Supplements to Mathematical Methods in Science and Engineering
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CHAPTER 11: CONTINUOUS GROUPS and REPRESENTATIONS

I. Solutions or Hints to Selected Problems:

1. **(Problem 11.6, 11.7, 11.8)** Solve the differential equation for $d^l_{mm'}(\beta)$ by the factorization method.

   i) Considering m as a parameter, find the normalized step-up and step-down operators $\pounds_+(m+1)$ and $\pounds_-(m)$, which change the index m while keeping the index m' fixed.

   ii) Considering m' as a parameter, find the normalized step-up and step-down operators $\pounds'_+(m'+1)$ and $\pounds'_-(m')$, which change the index m' while keeping the index m fixed. Show that $|m| \leq l$ and $|m'| \leq l$.

   iii) Find the normalized functions with $m = m' = l$.

   d) For $l = 2$, construct the full matrix $d^2_{m'm}(\beta)$.

   iv) By transforming the differential equation for $d^l_{mm'}(\beta)$ into an appropriate form, find the step-up and step-down operators that shift the index l for fixed m and m', giving the **normalized** functions $d^l_{mm'}(\beta)$.

   v) Using the result of the previous part, derive a recursion relation for $(\cos\beta)\, d^l_{mm'}(\beta)$. That is, express this as a combination of $d^{l'}_{mm'}(\beta)$ with $l' = l \pm 1, \ldots$.

   Solution:

   i) We first write the differential equation that $d^l_{mm'}(\beta)$ satisfies:

$$\left\{\frac{d^2}{d\beta^2} + \cot\beta\frac{d}{d\beta} + \left[l(l+1) - \left(\frac{m^2 + m'^2 - 2mm'\cos\beta}{\sin^2\beta}\right)\right]\right\} d^l_{m'm}(\beta) = 0, \tag{0.1}$$

and then put it into self-adjoint form [Eq. (8.6)]:

$$\frac{d}{d\beta}\left[\sin\beta\frac{d}{d\beta}d^l_{m'm}(\beta)\right]-\frac{m^2+m'^2-2mm'\cos\beta}{\sin^2\beta}d^l_{m'm}(\beta)$$
$$=-l(l+1)\sin\beta d^l_{m'm}(\beta). \tag{0.2}$$

We now identify the functions $p(\beta)$, $w(\beta)$, and $q(\beta)$ [Eqs. (8.6) and (8.16)] as

$$p(\beta)=\sin\beta, \tag{0.3}$$
$$w(\beta)=\sin\beta, \tag{0.4}$$
$$q(\beta)=-\frac{m^2+m'^2-2mm'\cos\beta}{\sin^2\beta}, \tag{0.5}$$

and make the substitutions [Eqs. (9.5) and (9.6)]

$$d^l_{m'm}(\beta)=\frac{y(\lambda_l,m',m,\beta)}{\sqrt{\sin\beta}}, \tag{0.6}$$
$$dz=d\beta, \tag{0.7}$$

to obtain the second canonical form [Eq. (9.7)]:

$$\frac{d^2y}{d\beta^2}+[\lambda_l+r(z,m)]y(z)=0, \tag{0.8}$$

where

$$\lambda_l=l(l+1)+\frac{1}{4}, \tag{0.9}$$
$$r(z,m)=-\frac{m^2+m'^2-2mm'\cos\beta-\frac{1}{4}}{\sin^2\beta}. \tag{0.10}$$

Next we use the table in Infeld and Hull (Bayin 2006) to determine $k(\beta,m)$ and $\mu(m)$ to write the ladder operators [Eq. (9.14)] as

$$O_\pm(m)=\pm\frac{d}{d\beta}-k(\beta,m) \tag{0.11}$$
$$=\pm\frac{d}{d\beta}-(m-\frac{1}{2})\cot\beta+\frac{m'}{\sin\beta}. \tag{0.12}$$

