SUPPLEMENTS TO MATHEMATICAL METHODS IN SCIENCE AND ENGINEERING:
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CHAPTER 10: COORDINATES and TENSORS

I. Cartesian Tensors and the Theory of Elasticity:

Strain Tensor:

All bodies deform under stress, where every point, $\overrightarrow{r}$, of the undeformed body is translated into another point, $\overrightarrow{r}'$ (Fig. 0.1):

$$\overrightarrow{r} \rightarrow \overrightarrow{r}', \tag{0.1}$$

$$\overrightarrow{r}' = \overrightarrow{r} + \overrightarrow{\eta}(\overrightarrow{r}). \tag{0.2}$$

We can also write

$$x_i' = x_i + \eta_i, \ \ i = 1, 2, 3. \tag{0.3}$$

The distance between two infinitesimally close points is given as

$$d\overrightarrow{r}^2 = (dx_1^2 + dx_2^2 + dx_3^2)^{1/2}, \tag{0.4}$$

which after deformation becomes

$$d\overrightarrow{r}'^2 = (dx_1'^2 + dx_2'^2 + dx_3'^2)^{1/2}. \tag{0.5}$$

Using Equation (0.3) we can write

$$dx_i' = dx_i + d\eta_i, \tag{0.6}$$

$$= dx_i + \sum_{k=1}^{3} \frac{\partial \eta_i}{\partial x_k} dx_k. \tag{0.7}$$

From hereafter we adopt the Einstein summation convention, where the repeated indices are summed over. We can now ignore the summation sign in Equation (0.7) to write

$$dx_i' = x_i + \frac{\partial \eta_i}{\partial x_k} dx_k, \tag{0.8}$$

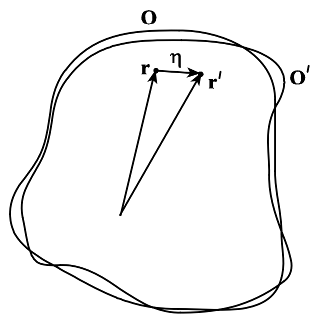

Fig. 0.1 In a general deformation, every point is displaced.

which allows us to write the distance between two infinitesimally close points after deformation as

$$\begin{aligned} d\overrightarrow{r}'^2 &= dx_i' dx_i' &(0.9)\\ &= \left(dx_i + \frac{\partial \eta_i}{\partial x_k} dx_k\right)\left(dx_i + \frac{\partial \eta_i}{\partial x_l} dx_l\right) &(0.10)\\ &= dx_i dx_i + \frac{\partial \eta_i}{\partial x_l} dx_i dx_l + \frac{\partial \eta_i}{\partial x_k} dx_i dx_k + \frac{\partial \eta_i}{\partial x_k}\frac{\partial \eta_i}{\partial x_l} dx_k dx_l. &(0.11)\end{aligned}$$

This can also be written as

$$d\overrightarrow{r}'^2 = d\overrightarrow{r}^2 + 2e_{kl} dx_k dx_l, \qquad (0.12)$$

where

$$e_{kl} = \frac{1}{2}\left(\frac{\partial \eta_k}{\partial x_l} + \frac{\partial \eta_l}{\partial x_k} + \frac{\partial \eta_i}{\partial x_k}\frac{\partial \eta_i}{\partial x_l}\right). \qquad (0.13)$$

For small deformations, $\eta_i \ll x_i$, we can ignore the second-order terms to define the **strain tensor** as

$$e_{kl} = \frac{1}{2}\left(\frac{\partial \eta_k}{\partial x_l} + \frac{\partial \eta_l}{\partial x_k}\right), \qquad (0.14)$$

which is a second-rank symmetric tensor:

$$e_{kl} = e_{lk}. \qquad (0.15)$$

Stress tensor:

Let $\overrightarrow{F}$ be the force per unit volume and $\overrightarrow{F} dV$ be the force acting on an infinitesimal portion of the body, which when integrated over a given volume:

$$\int_V \overrightarrow{F} dV, \qquad (0.16)$$

gives the total force acting on that volume of the body. We now assume that the force $\overrightarrow{F}$ can be written as the divergence of a second-rank tensor, σ_{ik}, as

$$F_i = \frac{\partial \sigma_{ik}}{\partial x_k}. \tag{0.17}$$

Using the divergence theorem we can write the ith component of the force as

$$\int_V F_i dV = \int_V \frac{\partial \sigma_{ik}}{\partial x_k} dV = \oint_S \sigma_{ik} ds_k, \tag{0.18}$$

where S is a surface that encloses the volume V and such that the area element, $d\overrightarrow{s}$, is oriented in the direction of the outward normal to S. The second-rank tensor, σ_{ik}, is called the **stress tensor**. In the above equation, $\sigma_{ik} ds_k$ gives the ith component of the force acting on the surface element when the normal to the surface points in the kth direction. In other words, σ_{ik} is the ith component of the force acting on a unit test area when the normal points in the kth direction.

We now write the torque, M_{ik}, acting on a volume V of the body due to $\overrightarrow{F}$ as the integral

$$M_{ik} = \int_V m_{ik} dV \tag{0.19}$$

$$= \int_V \left(\frac{\partial \sigma_{il}}{\partial x_l} x_k - \frac{\partial \sigma_{kl}}{\partial x_l} x_i \right) dV, \tag{0.20}$$

where the torque per unit volume, m_{ik}, is defined as

$$m_{ik} = F_i x_k - F_k x_i \tag{0.21}$$

$$= \left(\frac{\partial \sigma_{il}}{\partial x_l} x_k - \frac{\partial \sigma_{kl}}{\partial x_l} x_i \right). \tag{0.22}$$

We can also write M_{ik} as

$$M_{ik} = \int_V \frac{\partial(\sigma_{il} x_k - \sigma_{kl} x_i)}{\partial x_l} dV - \int \left(\sigma_{il} \frac{\partial x_k}{\partial x_l} - \sigma_{kl} \frac{\partial x_i}{\partial x_l} \right), \tag{0.23}$$

which after using the partial derivatives:

$$\frac{\partial x_k}{\partial x_l} = \delta_{kl}, \; \frac{\partial x_i}{\partial x_l} = \delta_{il} \tag{0.24}$$

and the divergence theorem for the first integral, yields

$$M_{ik} = \oint_S (\sigma_{il} x_k - \sigma_{kl} x_i) ds_l + \int_V (\sigma_{ki} - \sigma_{ik}) dV. \tag{0.25}$$

Assuming that the stress tensor is symmetric (Problem 0.2), we now obtain M_{ij} as

$$M_{ik} = \int_V m_{ij} dV, \tag{0.26}$$

$$= \oint_S (\sigma_{il} x_k - \sigma_{kl} x_i) ds_l. \tag{0.27}$$

Thermodynamics and Deformations:

Under external stresses all bodies deform. However, for sufficiently small strains, when the stresses are removed they all return to their original shapes. Such deformations are called **elastic**. When a body is strained beyond its elastic domain, there is always some residual deformation left when the stresses are removed, which is called **plastic** deformation.

