11.H APPENDIX: SHAPE OF OUTPUT POWER SPECTRUM

The phase power spectral density of Fig. 11.12*a* results in a power spectrum, shown as P_{sb} at Fig. 11.12*b*, which is given by Eq. (11.23) if the deviation is small enough for Eq. (11.29) to be satisfied. When the deviation increases to the point where that restriction is not well satisfied, we can compute the approximate changes in the spectral shape of P_{sb} by observing the effects of one component of S_φ in a narrow bandwidth.

While, for small modulation index, the central line has power P_c, which it has also in the absence of modulation, a more careful calculation can be made by considering first what happens if a single modulation component is accounted for. For small *m,* the power of the central line can be written

$$P_c = PJ_0^2(m_1) \approx P\left[1 - \frac{m_1^2}{4}\right]^2 \approx P\left[1 - \frac{m_1^2}{2}\right] \tag{11.H.1}$$

and the power of each of the two first sidebands created by the modulation (Fig. 11.H.1) is

$$P_1 = PJ_1^2(m_1) \approx P\frac{m_1^2}{4} \tag{11.H.2}$$

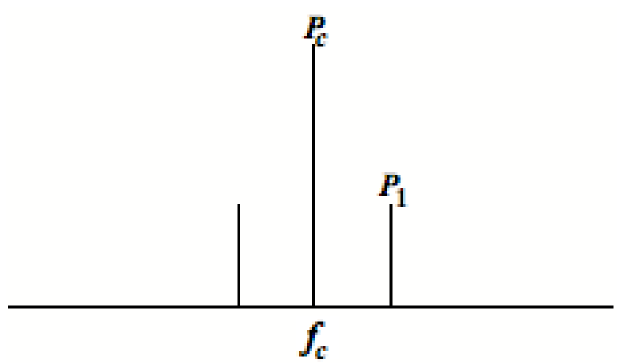

Fig. 11.H.1 One pair of noise sidebands.

When a second modulation component is accounted for (Fig. 11.H.2), the carrier power becomes

$$P_c \approx P\left[1-\frac{m_1^2}{2}\right]\left[1-\frac{m_2^2}{2}\right] \tag{11.H.3}$$

$$\approx P\left[1-\frac{m_1^2}{2}-\frac{m_2^2}{2}\right] \tag{11.H.4}$$

and the powers in each of the two first sidebands (on one side of the carrier) are

$$P_1 \approx P\frac{m_1^2}{4}\left[1-\frac{m_2^2}{2}\right] \tag{11.H.5}$$

and

$$P_2 \approx P\left[1-\frac{m_1^2}{2}\right]\frac{m_2^2}{4} \tag{11.H.6}$$

respectively.

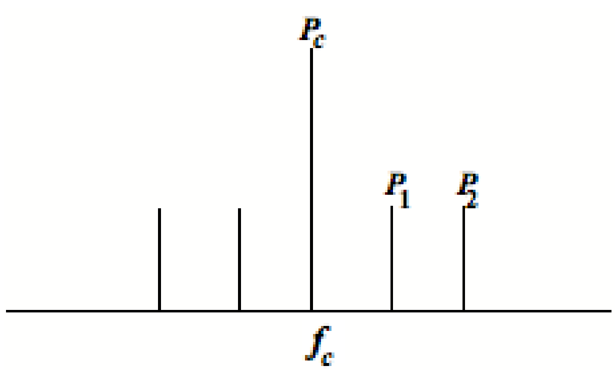

Fig. 11.H.2 Two pairs of noise sidebands.

As we continue this process with more modulation components we approach a condition similar to modulation by noise. The power in the central line [Eq. (11.H.4)] becomes[1]

[1] In decibels, the change in carrier power is
$10\log_{10}\exp\left(-\sigma^2_{\varphi,\mathrm{out}}\right) = -8.7\,\mathrm{dB}\left(\sigma_{\varphi,\mathrm{out}}/\mathrm{rad}\right)$.

$$P_c = P\left(1 - \frac{1}{2}\sum_i m_i^2\right) = P\left(1 - \sigma_{\varphi,\text{out}}^2\right) \approx P\exp\left(-\sigma_{\varphi,\text{out}}^2\right) \qquad (11.\text{H}.7)$$

and the sideband power [Eq. (11.H.5)] at an offset f_m becomes

$$P_{sb}\left(f_c \pm f_m\right) = P\left(1 - \sum_i \frac{m_i^2}{2}\right)\frac{m_m^2}{4}. \qquad (11.\text{H}.8)$$

In a differential bandwidth we can write the ratio of this sideband power to total power as

$$L\left(f_c \pm f_m\right) df_m = \left(1 - \sigma_{\varphi,\text{out}}^2\right)\left(\frac{S_\text{out}\left(f_m\right)}{2}\right) df_m, \qquad (11.\text{H}.9)$$

where we have again equated the sum of individual mean square phase deviations to the variance of the phase.

Thus the power of both the central line and the sideband density are reduced by the same factor, ($1 - \sigma^2_{\varphi,\text{out}}$), while the general shape remains that determined by the shape of $S_{\varphi,\text{out}}$. Note that, while this is intended to improve the estimate of sideband levels at higher values of *m,* it nevertheless depends on the small modulation index approximations used in Eqs. (11.H.2), (11.H.3), and so forth, so it is only a valid approximation for $m < 1$ or so.

The total power is not changed by phase modulation. It is still related to the sinusoid's amplitude A by $P = A^2/2$. One check on the accuracy of our approximations is to consider to what degree they have modified this relationship. Before we had taken these reductions into consideration, we could have written the total power as

$$P_{T0} \approx P\left[1 + \int_0^\infty S_\text{out}\left(f_m\right) df_m\right] = P\left[1 + \sigma_{\varphi,\text{out}}^2\right]. \qquad (11.\text{H}.10)$$

Whether the approximation $L_\varphi = S_\varphi/2$, on which this is based, is accurate or not, the integral of the sideband power produces some additional power and that represents an error. Here that error is $P\sigma^2_{\varphi,\text{out}}$. The question is, does the use of Eq. (11.H.7) and

(11.H.9) reduce this error? Repeating the above process, but using the reduced sideband levels from those equations, we obtain

$$P_{T1} \approx P\left[1-\sigma_{\varphi,\text{out}}^{2}\right]\left[1+\int_{0}^{\infty} S_{\text{out}}\left(f_m\right) df_m\right] \tag{11.H.11}$$

$$= P\left[1-\sigma_{\varphi,\text{out}}^{2}\right]\left[1+\sigma_{\varphi,\text{out}}^{2}\right] = P\left[1-\sigma_{\varphi,\text{out}}^{4}\right]. \tag{11.H.12}$$

This an improvement if $\sigma < 1$ and the smaller is σ the greater will be the relative improvement. For example, if $\sigma^2 = 0.1$ rad^2, the error in total power in Eq. (11.H.11) is 10% of the carrier power while in Eq. (11.H.12) it is only 1%.