SUPPLEMENTS TO MATHEMATICAL METHODS IN SCIENCE AND ENGINEERING
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CHAPTER 8: STURM-LIOUVILLE THEORY

I. Solutions or Hints to Selected Problems:

1. **(Problem 8.4)** A function $y(x)$ is to be a finite solution of the differential equation

$$x(1-x)\frac{d^2y}{dx^2} + (\frac{3}{2} - 2x)\frac{dy}{dx} + \left[\lambda - \frac{(2+5x-x^2)}{4x(1-x)}\right] y(x) = 0,$$

in the entire interval $x \in [0,1]$.

i) Show that this condition can only be satisfied for certain values of λ and write the solutions explicitly for the lowest three values of λ.

ii) Find the weight function.

iii) Show that the solution set $\{y_\lambda(x)\}$ is orthogonal with respect to $w(x)$ found above.

Solution:

This is a fun problem that involves some calculations. However, if you go through it on your own, it will clear a lot of questions for you.

If you think you need some directions, here is a list of the strategic steps that have to be taken towards the solution:

i) Analyze the singular points.

ii) Substitute a series solution of the form

$$y(x) = \sum_n a_n x^{n+r},$$

this will lead to a three-term recursion relation. You should be able to see this without this substitution, however, since we will use this recursion relation later, it is better to derive it now.

iii) To obtain a two-term recursion relation, derive a suitable transformation.

iv) Using this transformation, obtain a new differential equation, which will give you a two-term recursion relation.

v) Check the convergence of the series solution you found and find the allowed values of λ to satisfy the boundary conditions.

vi) Write the corresponding Sturm-Liouville operator and see whether it is self adjoint or not.

vii) Find the weight function and write the differential operator in first-canonical form [Eq. (8.6)].

viii) Show the orthogonality of the solution set.

Finally, As an extra calculation, go back to the three-term recursion relation found in part (ii) and show that the series solution it gives agrees with the one you obtained. Checking the first couple of terms is sufficient.

For additional features of the Frobenius method see Bayin (2008).

II. Useful Sites

Additional references and other useful information about the Sturm-Liouville theory can be found in the following site:

http://en.wikipedia.org/wiki/Sturm-Liouville_theory.

Selçuk Bayin (October, 2008)