In practice, we are interested in the **stress-strain relation**. To find such a relation we confine ourselves to the elastic domain. Furthermore, we assume that the deformation is performed sufficiently slowly, so that the entire process is reversible. Hence we can write the first law of thermodynamics as

$$dU = TdS - dW, \tag{0.28}$$

where the infinitesimal work done, dW, for infinitesimal deformations can be written as

$$dW = \left(\frac{\partial \sigma_{ik}}{\partial x_k} \right) \delta\eta_i dV. \tag{0.29}$$

For a finite deformation, we integrate over the region of interest:

$$\int_V dW = \int_V \left(\frac{\partial \sigma_{ik}}{\partial x_k} \right) \delta\eta_i dV, \tag{0.30}$$

which after integration by parts becomes

$$\int_V dW = \oint_S \sigma_{ik} \delta\eta_i ds_k - \int_V \sigma_{ik} \frac{\partial(\delta\eta_i)}{\partial x_k} dV. \tag{0.31}$$

We let the surface, S, be at infinity. Assuming that there are no stresses on the body at infinity, the surface term in the above integral vanishes. Also using the symmetry of the strain tensor, we can write

$$\int_V dW = -\frac{1}{2} \int_V \sigma_{ik} \left(\frac{\partial(\delta\eta_i)}{\partial x_k} + \frac{\partial(\delta\eta_k)}{\partial x_i} \right) dV \tag{0.32}$$

$$= -\frac{1}{2} \int_V \sigma_{ik} \delta \left(\frac{\partial \eta_i}{\partial x_k} + \frac{\partial \eta_k}{\partial x_i} \right) dV \tag{0.33}$$

$$= - \int_V \sigma_{ik} \delta e_{ik} dV. \tag{0.34}$$

In other words, the work done per unit volume, w, is

$$w = -\sigma_{ik}\delta e_{ik}. \tag{0.35}$$

From now on we consider all thermodynamic quantities like the entropy, s, work, w, internal energy, u, etc. in terms of their values per unit volume of the undeformed body and denote them with lower case letters. Now the first law of thermodynamics becomes

$$du(s, e_{ik}) = Tds + \sigma_{ik}de_{ik}, \tag{0.36}$$

where the scalar function $u(s, e_{ik})$ is called the **thermodynamic potential**. **Helmholtz free energy**, $f(T, e_{ik})$, is defined as

$$f(T, e_{ik}) = u - Ts, \tag{0.37}$$

which allows us to write the differential

$$df = -sdT + \sigma_{ik}de_{ik}. \tag{0.38}$$

Similarly, we write the **Gibbs free** energy, $g(T, \sigma_{ik})$, as

$$g(T, \sigma_{ik}) = u - Ts - \sigma_{ik}e_{ik} \tag{0.39}$$

$$= f - \sigma_{ik}e_{ik}, \tag{0.40}$$

which gives the differential

$$dg = -sdT - e_{ik}d\sigma_{ik}. \tag{0.41}$$

We can now obtain the stress tensor using the partial derivative

$$\sigma_{ik} = \left(\frac{\partial u(s, e_{ik})}{\partial e_{ik}}\right)_s, \tag{0.42}$$

or

$$\sigma_{ik} = \left(\frac{\partial f(T, e_{ik})}{\partial e_{ik}}\right)_T. \tag{0.43}$$

Similarly, the strain tensor can be obtained as

$$e_{ik} = -\left(\frac{\partial g(T, \sigma_{ik})}{\partial \sigma_{ik}}\right)_T. \tag{0.44}$$

In these expressions the subscripts outside the parentheses indicate the variables held constant.

Connection Between the Shear and the Strain Tensors :

Pure shear is a deformation that preserves the volume but alters the shape of the body. Since the fractional change in volume is (Problem 0.2)

$$\frac{\Delta V}{V} = tr\ (e_{ij}) = e_{ii}, \tag{0.45}$$

for pure shear the strain tensor is traceless. In hydrostatic compression, bodies suffer equal compression in all directions, hence the corresponding strain tensor is proportional to the identity tensor:

$$e_{ik} \propto \delta_{ik}, \tag{0.46}$$

and the stress tensor is given as

$$\sigma_{ik} = -P\delta_{ik}, \tag{0.47}$$

where P is the hydrostatic pressure. A general deformation can be written as the sum of pure shear and hydrostatic compression as

$$e_{ik} = \left(e_{ik} - \frac{1}{3}\delta_{ik}e_{ll}\right) + \frac{1}{3}\delta_{ik}e_{ll}. \tag{0.48}$$

Note that the first term on the right-hand side is traceless, hence represents pure shear while the second term corresponds to hydrostatic compression. We consider isotropic bodies deformed at **constant temperature**, thus eliminating the contribution due to thermal expansion. To obtain a relation between the shear and the stress tensors we first need to find the Helmholtz free energy, $f(T, e_{ik})$, and then expand it in powers of e_{ik} about the undeformed state of the body, that is, $e_{ik} = 0$. Since when the body is undeformed the stresses vanish,

$$\sigma_{ik}\big|_{e_{ik}=0} = \left(\frac{\partial f(T, e_{ik})}{\partial e_{ik}}\right)_T\bigg|_{e_{ik}=0} = 0, \tag{0.49}$$

there is no linear term in the expansion of $f(T, e_{ik})$. Also, since $f(T, e_{ik})$ is a scalar function, the most general expression for $f(T, e_{ik})$ valid up to second-order can be written as

$$f(T, e_{ik}) = \frac{1}{2}\lambda(e_{ii})^2 + \mu(e_{ik})^2, \tag{0.50}$$

where λ and μ are called the **Lamé coefficients** and e_{ii}^2 and $e_{ij}^2 = e_{ik}e_{ki}$ are the only second-order scalars composed of the strain tensor. We now write the differential of $f(T, e_{ik})$ as

$$df = \lambda e_{ii}de_{ii} + 2\mu e_{ik}de_{ik} \tag{0.51}$$

and substitute

$$de_{ii} = \delta_{ik}de_{ik} \tag{0.52}$$

to get

$$df = \lambda e_{ii}\delta_{ik}de_{ik} + 2\mu e_{ik}de_{ik} \tag{0.53}$$

$$= (\lambda e_{ii}\delta_{ik} + 2\mu e_{ik})\, de_{ik}. \tag{0.54}$$

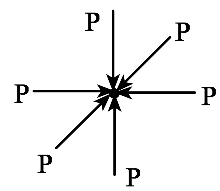

Fig. 0.2 Hydrostatic compression.

