Supplements to Mathematical Methods in Science and Engineering
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CHAPTER 9: STURM-LIOUVILLE SYSTEMS AND THE FACTORIZATION METHOD

I. Solutions or Hints to Selected Problems:

1. **(Problem 9.1)** Starting from the first canonical form of the Sturm-Liouville equation:

$$\frac{d}{dx}\left[p(x)\frac{d\Psi(x)}{dx}\right] + q(x)\Psi(x) + \lambda w(x)\Psi(x) = 0, \quad x \in [a,b]\,, \tag{0.1}$$

derive the second canonical form:

$$\frac{d^2 y_\lambda^m(z)}{dz^2} + \{\lambda + r(z,m)\}\, y_\lambda^m(z) = 0, \tag{0.2}$$

where

$$\begin{aligned} r(z,m) = {} & \frac{q}{w} + \frac{3}{16}\left[\frac{1}{w}\frac{dw}{dz} + \frac{1}{p}\frac{dp}{dz}\right]^2 \\ & - \frac{1}{4}\left[\frac{2}{pw}\frac{dp}{dz}\frac{dw}{dz} + \frac{1}{w}\frac{d^2w}{dz^2} + \frac{1}{p}\frac{d^2p}{dz^2}\right], \end{aligned} \tag{0.3}$$

by using the transformations

$$y(z) = \Psi(x)\left[w(x)p(x)\right]^{1/4} \tag{0.4}$$

and

$$dz = dx\left[\frac{w(x)}{p(x)}\right]^{1/2}. \tag{0.5}$$

Solution:

First write the differential operator

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] \tag{0.6}$$

in terms of z as

$$\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] = w\frac{d^2}{dz^2} + \frac{1}{2}\left[\sqrt{\frac{w}{p}}\frac{dp}{dx} + \sqrt{\frac{p}{w}}\frac{dw}{dx}\right]\frac{d}{dz}. \tag{0.7}$$

Next operate with this on $\Psi(z)$:

$$w\frac{d^2\Psi}{dz^2} + \frac{1}{2}\left[\sqrt{\frac{w}{p}}\frac{dp}{dx} + \sqrt{\frac{p}{w}}\frac{dw}{dx}\right]\frac{d\Psi}{dz}. \tag{0.8}$$

Using

$$\Psi(z) = [w(x)p(x)]^{-1/4}\, y(z) \tag{0.9}$$

and considering p, q, w and y as functions of z, evaluate the derivatives and simplify. Finally, use these in Equation (0.1) to obtain the desired result as

$$\frac{d^2 y_\lambda^m(z)}{dz^2} + \{\lambda + r(z,m)\}\, y_\lambda^m(z) = 0. \tag{0.10}$$

2. **(Problem 9.2)** Derive the normalization constants in

$$Y_l^m(\theta,\phi) = \sqrt{\frac{2l+1}{2}\frac{(l-m)!}{(l+1)!}\frac{1}{2\pi}}\,[L_+]^m\, P_l(\cos\theta) \tag{0.11}$$

and

$$Y_l^{-m}(\theta,\phi) = \sqrt{\frac{2l+1}{2}\frac{(l-m)!}{(l+1)!}\frac{1}{2\pi}}\,[L_-]^m\, P_l(\cos\theta). \tag{0.12}$$

Check these formulas by writing $Y_{2m}(\theta,\phi)$, where $m = -2,-1,0,1,2$, explicitly.

Solution:

We show only the first formula [Eq. (0.11)]. First write [Eq. (9.173)]

$$Y_{l,m+1}(\theta,\phi) = \frac{1}{\sqrt{(l-m)(l+m+1)}}L_+ Y_{lm}(\theta,\phi), \tag{0.13}$$

where [Eq. (9.171)]

$$L_+ = e^{i\phi}\left[\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right]. \tag{0.14}$$

