14.A Appendix: Integration of Eq. 6.4a

We will use indefinite integrals from a table by Bois (1961, pp. 17 and 20) and evaluate them from zero to infinite frequency. (A shorter solution can probably be obtained using complex integration in the *s* plane.) Starting with Eq. (6.4a) we obtain

$$|H(\omega)|^2 = \left|\omega_n^2 \frac{1 + j(2\alpha\zeta\omega/\omega_n)}{-\omega^2 + j2\zeta\omega_n\omega + \omega_n^2}\right|^2 = \omega_n^4 \frac{1 + (2\alpha\zeta\omega/\omega_n)^2}{\omega^4 + 2(2\zeta^2 - 1)\omega_n^2\omega^2 + \omega_n^4} . \quad (14.A.1)$$

$$B_n = \int_0^\infty |H(\omega)|^2 d\omega = \omega_n^4 I_1 + (2\alpha\zeta\omega_n)^2 I_2 . \quad (14.A.2)$$

For $\zeta<1$, I_1 and I_2 are each of the form[1]

$$I = \left[k_1 \ln\left\{\frac{\omega^2 + k_2\omega + k_3}{\omega^2 - k_2\omega + k_3}\right\} + \frac{1}{k_4}\tan^{-1}\frac{k_5\omega}{k_6^2 - \omega^2}\right]_0^\infty . \quad (14.A.3)$$

$$= \lim_{\varepsilon\to 0} \frac{1}{k_4}\left\{\tan^{-1}0 - \tan^{-1}\left(\frac{k_5(k_6+\varepsilon)}{-k_6\varepsilon}\right) + \tan^{-1}\left(\frac{k_5(k_6-\varepsilon)}{k_6\varepsilon}\right) + \tan^{-1}0\right\} \quad (14.A.4)$$

$$= \frac{1}{k_4}\left\{0 + \frac{\pi}{2} + \frac{\pi}{2} + 0\right\} = \frac{\pi}{k_4} \quad (14.A.5)$$

k_4 equals k_{41} or k_{42} for I_1 or I_2 respectively. For I_2,

$$k_{42} = 4\omega_n \sin\frac{\cos^{-1} k_7}{2} = 4\omega_n\sqrt{\frac{1 - k_7}{2}} = 4\omega_n\zeta , \quad (14.A.6)$$

where

$$k_7 = 1 - 2\zeta^2 . \quad (14.A.7)$$

For I_1,

$$k_{41} = \omega_n^2 k_{42} . \quad (14.A.8)$$

Combining Eq. (14.A.2) through (14.A.8), we obtain

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[1] Because of the pole at $\omega = k_6$, evaluate the $\tan^{-1}(\,)$ from $\omega = 0$ to $\omega = k_6 - \delta$ and from $\omega = k_6 + \delta$ to $\omega = \infty$, where δ approaches 0.

$$B_n = \pi\left[\frac{\omega_n}{4\zeta} + \alpha^2\omega_n\zeta\right] = \frac{\pi\omega_n}{2}\left[\frac{1}{2\zeta} + \alpha^2 2\zeta\right] = \frac{\omega_n}{4}\left[\frac{1}{2\zeta} + \alpha^2 2\zeta\right]\frac{\text{cycle}}{\text{rad}}. \tag{14.A.9}$$

This is Eq. (14.10).

Bois (1961) does not give Eq. (14.A.2) for $\zeta \geq 1$. We will not bother with $\zeta = 1$, since it applies to a zero-width interval and we expect continuity, but we will demonstrate that Eq. (14.A.9) is valid also for $\zeta > 1$. For $\zeta > 1$, Bois gives

$$I = \left[k_1 \tan^{-1}(k_2\omega) + k_3 \tan^{-1}(k_4\omega)\right]_0^\infty = \frac{\pi}{2}(k_1 + k_3), \tag{14.A.10}$$

where, for I_1,

$$k_{11} = \frac{2}{k_5\sqrt{2(k_6 - k_5)}} \ ; \ k_{31} = \frac{-2}{k_5\sqrt{2(k_6 + k_5)}} \tag{14.A.11}$$

so

$$k_{11} + k_{31} = \sqrt{2}\frac{\sqrt{k_6 + k_5} - \sqrt{k_6 - k_5}}{k_5\sqrt{k_6^2 - k_5^2}} = \frac{1}{\sqrt{2}\,\omega_n^2}\frac{\sqrt{k_6 + k_5} - \sqrt{k_6 - k_5}}{k_5}. \tag{14.A.12}$$

Here,

$$k_5 = 4\zeta\omega_n^2\sqrt{\zeta^2 - 1} \ ; \ k_6 = 2\left(2\zeta^2 - 1\right)\omega_n^2. \tag{14.A.13}$$