This gives the partial derivative

$$\left(\frac{\partial f(T, e_{ik})}{\partial e_{ik}}\right)_T = \lambda e_{ii}\delta_{ik} + 2\mu e_{ik}, \tag{0.55}$$

which is also equal to the stress tensor [Eq. (0.43)]:

$$\sigma_{ik} = \lambda e_{ii}\delta_{ik} + 2\mu e_{ik}. \tag{0.56}$$

We can also obtain a formula that expresses the strain tensor in terms of the stress tensor. Using Equation (0.56) we first write the following relation between the traces:

$$\sigma_{ii} = 3\lambda e_{ii} + 2\mu e_{ii} \tag{0.57}$$
$$= (3\lambda + 2\mu)e_{ii}, \tag{0.58}$$

which when substituted back into Equation (0.56) gives

$$\sigma_{ik} = \lambda \frac{\sigma_{ii}}{(3\lambda + 2\mu)}\delta_{ik} + 2\mu e_{ik} \tag{0.59}$$

and then yields the desired expression as

$$e_{ik} = \frac{1}{2\mu}\sigma_{ik} - \frac{\lambda\sigma_{ii}}{2\mu(3\lambda + 2\mu)}\delta_{ik}. \tag{0.60}$$

By considering specific deformations it is possible to relate the Lamé coefficients to the directly measurable quantities like the **bulk modulus**, K, **shear modulus**, G, **Young's modulus**, Y, etc. For example, the bulk modulus is defined as (Fig. 0.2)

$$P = -K\frac{\Delta V}{V}, \tag{0.61}$$

where P is the hydrostatic pressure and $\frac{\Delta V}{V}$ is the fractional change in volume. Using $\frac{\Delta V}{V} = e_{ii}$ and $\sigma_{ii} = -3P$, which follows from the stress tensor, $\sigma_{ik} =$

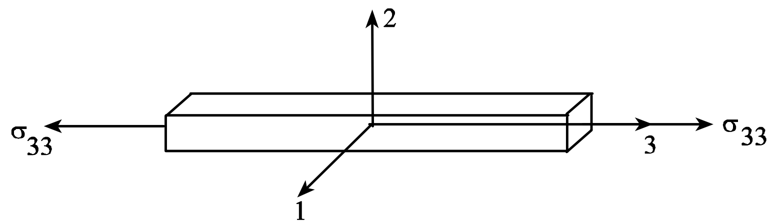

Fig. 0.3 Longitudinally stretched bar.

$-P\delta_{ik}$, for hydrostatic compressions, we can write [Eq. (0.58)]

$$-3P = (3\lambda + 2\mu)\frac{\Delta V}{V}, \tag{0.62}$$

$$P = -(\lambda + \frac{2}{3}\mu)\frac{\Delta V}{V}, \tag{0.63}$$

thus obtaining the relation

$$K = (\lambda + \frac{2}{3}\mu). \tag{0.64}$$

We now consider a long bar of length L with the cross sectional area A pulled longitudinally with the force (Fig. 0.3)

$$T = \sigma_{33}A. \tag{0.65}$$

Note that σ_{33} is the only non-zero component of the stress tensor. Young's modulus is defined as

$$\sigma_{33} = Y\frac{\Delta L}{L}. \tag{0.66}$$

As the bar stretches along the longitudinal direction, it gets thinner along the transverse directions by the relation

$$\begin{pmatrix} \Delta x_1/x_1 \\ \text{or} \\ \Delta x_2/x_2 \end{pmatrix} = -\sigma\frac{\Delta L}{L}, \tag{0.67}$$

where σ is called the **Poisson's ratio**. We now write the displacements as

$$\eta_1 = x_1\left(-\sigma\frac{\Delta L}{L}\right), \tag{0.68}$$

$$\eta_2 = x_2\left(-\sigma\frac{\Delta L}{L}\right), \tag{0.69}$$

$$\eta_3 = x_3\left(\frac{\Delta L}{L}\right), \tag{0.70}$$

which yields the nonzero components of the strain tensor as

$$e_{11} = e_{22} = -\sigma\frac{\Delta L}{L} \tag{0.71}$$

$$= -\sigma e_{33}, \tag{0.72}$$

$$e_{33} = \frac{\Delta L}{L}. \tag{0.73}$$

Using Equation (0.56) for σ_{33} :

$$\sigma_{33} = \lambda e_{kk} + 2\mu e_{33}, \tag{0.74}$$

we obtain the relation

$$Y = (-2\sigma + 1)\lambda + 2\mu. \tag{0.75}$$

Similarly, using $\sigma_{11} = \sigma_{22} = 0$, Equation (0.56) gives another relation as

$$0 = (-2\sigma + 1)\lambda + 2\mu(-\sigma). \tag{0.76}$$

We now consider a metal plate sheared as shown in Figure 0.4 (left), where the deformations are given as

$$\eta_1 = \frac{\theta}{2}x_2,\ \eta_2 = \frac{\theta}{2}x_1,\ \eta_3 = 0. \tag{0.77}$$

In this case the only nonvanishing components of the strain tensor [Eq. (0.14)] are

$$e_{12} = e_{21} = \theta/2. \tag{0.78}$$

Inserting these into Equation (0.56) we obtain

$$\sigma_{12} = \mu\left(\frac{\theta}{2} + \frac{\theta}{2}\right) = \mu\theta. \tag{0.79}$$

In engineering shear modulus, G, is defined in terms of the total angle of deformation (Fig. 0.4 (right)) as $\sigma_{12} = G\theta$, hence

$$\mu = G. \tag{0.80}$$

Using Equations (0.64), (0.75) and (0.76) we can express the Lamé coefficients, λ and μ, and the Poisson's ratio, σ, in terms of the Bulk modulus, K, Young's modulus, Y, and the shear modulus, G, which are experimentally easy to measure.

Hook's law:

We can also obtain the relation between the stress and the strain tensors [Eq. (0.56)] by writing the **Hook's law in covariant** form as

$$\sigma_{ij} = E_{ijkl}e_{kl}, \tag{0.81}$$

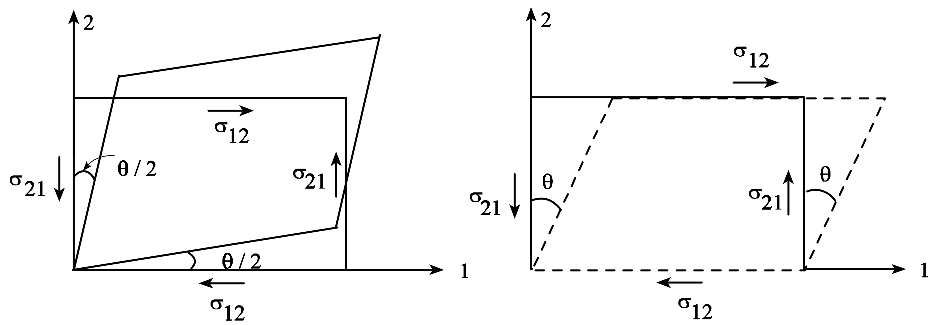

Fig. 0.4 Pure shear.

where E_{ijkl} is a fourth-rank tensor called the **elasticity tensor.** It obeys the following symmetry properties:

$$E_{ijkl} = E_{klij} = E_{jikl} = E_{ijlk}. \tag{0.82}$$

For an isotropic body the most general tensor with the required symmetries can be written as

$$E_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{0.83}$$

where λ and μ are the Lamé coefficients. Substituting Equation (0.83) into (0.81) gives

$$\sigma_{ij} = \lambda\delta_{ij}\delta_{kk} + 2\mu e_{ij}, \tag{0.84}$$

which is Equation (0.56).