Next we operate $m-$fold on Y_{l0} with L_+ to write

$$Y_{lm}(\theta,\phi) = N_{lm}(L_+)^m Y_{l0}(\theta,\phi). \tag{0.15}$$

We finally evaluate N_{lm} as

$$N_{nm} = \frac{1}{\sqrt{l(l-1)\cdots(l-m+1)(l+1)(l+2)\cdots(l+m)}} \tag{0.16}$$

$$= \sqrt{\frac{(l-m)!l!}{(l+m)!l!}} \tag{0.17}$$

$$= \sqrt{\frac{(l-m)!}{(l+m)!}}. \tag{0.18}$$

Note that

$$Y_{l0} = \sqrt{\frac{2l+1}{2}} P_l(\cos\theta)\frac{e^{i(0)\phi}}{\sqrt{2\pi}} \tag{0.19}$$

$$= \sqrt{\frac{2l+1}{2}\frac{1}{2\pi}} P_l(\cos\theta). \tag{0.20}$$

For the second equation [Eq. (0.12)], use the relations given in Equations (9.177) and (9.178).

3. **(Problem 9.4)** Derive Equation (9.195), which is given as

$$\frac{d^2V(z)}{dz^2} + \left[\lambda + \frac{l(l+1)}{\cosh^2\theta}\right] V(z) = 0. \tag{0.21}$$

Solution:

Start with Equation (9.193):

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta} + \left[l(l+1) - \frac{m^2}{\sin^2\theta}\right]\Theta(\theta) = 0 \tag{0.22}$$

and make the transformations

$$z = \ln\left[\tan(\theta/2)\right], \quad \Theta(\theta) \to V(z), \tag{0.23}$$

to write the derivatives

$$\frac{d}{d\theta} = \frac{1}{\sin\theta}\frac{d}{dz}, \tag{0.24}$$

$$\frac{d^2}{d\theta^2} = \frac{1}{\sin^2\theta}\frac{d^2}{dz^2} - \frac{1}{\sin^2\theta\cos^{-1}\theta}\frac{d}{dz}. \tag{0.25}$$

Finally, substitute into Equation (0.22) and simplify.

4. **(Problem 9.11)** Use the factorization method to show that the spherical Hankel functions of the first kind:
$$h_l^{(1)} = j_l + in_l \tag{0.26}$$
can be expressed as
$$h_l^{(1)}(x) = (-1)^l x^l \left[\frac{1}{x}\frac{d}{dx}\right]^l h_0^{(1)}(x) \tag{0.27}$$
$$= (-1)^l x^l \left[\frac{1}{x}\frac{d}{dx}\right]^l \left(\frac{-ie^{ix}}{x}\right). \tag{0.28}$$
Hint: Introduce
$$u_l(x) = y_l(x)/x^{l+1} \tag{0.29}$$
in
$$y_l'' + \left[1 - \frac{l(l+1)}{x^2}\right] y_l = 0. \tag{0.30}$$

Solution:

Helmholz equation:
$$\nabla^2\Psi + k^2\Psi = 0, \tag{0.31}$$
in spherical polar coordinates can be separated by the substitution
$$\Psi(r,\theta,\phi) = R_l(r)Y_{lm}(\theta,\phi), \tag{0.32}$$
where $Y_{lm}(\theta,\phi)$ are the spherical harmonics and $R_l(r)$ satisfies
$$\frac{d^2R_l}{dr^2} + \frac{2}{r}\frac{dR_l}{dr} + \left[k^2 - \frac{l(l+1)}{r^2}\right] R_l(r) = 0. \tag{0.33}$$
If we substitute
$$R_l(r) = \frac{g_l(r)}{\sqrt{r}} \tag{0.34}$$
we obtain
$$r^2 g_l'' + rg_l' + \left[k^2r^2 - (l+\frac{1}{2})^2\right] g_l(r) = 0. \tag{0.35}$$
Since the spherical Bessel functions, $j_l(kr)$, $n_l(kr)$, (kr), satisfy the same differential equation with $R_l(kr)$ [Eq. (6.43) and Eq. (0.33)], we can write
$$R_l(r) = \frac{g_l(r)}{\sqrt{r}} = \sqrt{\frac{\pi}{2}}\frac{J_{l+1/2}(kr)}{\sqrt{kr}} = j_l(kr). \tag{0.36}$$

