SUPPLEMENTS TO MATHEMATICAL METHODS IN SCIENCE AND ENGINEERING
http://www.wiley.com/WileyCDA/WileyTitle/productCd-0470041420.html

CHAPTER 18: INTEGRAL EQUATIONS

I. Neumann Series:

We often encounter cases where a given second-order linear differential operator,

$$\pounds = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q^{(0)}(x) + q^{(1)}(x),\ x \in [a,b], \tag{0.1}$$

differs from an exactly solvable Sturm-Liouville operator,

$$\pounds_0 = \frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q^{(0)}(x), \tag{0.2}$$

by a small term, $q^{(1)}(x)$, compared to $q^{(0)}(x)$. Since the eigenvalue problem for $\pounds_0$ is exactly solvable, it yields a complete and orthonormal set of eigenfunctions, u_i, which satisfy the eigenvalue equation

$$\pounds_0 u_i + \lambda_i u_i = 0, \tag{0.3}$$

where λ_i are the eigenvalues. We now consider the eigenvalue equation for the general operator $\pounds$:

$$\pounds\Psi(x) + \lambda\Psi(x) = 0, \tag{0.4}$$

and write it as

$$\pounds_0\Psi(x) + \lambda\Psi(x) = -q^{(1)}\Psi(x). \tag{0.5}$$

In general, the above equation is given as

$$\pounds_0\Psi(x) + \lambda\Psi(x) = f(x, \Psi(x)), \tag{0.6}$$

where the inhomogeneous term usually corresponds to sources or interactions. We confine our discussion to cases where $f(x, \Psi)$ is separable:

$$f(x, \Psi(x)) = h(x)\Psi(x). \tag{0.7}$$

This covers a wide range of physically interesting cases. For example, in scattering problems the time independent Schrödinger equation is written as

$$\overrightarrow{\nabla}^2\Psi(\overrightarrow{r}) + \frac{2mE}{\hbar^2}\Psi(\overrightarrow{r}) = \frac{2m}{\hbar^2}V(\overrightarrow{r})\Psi(\overrightarrow{r}), \tag{0.8}$$

where $V(\overrightarrow{r})$ is the scattering potential.

As we discussed in Chapters 18 and 19 of Bayin (2006), the general solution of Equation (0.5) can be written as

$$\begin{aligned}\Psi(x) &= \Psi^{(0)}(x) + \int dx' G(x,x')h(x')\Psi(x') \\ &= \Psi^{(0)}(x) + \int dx' K(x,x')\Psi(x'),\end{aligned} \tag{0.9}$$

where $G(x,x')$ is the Green's function, $K(x,x') = G(x,x')h(x')$ is called the kernel of the integral equation and $\Psi^{(0)}(x)$ is the known solution of the homogeneous equation:

$$\pounds_0\Psi(x) + \lambda\Psi(x) = 0. \tag{0.10}$$

Introducing the linear integral operator $\mathbb{K}$,

$$\mathbb{K} = \int_a^b dx' K(x,x'), \tag{0.11}$$

$$\mathbb{K}(\alpha\Psi_1 + \beta\Psi_2) = \alpha\mathbb{K}\Psi_1 + \beta\mathbb{K}\Psi_2, \tag{0.12}$$

where α, β are constants, we can write Equation (0.9) as

$$(\mathbf{I} - \mathbb{K})\Psi(x) = \Psi^{(0)}(x), \tag{0.13}$$

where $\mathbf{I}$ is the identity operator. Assuming that $K(x,x')$ is small, we can write

$$\Psi(x) = \frac{\Psi^{(0)}(x)}{(\mathbf{I} - \mathbb{K})} \tag{0.14}$$

$$= (\mathbf{I} + \mathbb{K} + \mathbb{K}^2 + \cdots)\Psi^{(0)}(x). \tag{0.15}$$

A similar expansion can be written for $\Psi(x)$ as

$$\Psi(x) = \Psi^{(0)}(x) + \Psi^{(1)}(x) + \cdots, \tag{0.16}$$

which when substituted into Equation (0.15) yields the zeroth-order term of the approximation as

$$\Psi(x) \simeq \Psi^{(0)}(x), \tag{0.17}$$

and the subsequent terms of the expansion as

$$\Psi^{(1)}(x) = \int dx' K(x,x')\Psi^{(0)}(x), \tag{0.18}$$

$$\Psi^{(2)}(x) = \mathbb{K}\Psi^{(1)}(x) = \mathbb{K}^2\Psi^{(0)}(x) = \int dx'' K(x,x'') \int dx' K(x'',x')\Psi^{(0)}(x'), \tag{0.19}$$

$$\vdots$$

We can now write the following Neumann series [Eq. (18.51)]:

$$\Psi(x) = \Psi^{(0)}(x) + \int dx' K(x,x')\Psi^{(0)}(x') + \int dx' K(x,x')\Psi^{(1)}(x') + \cdots, \tag{0.20}$$

that is,

$$\begin{aligned} \Psi(x) = {} & \Psi^{(0)}(x) \\ & + \int dx' K(x,x')\Psi^{(0)}(x') \\ & + \int dx' K(x,x') \int dx'' K(x',x'')\Psi^{(0)}(x'') \\ & + \int dx' K(x,x') \int dx'' K(x',x'') \int dx''' K(x'',x''')\Psi^{(0)}(x''') \\ & + \cdots . \end{aligned} \tag{0.21}$$