Problem 0.1: Show that

$$\frac{\Delta V}{V} = e_{ii} = tr(e_{ij}). \tag{0.85}$$

Problem 0.2: We have written the moment (torque) of the force acting on a portion of a body as

$$M_{ik} = \oint_S (\sigma_{il}x_k - \sigma_{kl}x_i)ds_l + \int_V (\sigma_{ki} - \sigma_{ik})dV. \tag{0.86}$$

When the stress tensor is symmetric, $\sigma_{ki} = \sigma_{ik}$, obviously the volume integral on the right-hand side is zero. However, even when the stress tensor is not symmetric, under certain conditions it can be made symmetric.

Show that if a stress tensor can be written as the divergence of a third-rank tensor antisymmetric in the first pair of indices:

$$\sigma_{ik} - \sigma_{ki} = 2\frac{\partial \Psi_{ikl}}{\partial x_l}, \;\; \Psi_{ikl} = -\Psi_{kil}, \tag{0.87}$$

then a third-rank tensor, χ_{ikl}:

$$\chi_{ikl} = \Psi_{kli} + \Psi_{ilk} - \Psi_{ikl}, \tag{0.88}$$

antisymmetric in the last pair of indices, $\chi_{ikl} = -\chi_{ilk}$, can be found to transform σ_{ik} into symmetric form via the transformation

$$\sigma' = \sigma + \frac{\partial \chi_{ikl}}{\partial x_l} \tag{0.89}$$

as

$$\sigma'_{ik} = \frac{1}{2}(\sigma_{ik} + \sigma_{ki}) + \left(\frac{\partial \Psi_{ilk}}{\partial x_l} + \frac{\partial \Psi_{kli}}{\partial x_l} \right), \ \sigma'_{ik} = \sigma'_{ki}. \tag{0.90}$$

Also, show that the forces corresponding to the two stress tensors, $\sigma'$and σ, are identical.

Problem 0.3: Write the components of the strain tensor in cylindrical and spherical coordinates.

II. Interpretation of the Metric Tensor:

In classical physics space is an endless continuum, in which everything in the universe exists. In other words, space is the *arena* on which all processes take place. We use coordinate systems to assign numbers called coordinates to every point in space, which in turn allow us to study physical processes in terms of separations and directions. Obviously, there are infinitely many possibilities for the coordinate system that one may choose to use. In this regard, tensors, which are defined in terms of their transformation properties under coordinate transformations, have proven to be very useful in physics. Since it contains crucial information regarding the intrinsic properties of the physical space, the **metric tensor** plays a fundamental role in physics. However, this information is not easily revealed by the metric tensor.

Let us consider a two dimensional universe with two dimensional intelligent bugs living in it. Some of the bugs in this universe use a coordinate system, which allows them to write the line element as

$$ds^2 = dx^2 + dy^2, \ (x, y) \in (-\infty, \infty), \tag{0.91}$$

while the others prefer to work with a different coordinate system, where the line element is

$$ds^2 = dr^2 + r^2 d\theta^2, \ r \in [0, \infty), \ \theta \in [0, 2\pi]. \tag{0.92}$$

In the first coordinate system the metric tensor is obviously the identity tensor:

$$g_{ij} = \mathbf{I} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \tag{0.93}$$

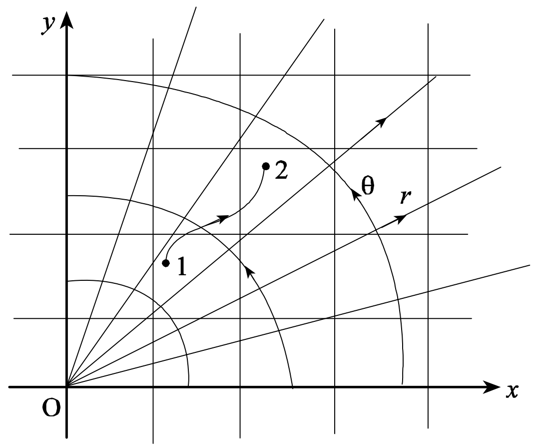

Fig. 0.5 Cartesian and plane polar coordinates.

while in the second coordinate system it is given as

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}. \tag{0.94}$$

A path connecting two points in space can be written in the first coordinate system as $y = y(x)$, while in the second coordinate system it can be expressed as $r = r(\theta)$. Since the **path length**, $l = \int_1^2 ds$, is a scalar, its value does not depend on the coordinate system used. Since l is basically the length that the bugs will measure by laying their rulers end to end along the path, it is also called the **proper length**. One can also calculate l. The first group of bugs using the first coordinate system [Eq. (0.91)] will use the formula

$$l = \int_1^2 ds = \int_1^2 dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \tag{0.95}$$

while the second group of bugs using the second coordinate system [Eq. (0.92)] will use

$$l = \int_1^2 ds = \int_1^2 d\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}. \tag{0.96}$$

A group of bugs immediately set out to investigate the properties of their space by taking direct measurements using rulers and protractors. They draw circles of various sizes at various locations in their universe and measure their circumference to radius ratios. Operationally, this is a well defined procedure; first they pick a point and connect all points equidistant from that point and then they measure the (proper) circumference by laying their rulers end to end along the periphery. To measure the (proper) radius, they lay their rulers from the center onwards along one of the axes. Their measurements turn out to be in perfect agreement with their calculations. For the first group of bugs using the first coordinate system [Eq. (0.91)], equation of a circle is given by

$$x^2 + y^2 = r_0^2, \ r_0 = \text{radius}. \tag{0.97}$$

For the second group using Equation (0.92), a circle is simply written as

$$r = r_0. \tag{0.98}$$

In the first coordinate system, the circumference is calculated as

$$\begin{align} c &= \int ds_{(x^2+y^2=r_0^2)} \tag{0.99} \\ &= \int dx \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{0.100} \\ &= \int \frac{dx}{\left(1 - \frac{x^2}{r_0^2}\right)^{1/2}} \tag{0.101} \\ &= 2\pi r_0, \tag{0.102} \end{align}$$

while in the second coordinate system it is found as

$$\begin{align} c &= \int ds_{(r=r_0)} \tag{0.103} \\ &= \int d\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \tag{0.104} \\ &= r_0 \int_0^{2\pi} d\theta \tag{0.105} \\ &= 2\pi r_0. \tag{0.106} \end{align}$$

On the other hand, in the first coordinate system the radius is calculated as

$$\begin{align} \text{radius} &= \int ds_{(y=0)} \tag{0.107} \\ &= \int_0^{r_0} dx \sqrt{1 + \frac{dy}{dx}} \tag{0.108} \\ &= \int_0^{r_0} dx \tag{0.109} \\ &= r_0, \tag{0.110} \end{align}$$

while in the second coordinate system it is found as

$$\text{radius} = \int ds_{(\theta=\theta_0)} \tag{0.111}$$

$$= \int_0^{r_0} dr \sqrt{1 + r^2 \frac{d\theta}{dr}} \tag{0.112}$$

$$= \int_0^{r_0} dr \tag{0.113}$$

$$= r_0. \tag{0.114}$$

In conclusion, no matter how large or small a circle the bugs draw and regardless of the location of these circles, they always find the same number for the circumference to radius ratio, c/r_0, which is twice a mysterious number they called π. Furthermore, when they draw triangles of various sizes and orientations, regardless of the location of these triangles, they always find the interior angles of the triangles add up to the same mysterious number π.