Since they all satisfy the same differential equation, similar expressions for the other spherical Bessel functions [Eq. (6.41)] can be written. We now substitute

$$R_l(kr) = \frac{y_l(kr)}{kr} = \frac{y_l(x)}{x}, \tag{0.37}$$

where $x = kr$, and obtain the differential equation that $y_l(x)$ satisfies:

$$\frac{d^2 y_l}{dx^2} + \left[1 - \frac{l(l+1)}{x^2}\right] y_l(x) = 0. \tag{0.38}$$

This is in second canonical form. Using the table given in Infeld and Hull (Bayin, 2006) we can write the normalized ladder operators [Eq. (9.48)] as

$$\pounds_{\pm} = \pm\frac{d}{dx} - \frac{l}{x}, \tag{0.39}$$

which allows us to write

$$y_{l+1}(x) = \left[\frac{d}{dx} - \frac{l+1}{x}\right] y_l(x), \tag{0.40}$$

$$y_{l-1}(x) = \left[-\frac{d}{dx} - \frac{l}{x}\right] y_l(x). \tag{0.41}$$

We now use the substitution

$$u_l(x)x^{l+1} = y_l(x) \tag{0.42}$$

to obtain

$$u_{l+1} = \left[\frac{1}{x}\frac{d}{dx}\right] u_l(x). \tag{0.43}$$

For $l = 0$, this gives

$$u_1 = \left[\frac{1}{x}\frac{d}{dx}\right] u_0(x). \tag{0.44}$$

Iterating this formula l times, we obtain

$$u_l = \left[\frac{1}{x}\frac{d}{dx}\right]^l u_0(x). \tag{0.45}$$

Since we can write

$$R_l(x) = h_l^{(1)}(x), \tag{0.46}$$

Equation (0.37) gives

$$xh_l^{(1)}(x) = y_l(x). \tag{0.47}$$

Also using Equation (0.42) we have

$$\begin{aligned} u_l(x) &= \frac{xh_l^{(1)}(x)}{x^{l+1}} \\ &= \frac{h_l^{(1)}(x)}{x^l}, \end{aligned} \tag{0.48}$$

which for $l = 0$ becomes

$$u_0(x) = h_0^{(1)}(x). \tag{0.49}$$

Finally, substituting Equations (0.48) and (0.49) into Equation (0.45) we obtain the desired expression:

$$h_l^{(1)}(x) = (-1)^l x^l \left[\frac{1}{x}\frac{d}{dx}\right]^l h_0^{(1)}(x). \tag{0.50}$$

We have introduced the factor $(-1)^l$ to match the conventional phase.

5. **(Problem 9.12)** Using the factorization method, find a recursion relation relating the normalized eigenfunctions $y(n, l, r)$ of the differential equation

$$\frac{d^2y}{dr^2} + \left[\frac{2}{r} - \frac{l(l+1)}{r^2}\right] y - \frac{1}{n^2} y = 0 \tag{0.51}$$

to the eigenfunctions with $l \pm 1$.

Hint: First show that

$$l = n-1, n-2, ..., \quad l = \text{integer}$$

and the normalization is

$$\int_0^\infty y^2(n, l, r) dr = 1.$$

Solution:

First convert to second canonical form and then use the table given in Infeld and Hull (Bayin, 2006) to find the normalized ladder operators.

6. **(Problem 9.15)** The spherical Bessel functions $j_l(x)$ are related to the solutions of

$$\frac{d^2y_l}{dx^2} + \left[1 - \frac{l(l+1)}{x^2}\right] y_l(x) = 0,$$

(regular at $x = 0$) by

$$j_l(x) = \frac{y_l(x)}{x}.$$

Using the factorization technique, derive recursion formulae

i) Relating $j_l(x)$ to $j_{l+1}(x)$ and $j_{l-1}(x)$.

ii) Relating $j_l'(x)$ to $j_{l+1}(x)$ and $j_{l-1}(x)$.

Solution:

This is already in second canonical form. First use the table in Infeld and Hull (Bayin, 2006) to find the normalized ladder operators and then generate the desired recursion relations.

II. Useful Sites

Additional references and other useful information about the Sturm-Liouville theory can be found in the following sites:

`http://en.wikipedia.org/wiki/Ladder_operators`,
`http://scienceworld.wolfram.com/physics/LadderOperator.html`.

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