For I_2,

$$k_{12} = -\frac{k_6 - k_5}{k_5\sqrt{2(k_6 - k_5)}} = -\frac{\sqrt{k_6 - k_5}}{\sqrt{2}k_5}, \ k_{32} = \frac{k_6 + k_5}{k_5\sqrt{2(k_6 + k_5)}} = \frac{\sqrt{k_6 + k_5}}{\sqrt{2}k_5} \tag{14.A.14}$$

and

$$k_{12} + k_{32} = \frac{\sqrt{k_6 + k_5} - \sqrt{k_6 - k_5}}{\sqrt{2}k_5}. \tag{14.A.15}$$

Substituting Eq. (14.A.12) and (14.A.15) each into Eq. (14.A.10) and then the two resulting versions of (14.A.10) into (14.A.2), we obtain

$$B_n = \frac{\pi}{2}\frac{\omega_n^2}{k_5}\left\{\frac{1}{\sqrt{2}}\left[\sqrt{k_6+k_5}-\sqrt{k_6-k_5}\right]+(\alpha\zeta)^2 2\sqrt{2}\left[\sqrt{k_6+k_5}-\sqrt{k_6-k_5}\right]\right\} \tag{14.A.16}$$

$$= \frac{\omega_n^2}{4k_5}\left\{\frac{1}{\sqrt{2}}\left[\sqrt{k_6+k_5}-\sqrt{k_6-k_5}\right]+(\alpha\zeta)^2 2\sqrt{2}\left[\sqrt{k_6+k_5}-\sqrt{k_6-k_5}\right]\right\}\frac{\text{cycle}}{\text{rad}} \ . \tag{14.A.17}$$

Although the equivalence between Eq. (14.A.9) and (14.A.17) is not apparent, evaluation for $1 < \zeta \leq 3$ showed the same values from each to within a relative error of one part in 10^{15}. Therefore, Eq. (14.A.9) will be used for all ζ.

14.B APPENDIX: LOOP OPTIMIZATION IN THE PRESENCE OF NOISE

This appendix endeavors to provide a theoretical basis for paragraph 14.3.

Rather than attempting to repeat Jaffe and Rechtin's (1955) rather lengthy mathematical development, we will base our discussion on two other procedures, minimization using Lagrange multipliers and the Weiner filter , which will be summarized but not derived here. In that way the results will be made available and a theoretical basis provided. However, those who are most familiar with these procedures will undoubtedly achieve a better understanding of the process.

14.B.1 Background

14.B.1.1 Minimization Under Constraint — Use of Lagrange Multipliers

To minimize a function $f(x,y,...)$ subject to the constraint that another function has a given value, $g(x,y,...) = g'$, minimize $[f(x,y,...) + \lambda^2 g(x,y,...)]$, where λ^2 is called the Lagrange multiplier. And λ^2 can be selected, perhaps subsequent to the minimization and perhaps implicitly, such that $g(x,y,...) = g'$. Then both the constraint and the minimization will occur simultaneously.

14.B.1.2 Weiner Filter. The Weiner filter is the name sometimes given to a filter that minimizes the mean square error between the actual output and the desired output. The procedure includes safeguards to insure that the filter is theoretically realizable in that it has no right-half-plane (RHP) poles.

Let $S(f)$ be the phase power spectral density (PSD) of the input (signal plus noise). Since PSD is an absolute value, it will be of the form $S(f) = N_0/P_c + X(f)X^*(f)$, where $X(s)$ is the desired input (equal to the desired output in our development). It is possible to write $S(f)$ in the form $S(f) = [\Psi(s)\Psi(-s)]_{s=j2\pi f}$ where $\Psi(s)$ has only left-half-plane (LHP) poles and zeroes while $\Psi(-s)$ has only RHP poles and zeros[1]. Then the optimum filter for minimum mean square error in reproduction of the desired input has transfer function

[1] Symmetry about the real axis is necessary for transforms of real functions of time and symmetry about the imaginary axis corresponds to zero phase shift, which is necessary for an absolute value.

$$H(s)\big|_{\text{opt}} = \frac{1}{\psi(s)}\left[\frac{|X(s)|^2}{\psi(-s)}\right]_+ , \tag{14.B.1}$$

where $[\ ]_+$ indicates the realizable parts from the Heaviside expansion of [ ], that is, those parts having LHP poles. In other words, the term in the [ ] is formed by taking the indicated ratio and removing any RHP poles. Here $|X(s)|^2$ means $X(s)\,X(-s)$ [Truxal, 1955, pp. 469-471].