In fact, nothing would have changed even if some of the bugs had used a coordinate system where the line element and the metric are given, respectively, as

$$ds^2 = (y^2 + 1)dx^2 + 2(xy + 1)dxdy + (x^2 + 1)dy^2, \tag{0.115}$$

$$g_{ij} = \begin{pmatrix} y^2 + 1 & xy + 1 \\ xy + 1 & x^2 + 1 \end{pmatrix}. \tag{0.116}$$

Being intelligent creatures capable of abstract thought, these bugs immediately notice that they are living in a flat universe. In fact, the first metric is nothing but the Pythagorean theorem in Cartesian coordinates, while the second metric is the same metric written in plane polar coordinates. The two coordinate systems are related by the transformation equations

$$x = r \cos\theta, \tag{0.117}$$

$$y = r \sin\theta. \tag{0.118}$$

Similarly, a transformation can be written between the metric in Equation (0.115) and the first two metrics [Eqs. (0.93) and (0.94)].

Now consider another two dimensional universe, where this time the metric given as

$$ds^2 = \frac{dr^2}{\left(1 - \frac{r^2}{R^2}\right)} + r^2 d\phi^2, \; r \in [0, R], \; \phi \in [0, 2\pi]. \tag{0.119}$$

A circle in this universe is defined by

$$r = r_0. \tag{0.120}$$

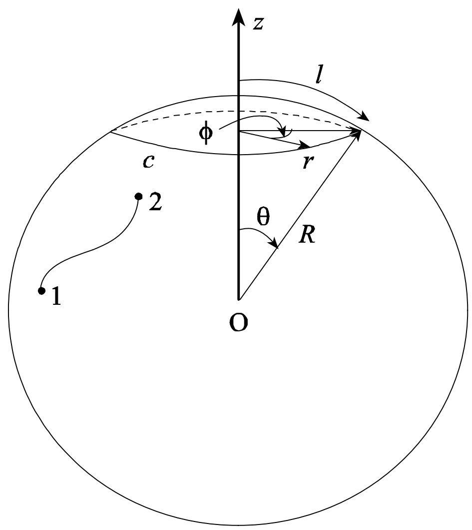

Fig. 0.6 Bugs living on a sphere.

The proper radius, r_p, and the proper circumference, c_p, that the bugs will measure in this universe are calculated as

$$r_p = \int ds_{(\phi=\phi_0)} = \int_0^{r_0} \frac{dr}{\left(1 - \frac{r^2}{R^2}\right)^{1/2}} \tag{0.121}$$

$$= R \sin^{-1}(r_0/R). \tag{0.122}$$

$$c_p = \int ds_{(r=r_0)} \tag{0.123}$$

$$= r_0 \int_0^{2\pi} d\phi \tag{0.124}$$

$$= 2\pi r_0, \tag{0.125}$$

thus yielding the ratio

$$\frac{c_p}{r_p} = \frac{2\pi r_0}{R \sin^{-1}(r_0/R)}. \tag{0.126}$$

Clearly, this ratio depends on the size of the circle and only in the limit as the radius of the circle goes to zero, $r_0 \to 0$, or as $R \to \infty$, goes to 2π. Expansion of c_p/r_p in powers of r_0/R :

$$\frac{c_p}{r_p} = 2\pi \left[1 - \frac{1}{6}\left(\frac{r_0}{R}\right)^2 + \cdots\right], \tag{0.127}$$

shows that in general these bugs will measure a c_p/r_p ratio lower than 2π.

To the bugs this looks rather strange and they argue that there must be a force field that effects the rulers to give this $c_p/r_p < 2\pi$ ratio. In fact, one of the bugs uses the transformation

$$r = \frac{\rho}{1 + \frac{\rho^2}{4R^2}} \tag{0.128}$$

to write the line element [Eq. (0.119)] as

$$ds^2 = \frac{1}{\left(1 + \frac{\rho^2}{4R^2}\right)^2} \left[d\rho^2 + \rho^2 d\phi^2\right], \tag{0.129}$$

which demonstrates that the proper lengths, hence the rulers, in their universe are indeed shortened by the factor

$$\frac{1}{\left(1 + \frac{\rho^2}{4R^2}\right)}, \tag{0.130}$$

with respect to a flat universe.

They even develop a field theory, where there is a force field that shortens the rulers by the factor $1/\left(1 + \frac{\rho^2}{4R^2}\right)$. However, like the electric fields, which effect only electrically charged objects, this field may also effect only certain types of matter possessing a new type of charge. To check this, they repeat their measurements with different rulers made from all kinds of materials they could find. No matter how hard they try and how precise their measurements are made, to their surprise, they always find the same circumference to radius ratio [Eq. (0.126)]. Whatever this field is, apparently effecting everything precisely the same way. In other words, it is a universal force field. This fact continues to intrigue them, but not knowing what to do with it, they continue with the force field concept, which after all appears to work fine in terms of their existing data.

Then comes a brilliant bug and says that all these years they have been mesmerized by the beauty and the simplicity of the geometry on flat space, but the measurements they have been getting could actually indicate that they may be living on the surface of a sphere. Then the brilliant bug shows them that the transformation

$$\frac{r}{R} = \sin\theta \tag{0.131}$$

transforms their line element [Eq. (0.119)] into the form

$$ds^2 = R^2 d\theta^2 + R^2 \sin^2\theta d\phi^2, \tag{0.132}$$

which when compared with line element in three-dimensional space in spherical coordinates:

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2, \tag{0.133}$$

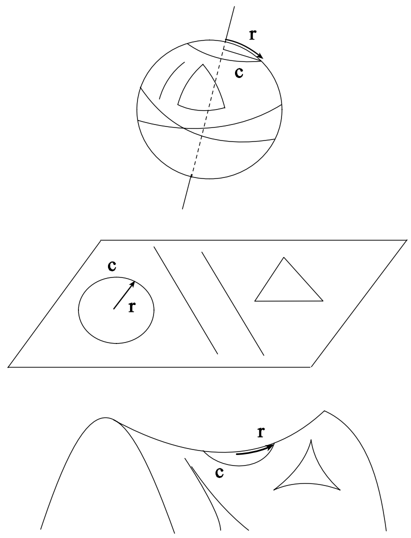

Fig. 0.7 Geometry of space is an experimental science.

corresponds to the line element on the surface of a sphere with the radius R (Fig. 0.6):

$$r = R. \tag{0.134}$$

In summary, these bugs do not need a force field to explain their observations. All they have to do is to accept that they are living on the two dimensional surface of a sphere in three dimensions. Since the geometry of space is something experimentally detectable, the fact that they have been getting the same geometry regardless of the internal structure of their measuring instruments; rulers, protractors, etc., indicates that this *new* geometry is universal. That is, it is the geometry of the physical space that everything exists in.