If we approximate $\Psi(x)$ with the first N terms,

$$\Psi(x) \simeq \Psi^{(0)}(x) + \Psi^{(1)}(x) + \cdots + \Psi^{(N)}(x), \tag{0.22}$$

we can write Equation (0.13) as

$$(\mathbf{I} - \mathbb{K})(\Psi^{(0)} + \Psi^{(1)} + \cdots + \Psi^{(N)}) \simeq \Psi^{(0)}, \tag{0.23}$$

$$\Psi^{(0)} - \Psi^{(N+1)} \simeq \Psi^{(0)}. \tag{0.24}$$

For the convergence of Neumann series, for a given small positive number ε_0, we should be able to find a number N_0, independent of x,such that for

$$N + 1 > N_0, \tag{0.25}$$

$$\left|\Psi^{(N+1)}\right| < \varepsilon_0. \tag{0.26}$$

To obtain the sufficient condition for convergence let

$$\max |K(x,x')| = M \tag{0.27}$$

for

$$x, x' \in [a, b] \tag{0.28}$$

and take

$$\int_a^b dx' \left| \Psi^{(0)}(x) \right| = C. \tag{0.29}$$

We can now write the inequality

$$\left| \Psi^{(n+1)}(x) \right| < CM^{N+1}(b-a)^N = CM[M(b-a)]^N, \tag{0.30}$$

which yields the error committed by approximating $\Psi(x)$ with the first $N+1$ terms of the Neumann series [Eq. (0.21)] as

$$\left| \Psi(x) - \sum_{n=0}^{N} \Psi^{(n)}(x) \right| \leq \left| \Psi^{N+1}(x) \right| + \left| \Psi^{N+2}(x) \right| + \cdots \tag{0.31}$$

$$\leq CM[M(b-a)]^N \{1 + M(b-a) + M^2(b-a)^2 + \cdots\}. \tag{0.32}$$

If

$$M(b-a) < 1, \tag{0.33}$$

which is sufficient but not necessary, we can write

$$\left| \Psi(x) - \sum_{n=0}^{N} \Psi^{(n)}(x) \right| \leq \frac{CM[M(b-a)^N}{[1 - M(b-a)]} < \varepsilon_0, \tag{0.34}$$

which is true for all $N > N_0$ independent of x.

In scattering problems Schrödinger equation:

$$\overrightarrow{\nabla} \Psi(\overrightarrow{r}) + k^2 \Psi(\overrightarrow{r}) = \frac{2m}{\hbar^2} V(\overrightarrow{r}) \Psi(\overrightarrow{r}), \tag{0.35}$$

can be written as the integral equation

$$\Psi(\overrightarrow{r}) = e^{i \overrightarrow{k}_0 \cdot \overrightarrow{r}} - \frac{2m}{4\pi\hbar^2} \int d^3 \overrightarrow{r} \frac{e^{ik\left| \overrightarrow{r} - \overrightarrow{r}' \right|}}{|\overrightarrow{r} - \overrightarrow{r}'|} V(\overrightarrow{r}') \Psi(\overrightarrow{r}'), \tag{0.36}$$

where $e^{i \overrightarrow{k}_0 \cdot \overrightarrow{r}}$ is the solution of the homogeneous equation representing the incident plane wave and $k^2 = k_0^2 = 2mE/\hbar^2$. Wave vector of the incident wave is $\overrightarrow{k}_0$, while $\overrightarrow{k}$ is the wave vector of the outgoing wave as $\overrightarrow{r} \to \infty$. The first two terms of Equation (0.21) already gives the important result

$$\Psi(\overrightarrow{r}) \simeq e^{i \overrightarrow{k}_0 \cdot \overrightarrow{r}} - \frac{2m}{4\pi\hbar^2} \int d^3 \overrightarrow{r} \frac{e^{ik\left| \overrightarrow{r} - \overrightarrow{r}' \right|}}{|\overrightarrow{r} - \overrightarrow{r}'|} V(\overrightarrow{r}') e^{i \overrightarrow{k}_0 \cdot \overrightarrow{r}}, \tag{0.37}$$

called the *Born approximation*.

III. Useful Sites

Additional references and other useful information about the integral equations can be found in the following sites:

http://en.wikipedia.org/wiki/Integral_equations.

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