14.B.2 Explanation of Jaffe and Rechtin's Procedure

We wish to minimize $\sigma^2_{\varphi,\text{out}}$ under the constraint that $E = E_0$, the value that we are willing to accept for the integrated square error. Therefore, according to paragraph 14.B.1.1, we minimize $[\sigma_{\varphi,\text{out}}{}^2 + \lambda^2 E^2]$, choosing λ such that $E = E_0$ with the values of the parameters that occur at the minimum. But this is what the Wiener filter would do if the input were $\lambda\Phi(s)$, where $\Phi(s)$ is the Laplace transform of $\varphi_{\text{in}}(t)$. It would minimize the total square error, consisting of the mean square filter response to the input noise plus the integrated square difference between output and input, $\lambda^2 E^2$.[2] The solution is given by Eq. (14.B.1) with $X(s) \to \lambda\Phi(s)$.

$$H(s)\big|_{\text{opt}} = \frac{\lambda^2}{\psi(s)}\left[\frac{|\Phi(s)|^2}{\psi(-s)}\right]_+ . \tag{14.B.2}$$

This will give the optimum filter shape, but not all the parameters are determined; λ is still to be chosen. We will probably do that implicitly by writing $\sigma_{\varphi,\text{out}}{}^2$ and E^2 for the loop and choosing a parameter (e.g., ω_n) to give the allowed value of E^2 or $\sigma_{\varphi,\text{out}}{}^2$.

14.B.3 Detailed Calculation for a Phase Step

For a Phase Step, θ/s

$$X(s) = \lambda\theta/s\ . \tag{14.B.3}$$

[2] Jaffe & Rechtin (1955) use a more fundamental development, which is related to Wiener filter theory, whereas this presentation attempts to make use of the developed filter theory more directly. Here E is an energy related to the Fourier transform of a single event (e.g., a step). In the usual Wiener filter, E is a power related to the Fourier transform-in-the-limit of a continuous process. In both cases noise power is minimized.

The two-sided PSD of a signal plus noise is

$$S_2(s) = \frac{N_0}{2P_c} + \frac{\lambda^2\theta^2}{(s)(-s)} = \left(\sqrt{\frac{N_0}{2P_c}} + \frac{\lambda\,\theta}{s}\right)\left(\sqrt{\frac{N_0}{2P_c}} + \frac{\lambda\,\theta}{-s}\right). \qquad (14.B.4)$$

The 2 in the denominator of the noise term is because we use two-sided density with Fourier or Laplace transforms. From this we obtain

$$\psi(s) = \sqrt{\frac{N_0}{2P_c}} + \frac{\lambda\theta}{s}. \qquad (14.B.5)$$

We begin substituting into Eq. (14.B.2).

$$\lambda^2\frac{|\Phi(s)|^2}{\psi(-s)} = -\lambda^2\frac{\theta^2}{s^2}\frac{1}{\sqrt{\frac{N_0}{2P_c}} - \lambda\frac{\theta}{s}} = \frac{\lambda^2\theta^2}{s\left(\lambda\,\theta - s\sqrt{\frac{N_0}{2P_c}}\right)} = \lambda\frac{\theta}{s} + \frac{b}{s - \lambda\theta\Big/\sqrt{\frac{N_0}{2P_c}}} \qquad (14.B.6)$$

The expression on the right, obtained by a Heaviside expansion, has an RHP pole, which is dropped in Eq. (14.B.2),

$$\lambda^2\left[\frac{|\Phi(s)|^2}{\psi(-s)}\right]_+ = \lambda\frac{\theta}{s}; \qquad (14.B.7)$$

$$H(s)\big|_{\text{opt}} = \frac{1}{\sqrt{\frac{N_0}{2P_c}} + \lambda\frac{\theta}{s}}\left(\lambda\frac{\theta}{s}\right) = \frac{\lambda\theta\Big/\sqrt{\frac{N_0}{2P_c}}}{s + \lambda\theta\Big/\sqrt{\frac{N_0}{2P_c}}} = \frac{K}{s+K}, \qquad (14.B.8)$$

where

$$K = \lambda\,\theta\Big/\sqrt{\frac{N_0}{2P_c}}. \qquad (14.B.9)$$

Having obtained this general form, we then write the mean square error E and the phase variance $\sigma_{\varphi,\text{out}}$ in terms of K and choose K to give the best tradeoff between E and $\sigma_{\varphi,\text{out}}$. One could solve for λ but it would only be of value in determining K in terms of E and

$\sigma_{\varphi,\mathrm{out}}$ and it is simpler to do that directly.

14.B.4 A Simplified Formula for $H(j\omega)|_{\mathrm{opt}}$

Blanchard (1976, p. 162) gives

$$H\left(j\omega\right)\Big|_{\mathrm{opt}} = 1 - \frac{\sqrt{N_0/A^2}}{\psi\left(j\omega\right)} \ . \tag{14.B.10}$$

Apparently this is equivalent to Eq. (14.B.2) for the group of waveforms that are considered here, which have Laplace transforms of the form $(k/s)^n$ with $n \le 3$.