There is actually another possible geometry for the two dimensional bugs, where the line element is this time given as

$$ds^2 = \frac{dr^2}{(1 + \frac{r^2}{R^2})} + r^2 d\phi^2,\ r \in [0, \infty],\ \phi \in [0, 2\pi]. \tag{0.135}$$

In this case the ratio of the circumference to the radius of a circle is larger than 2π :

$$\frac{c_p}{r_p} = \frac{2\pi r_0}{R \sinh^{-1}(r_0/R)} = 2\pi \left[1 + \frac{1}{6}\left(\frac{r_0}{R}\right)^2 + \cdots\right], \tag{0.136}$$

and the interior angle of triangles are less than π. Such surfaces can be visualized as the surface of a saddle (Fig. 0.7). These are the three basic geometries for the surfaces in three dimensions. Of course, in general the surfaces could be something rather arbitrary with lots of bumps and dimples like the surface of an orange or an apple.

III. Curvature:

We have seen that from the appearance of a metric tensor one could not tell whether the underlying space is curved or not. A complicated looking metric with all or some of its components depending on position may very well be due to an unusual choice of coordinates. Still, the metric tensor possesses all the necessary information regarding the intrinsic properties of the underlying space. **Intrinsic curvature** is defined entirely interms of measurments that can be carried out in the space itself and not on how the space is embedded in a higher dimension. Our task is now to find a way to extract this information from the metric tensor. Furthermore, we would like to find a way that works not just for two-dimensional surfaces, but also for surfaces with any number of dimensions and for any shape. Hence, we need a criteria more sophisticated than the just the circumference to radius ratio of a circle.

Let the intelligent bugs living on the two-dimensional surface of a sphere transport a small vector over a closed path always pointing in the same direction so that it remains parallel to itself. This is called **parallel transport**. When the vector comes back to its starting point, the bugs will see that the vector has turned a certain angle ϑ (Fig.0.8). This angle, which is zero in flat space, for a sufficiently small area enclosed by the path, δA, is proportional to the area:

$$\delta\vartheta = K\delta A. \tag{0.137}$$

The proportionality constant K is called the **Gaussian curvature.** For a sphere $K = 1/R^2$. In fact, for a triangular path this angle is precisely the excess over π for the sum of the interior angles of the triangle. For a flat space we can take R as infinity, thus obtaining $K = 0$. For the saddle like surface in Figure 0.7, the Gaussian curvature is negative: $K = -1/R^2$. Gaussian curvature can be defined locally in terms of the radii of curvature in two perpendicular planes as $K = 1/R_1R_2$, where for a sphere $R_1 = R_2 = R$, hence $K = 1/R^2$. For a cylinder $K = 0$, since $R_1 = R$ and $R_2 = \infty$.

The general description of curvature in many-dimensional surfaces is still based on parallel transport over closed paths. However, this time $\delta\vartheta$ will also depend on the orientation of the path. The fact that the parallel transported vectors over closed paths in general do not coincide with themselves, is due to the fact that the covariant derivatives with respect to j and k in $v_{i;jk}$ do not commute, $v_{i;jk} \neq v_{i;kj}$, unless the space is flat. We have mentioned that the difference between $v_{i;jk}$ and $v_{i;kj}$ is given in terms of a fourth-rank tensor, R^l_{ijk}, called the **Riemann curvature tensor**, or in short, the **curvature**

tensor [Eqs. (10.227) and (10.228)]:

$$v_{i;jk} - v_{i;kj} = R^l_{ijk} v_l, \tag{0.138}$$

where

$$R^l_{ijk} = \begin{Bmatrix} l \\ mj \end{Bmatrix}\begin{Bmatrix} m \\ ik \end{Bmatrix} - \begin{Bmatrix} l \\ mk \end{Bmatrix}\begin{Bmatrix} m \\ ij \end{Bmatrix} + \frac{\partial}{\partial x^j}\begin{Bmatrix} l \\ ik \end{Bmatrix} - \frac{\partial}{\partial x^k}\begin{Bmatrix} l \\ ij \end{Bmatrix}. \tag{0.139}$$

To understand the properties of the curvature tensor we now discuss parallel transport in detail.

Parallel Transport:

Covariant differentiation [Eq. (10.208)] over the entire space is defined as

$$v^i_{;j} = \frac{\partial v^i}{\partial x^j} + \begin{Bmatrix} i \\ jk \end{Bmatrix} v^k, \tag{0.140}$$

where the Chrisfoffel symbols of the second kind are defined as [Eq. (10.202)]

$$\begin{Bmatrix} i \\ jk \end{Bmatrix} = \frac{g^{il}}{2}\left(\frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l}\right). \tag{0.141}$$

However, we are frequently interested in covariant differentiation along a path parametrized as $x^i(\tau)$. Along $x^i(\tau)$, we can also parametrize a vector in terms of τ as $v^i(\tau)$. Now the covariant derivative of v^i over the path $x^i(\tau)$ becomes

$$\frac{Dv^i}{D\tau} = v^i_{;j}\frac{dx^j}{d\tau} \tag{0.142}$$

$$= \frac{\partial v^i}{\partial x^j}\frac{dx^j}{d\tau} + \begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^j}{d\tau} v^k \tag{0.143}$$

$$= \frac{dv^i}{d\tau} + \begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^j}{d\tau} v^k. \tag{0.144}$$

Note that $\frac{Dv^i}{D\tau}$ is a covariant expression, hence valid in all coordinate systems. A vector parallel transported along a curve satisfies

$$\frac{Dv^i}{D\tau} = 0, \tag{0.145}$$

that is,

$$\frac{dv^i}{d\tau} = -\begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^j}{d\tau} v^k. \tag{0.146}$$

For a covariant vector the parallel transport equation becomes

$$\frac{dv_i}{d\tau} = \begin{Bmatrix} k \\ ij \end{Bmatrix}\frac{dx^j}{d\tau} v_k. \tag{0.147}$$

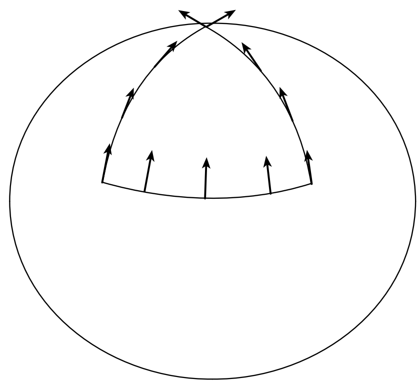

Fig. 0.8 Parallel transport.

