac{M}{\lambda+2\mu} \tag{2-75}$$

Suppose that the source of expansion is at the origin of a free sphere of radius R. In order to obtain a displacement field u_r consistent with the free surface condition [Eq. (2-72)], a term αr must be added to make σ_{rr} vanish at $r = R$:

$$u_r = \frac{\delta v}{4\pi r^2} + \alpha r \tag{2-76}$$

In spherical coordinates, the appropriate form of Eq. (2-7) is[13]

$$\sigma_{rr} = (\lambda + 2\mu)\frac{\partial u_r}{\partial r} + \frac{2\lambda u_r}{r}, \qquad \sigma_{\theta\theta} = \sigma_{\phi\phi} = 2(\lambda + \mu)\frac{u_r}{r} + \lambda\frac{\partial u_r}{\partial r} \tag{2-77}$$

Combining Eqs. (2-76) and (2-77) and setting $\sigma_{rr} = 0$ at $r = R$, one finds

$$\alpha = \frac{4\mu}{2\mu + 3\lambda}\,\frac{\delta v}{4\pi R^3} \tag{2-78}$$

The displacement at the surface is then

$$u_r(R) = \frac{3\lambda + 6\mu}{3\lambda + 2\mu}\,\frac{\delta v}{4\pi R^2} = \frac{3B + 4\mu}{3B}\,\frac{\delta v}{4\pi R^2}$$

This displacement produces an expansion of the total volume $V = (4\pi/3)R^3$, given by[14]

$$\delta V = \frac{3B + 4\mu}{3B}\,\delta v \tag{2-80}$$

[13] L. D. Landau and E. M. Lifshitz, "Theory of Elasticity," Pergamon, New York, 1959, p. 3.

[14] The relation between point force distributions and external volume changes is general, as shown by J. D. Eshelby, *Solid State Phys.*, 3: 79 (1956). We briefly outline the derivation. The total external volume change δV is related to the local dilatation ϵ_{ii} by

$$\delta V = \int \epsilon_{ii}\, dV = s_{iijk} \int \sigma_{jk}\, dV \tag{a}$$

where Hooke's law is used in the second step. An identity is

$$\frac{\partial \sigma_{jl} x_k}{\partial x_l} = x_k \frac{\partial \sigma_{jl}}{\partial x_l} + \sigma_{jl}\delta_{kl} = x_k \frac{\partial \sigma_{jl}}{\partial x_l} + \sigma_{jk} \tag{b}$$

Substituting (b) in (a) and using Eq. (2-2) gives

$$\delta V = s_{iijk}\left[\int \frac{\partial \sigma_{jl} x_k}{\partial x_l}\, dV + \int f_j x_k\, dV\right] \tag{c}$$

Recognizing $s_{iijk}\,\sigma_{jl} x_k$ as a vector v_l, one sees that the first integral is the volume integral of the divergence of **v**, so Stoke's theorem can be used to transform the integral to an integral over a closed surface A:

$$\delta V = s_{iijk}\left[\int \sigma_{jl} n_l x_k\, dA + \int f_j x_k\, dV\right] \tag{d}$$

Near the singularity, for $r \ll R$, the first term in Eq. (2-76) is dominant. In this region the dominant part of the internal stress is found from Eq. (2-77) to be

$$-\sigma_{rr} = \frac{\mu\,\delta v}{\pi r^3} \qquad \sigma_{\theta\theta} = \sigma_{\phi\phi} = \frac{\mu\,\delta v}{2\pi r^3} \tag{2-81}$$

The results of this section are useful in the treatment of point defects and are also applied in the next section. Other specific examples of the use of Eqs. (2-70) and (2-71) are given in Chap. 4. In closing this section we note that the analogy between electromagnetics and elasticity is more general than the special case invoked above. The general analogy is sometimes used in the continuum theory of dislocations.[14–17]

2-7. INTERACTION BETWEEN INTERNAL STRESS AND EXTERNAL STRESS

Approximations in Linear Elasticity

The main purpose of this section is to develop a very important theorem, valid in linear elasticity, about the elastic energy of a body containing internal stresses and subjected to external forces. The desired theorem follows directly from a discussion of the limitations of linear elasticity, without the complicated mathematical considerations of a formal development from the equations of linear elasticity.

In linear elasticity the superposition principle is assumed to hold true. The stresses and displacements caused by a set of forces acting on a body are assumed to be the sum of those caused by the individual forces. The linear-elasticity assumption clearly is invalid for large strains, where force-distance

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where **n** is a unit vector normal to A. If the surface is a free surface $\sigma_{ji}n_i$ must be zero and appropriate image terms can be incorporated into modified values of s'_{iijk}

$$\delta V = s'_{iijk} \int f_j x_k \, dV \tag{e}$$

for an isotropic elastic solid, $s'_{iijk} = 0$ unless $j = k$ and $s'_{iijj} = 1 + (4\mu/3B) = \delta/3B$, so

$$\delta V = \frac{\delta}{3B} \int f_j x_j \, dV \tag{2-79}$$

Thus a set of point forces **f** at positions **x** produce a volume change as in the special case of Eq. (2-80).

[15] R. deWit, *Solid State Phys.*, 10: 249 (1960).

[16] J. D. Eshelby, *Phys. Rev.*, 90: 248 (1953).

[17] F. R. N. Nabarro, *Phil. Mag.*, 42: 1224 (1951).