Parallel transport is what comes closest to a constant vector along a curve in curved space.

Round Trips via Parallel transport:

We have obtained the formula [Eq. (0.147)] that tells us how a vector changes when parallel transported along a curve. We now apply this result to see whether a given vector returns to its initial state when parallel transported along a small but closed path. If the curve is sufficiently small, we can expand the Christoffel symbols and the vector v_i around some point $X = x(\tau_0)$ as

$$\Gamma^k_{ij}(x) = \Gamma^k_{ij}(X) + (x^l(\tau) - X^l)\frac{\partial}{\partial X^l}\Gamma^k_{ij}(X) + \cdots , \tag{0.148}$$

$$v_i(\tau) = v_i(\tau_0) + \Gamma^k_{ij}(X)(x^j(\tau) - X^j)v_k(\tau_0) + \cdots , \tag{0.149}$$

where we have written the Christoffel symbols as

$$\Gamma^k_{ij}(x) = \begin{Bmatrix} k \\ ij \end{Bmatrix} \tag{0.150}$$

and used Equation (0.147) to first order in $(x^j(\tau) - X^j)$ to write Equation (0.149). Substituting Equations (0.148) and (0.149) into (0.147) and keeping terms of up to second-order we get

$$\begin{aligned} v_i(\tau) \simeq v_i(\tau_0) + \int_{\tau_0}^{\tau} &\left[\Gamma^k_{ij}(X) + (x^l(\tau) - X^l)\frac{\partial}{\partial X^l}\Gamma^k_{ij}(X) + \cdots\right] \\ &\times \left[v_k(\tau_0) + v_m(\tau_0)\Gamma^m_{kl}(X)(x^l(\tau) - X^l) + \cdots\right]\frac{dx^j(\tau)}{d\tau}d\tau. \end{aligned} \tag{0.151}$$

We could simplify this further to write

$$
\begin{aligned}
v_i(\tau) \simeq v_i(\tau_0) &+ \Gamma^k_{ij}(X) v_k(\tau_0) \int_{\tau_0}^{\tau} \frac{dx^j(\tau)}{d\tau} d\tau \\
&+ \left[\frac{\partial}{\partial X^l} \Gamma^m_{ij}(X) + \Gamma^k_{ij}(X) \Gamma^m_{kl}(X) \right] v_m(\tau_0) \int_{\tau_0}^{\tau} (x^l(\tau) - X^l) \frac{dx^j}{d\tau} d\tau.
\end{aligned}
\tag{0.152}
$$

Since for a closed path x^i returns to its initial value X^i for some τ_1,

$$
\int_{\tau_0}^{\tau_1} \frac{dx^j}{d\tau} d\tau = 0. \tag{0.153}
$$

This gives the change in value, Δv_i, of the vector v_i when parallel transported over a sufficiently small closed path as

$$
\Delta v_i = \left[\frac{\partial}{\partial X^l} \Gamma^m_{ij}(X) + \Gamma^k_{ij}(X) \Gamma^m_{kl}(X) \right] v_m(\tau_0) \int_{\tau_0}^{\tau_1} x^l(\tau) \frac{dx^j}{d\tau} d\tau, \tag{0.154}
$$

or as

$$
\Delta v_i = \left[\frac{\partial}{\partial X^l} \Gamma^m_{ij}(X) + \Gamma^k_{ij}(X) \Gamma^m_{kl}(X) \right] v_m(\tau_0) \oint x^l(\tau) dx^j. \tag{0.155}
$$

The integral, $\oint x^l(\tau) dx^j$, is in general nonzero and antisymmetric:

$$
\begin{aligned}
\oint x^l(\tau) dx^j &= \int_{\tau_0}^{\tau_1} \frac{d(x^l x^j)}{d\tau} d\tau - \int_{\tau_0}^{\tau_1} x^j \frac{dx^l}{d\tau} d\tau && (0.156) \\
&= -\oint x^j(\tau) dx^l, && (0.157)
\end{aligned}
$$

hence we can also write Δv_i as

$$
\begin{aligned}
\Delta v_i &= \left[\frac{\partial}{\partial X^j} \Gamma^m_{il}(X) + \Gamma^k_{il}(X) \Gamma^m_{kj}(X) \right] v_m(\tau_0) \oint x^j(\tau) dx^l && (0.158) \\
&= -\left[\frac{\partial}{\partial X^j} \Gamma^m_{il}(X) + \Gamma^k_{il}(X) \Gamma^m_{kj}(X) \right] v_m(\tau_0) \oint x^l(\tau) dx^j. && (0.159)
\end{aligned}
$$

Adding Equations (0.155) and (0.159) we write

$$
\begin{aligned}
2\Delta v_i = &\left[\frac{\partial}{\partial X^l} \Gamma^m_{ij}(X) \ - \frac{\partial}{\partial X^j} \Gamma^m_{il}(X) + \Gamma^k_{ij}(X) \Gamma^m_{kl}(X) - \Gamma^k_{il}(X) \Gamma^m_{kl}(X) \right] \\
&\times v_m(\tau_0) \oint x^l(\tau) dx^j.
\end{aligned}
\tag{0.160}
$$

The quantity inside the square brackets is nothing but R^m_{ilj}, that is, the curvature tensor, hence

$$
\Delta v_i = \frac{1}{2} R^m_{ilj} v_m(\tau_0) \oint x^l(\tau) dx^j. \tag{0.161}
$$

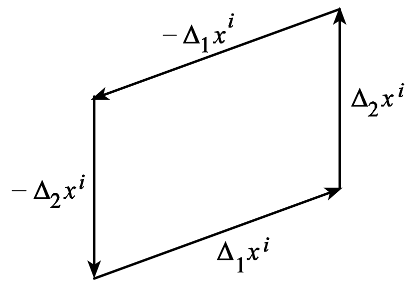

Fig. 0.9 Parallelogram.

This result indicates that a vector, v_i, parallel transported over a small closed path does not return to its initial value unless R^m_{ilj} vanishes at X. If we take our closed path as a small parallelogram with the sides $\Delta_1 x^i$ and $\Delta_2 x^j$, then $\oint x^l(\tau)dx^j$ are the components of the area of the parallelogram (Fig. 0.9):

$$\oint x^l(\tau)dx^j = \Delta_1 x^l \Delta_2 x^j - \Delta_1 x^j \Delta_2 x^l. \tag{0.162}$$

For a finite closed path C enclosing an area A, we can subdivide A into small cells each bounded by c_N. The change in v_i when parallel transported around C can then be written as the sum.

$$\Delta v_i = \sum_N \Delta_N v_i. \tag{0.163}$$

This follows from the fact that the change in v_i around the neighboring cells are cancelled, thus leaving only the outermost cell boundaries making up the path C.

Algebraic Properties of the Curvature Tensor :

To reveal the algebraic properties of the curvature tensor we write it as

$$R_{ijkl} = g_{im} R^m_{jkl}. \tag{0.164}$$

Using Equation (0.139) this can be written as

$$R_{ijkl} = \frac{1}{2}\left[\frac{\partial^2 g_{jk}}{\partial x^l \partial x^i} - \frac{\partial^2 g_{ik}}{\partial x^l \partial x^j} + \frac{\partial^2 g_{il}}{\partial x^k \partial x^j} - \frac{\partial^2 g_{jl}}{\partial x^k \partial x^i}\right] + g_{nm}\left[\Gamma^n_{li}\Gamma^m_{jk} - \Gamma^n_{ki}\Gamma^m_{jl}\right]. \tag{0.165}$$

From this equation the following properties are evident:

i) **Symmetry**:

$$R_{ijkl} = R_{klij} \tag{0.166}$$

ii) **Antisymmetry**:

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{jilk} \tag{0.167}$$

iii) **Cyclicity**:

$$R_{ijkl} + R_{iljk} + R_{iklj} = 0. \tag{0.168}$$

There is one more symmetry called the **Bianchi identity**, which is not obvious but could be shown by direct substitution:
iv) **Bianchi identity**:

$$R_{ijkl;m} + R_{ijmk;l} + R_{ijlm;k} = 0. \tag{0.169}$$

Contractions of the Curvature Tensor:
Using the symmetry property we can contract the first and the third indices to get a very important symmetric second-rank tensor called the **Ricci tensor**:

$$g^{ik} R_{ijkl} = R_{jl}, \tag{0.170}$$
$$R_{jl} = R_{lj}. \tag{0.171}$$

The antisymmetry property indicates that this is the only second-rank tensor that can be constructed by contracting the indices of the curvature tensor. Contracting the first and the third, and then the second and the fourth indices of the curvature tensor gives us the only scalar, R, that can be constructed from the curvature tensor as

$$g^{jl} g^{ik} R_{ijkl} = g^{jl} R_{jl} = R^j_j = R. \tag{0.172}$$

Finally, contracting the Bianchi identity gives

$$g^{ik} R_{ijkl;m} + g^{ik} R_{ijmk;l} + g^{ik} R_{ijlm;k} = 0, \tag{0.173}$$

$$R_{jl;m} - R_{jm;l} + R^k_{jlm;k} = 0. \tag{0.174}$$

Contracting once more yields

$$R_{;m} - R^j_{m;j} - R^k_{m;k} = 0, \tag{0.175}$$
$$\left(R^j_m - \frac{1}{2} R \delta^j_m \right)_{;j} = 0, \tag{0.176}$$

which can also be written as

$$\left(R^{ij} - \frac{1}{2} R g^{ij} \right)_{;j} = 0. \tag{0.177}$$

Curvature in n dimensions:

The curvature tensor, R_{ijkl}, in n dimensions has n^4 components. In four dimensions it has 256, in three dimensions 81 and in two dimensions 16 components. However, due to its large number of symmetries expressed in Equations (0.166-0.168) it has only

$$C_n = \frac{1}{12}N^2(N^2 - 1) \tag{0.178}$$

independent components. In four dimensions this gives the number of independent components as 20, in three dimensions as 6 and in two dimensions as 1. In one dimension, the curvature tensor has only one component, R_{1111}, which due to Equation (0.167) or (0.168) is always zero. In other words, in one dimension we can not have intrinsic curvature. It sounds odd that a curved wire has zero curvature. However, curvature tensor reflects the inner properties of the space and not how it is embedded or viewed from a higher dimension. Indeed, in one dimension we can always transform the line element

$$ds^2 = g_{11}(x)dx^2 \tag{0.179}$$

everywhere into the form

$$ds^2 = dx'^2, \tag{0.180}$$

via the coordinate transformation

$$x' = \int \sqrt{g_{11}}dx. \tag{0.181}$$

Another way to see this is that we can always straighten a bent wire without cutting it. In two dimensions, R_{ijkl} has only one independent component, which can be taken as R_{1212}. Using Equations (0.165)-(0.168) we can write all the components of R_{ijkl} as

$$R_{1212} = -R_{2112} = -R_{1221} = R_{2121}, \tag{0.182}$$
$$R_{1111} = R_{1112} = R_{1121} = R_{1122} = 0, \tag{0.183}$$
$$R_{1211} = R_{1222} = R_{2111} = R_{2122} = 0, \tag{0.184}$$
$$R_{2211} = R_{2212} = R_{2221} = R_{2222} = 0. \tag{0.185}$$

These can be conveniently expressed as

$$R_{ijkl} = (g_{ik}g_{jl} - g_{il}g_{jk})\frac{R_{1212}}{g}, \tag{0.186}$$

where g is the determinant:

$$g = \left(g_{11}g_{22} - g_{12}^2\right). \tag{0.187}$$

If we contract i and k in R_{ijkl} we get

$$R_{jl} = g_{jl}\frac{R_{1212}}{g}. \tag{0.188}$$

Contracting j and l in R_{jl} gives the curvature scalar

$$R = \frac{2R_{1212}}{g}. \tag{0.189}$$

We can now write the curvature tensor as

$$R_{ijkl} = \frac{R}{2}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right). \tag{0.190}$$

The Gaussian curvature, K, introduced in Equation (0.137) is related to the curvature scalar R as

$$K = \frac{R}{2} \tag{0.191}$$

$$= \frac{R_{1212}}{g}, \tag{0.192}$$

which for a sphere becomes

$$K = \frac{1}{a^2}, \tag{0.193}$$

where a is the radius of the sphere.

Problem 0.4:

Verify Equation (0.165).

Problem 0.5:

Show by direct substitution that the Bianchi identities are true.

Problem 0.6:

Verify Equations (0.182)−(0.190).

Problem 0.7:

For the surface of a sphere the metric can be written as [Eq. (0.132)]

$$ds^2 = a^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{0.194}$$

where a is the radius of the sphere. Using the definition of the Riemann curvature tensor [Eq. (0.165)] evaluate R_{1212} and R, and verify that

$$K = \frac{1}{a^2}. \tag{0.195}$$

IV. References and Useful Links

Eisenhart, L.P., An Introduction to Differential Geometry With Use of the Tensor Calculus, Princeton, sixth printing, 1964.
Eisenhart, L.P., Riemann Geometry, Princeton, fifth printing, 1964.

Additional references and other useful information can be found in the following sites:

Elasticity:
http://en.wikipedia.org/wiki/Theory_of_elasticity,
http://en.wikipedia.org/wiki/Elastic_moduli,
http://en.wikipedia.org/wiki/Elastic_compliance_tensor,
Curvature:
http://en.wikipedia.org/wiki/Riemann_space,
http://en.wikipedia.org/wiki/Gaussian_curvature,
http://en.wikipedia.org/wiki/Riemann_curvature_tensor,

Selcuk Bayin
(November, 2009)