Using the $\mu(m)$ found from the table:

$$\mu(m)=(m-\frac{1}{2})^2, \tag{0.13}$$

and theorem I, we show that

$$|m|\le l. \tag{0.14}$$

We now construct the eigenfunctions starting from the top of the ladder, that is, $m = l$, as

$$y(\lambda_l, m', m = l, \beta) = \sin^{l+1/2}\beta \ \tan^{-m'}\left(\frac{\beta}{2}\right) \tag{0.15}$$

or

$$d^l_{m'l} = \sin^l \beta \ \tan^{-m'}\left(\frac{\beta}{2}\right). \tag{0.16}$$

ii) Following similar steps, but this time keeping m as fixed and treating m' as a parameter and utilizing the table in Infeld and Hull, we first find $k(\beta, m')$ and $\mu(m')$ to write the ladder operators, and then show that m' satisfies

$$|m'| \le l. \tag{0.17}$$

Note that

$$r(z, m, m', \beta) \tag{0.18}$$

is symmetric in m and m'. From the definition of $d^l_{m'm}(\beta)$ [Eq. (11.127)]:

$$d^l_{m'm}(\beta) = \int\int d\Omega Y^*_{lm'}(\theta,\phi) e^{-i\beta L_y} Y_{lm}(\theta,\phi), \tag{0.19}$$

it is seen that the $d^l_{m'm}(\beta)$ are the elements of unitary matrices, furthermore, they are real; hence,

$$d^l_{m'm}(\beta) = d^l_{mm'}(-\beta) \tag{0.20}$$

is true (show this). In order to satisfy this relation, we introduce a factor of -1 into the definition of the ladder operators $O_\pm(m')$ as

$$O_\pm(m') = -\left[\pm\frac{d}{d\beta} - K(\beta, m')\right] \tag{0.21}$$

$$= -\left[\pm\frac{d}{d\beta} - (m' - \frac{1}{2})\cot\beta + \frac{m}{\sin\beta}\right]. \tag{0.22}$$

iii) We have found that [Eq. (0.15)]

$$y(l, m', m = l, \beta) = \sin^{l+1/2}\beta \tan^{-m'}(\frac{\beta}{2}). \tag{0.23}$$

Using the definition

$$d^l_{m'm}(\beta) = \frac{y(l, m, m', \beta)}{\sqrt{\sin\beta}}, \tag{0.24}$$

we can write

$$d^l_{ll}(\beta) = (1+\cos\beta)^l. \tag{0.25}$$

Using the weight function [Eq. (0.4)] we evaluate the normalization constant from

$$\int_0^\pi w(\beta)\left[d^l_{ll}(\beta)\right]^2 d\beta = \frac{2^{2l+1}}{2l+1} \tag{0.26}$$

and write the normalized $d^l_{ll}(\beta)$ as

$$d^l_{ll}(\beta) = \left(\frac{2l+1}{2^{2l+1}}\right)^{1/2} (1+\cos\beta)^l. \tag{0.27}$$

iv) For $l = 2$, using the eigenfunctions

$$d^l_{m'l}(\beta) = C^l_{m'l}\sin^l\beta\ \tan^{-m'}(\frac{\beta}{2}), \tag{0.28}$$

where $C^l_{m'l}$ are the normalization constants, we write

$$d^2_{m'2}(\beta) = C^2_{m'2}\sin^2\beta\ \tan^{-m'}\left(\frac{\beta}{2}\right), \tag{0.29}$$

hence

$$\begin{aligned}
d^2_{22}(\beta) &= \left(\frac{5}{2}\right)^{1/2}\left(\frac{1+\cos\beta}{2}\right)^2, \\
d^2_{12}(\beta) &= -\left(\frac{5}{2}\right)^{1/2}\frac{\sin\beta}{2}(1+\cos\beta), \\
d^2_{02}(\beta) &= \left(\frac{15}{16}\right)^{1/2}\sin^2\beta, \\
d^2_{-1,2}(\beta) &= -\left(\frac{5}{2}\right)^{1/2}\frac{\sin\beta}{2}(1-\cos\beta), \\
d^2_{-2,2}(\beta) &= \left(\frac{5}{2}\right)^{1/2}\left(\frac{1-\cos\beta}{2}\right)^2,
\end{aligned} \tag{0.30}$$

As the reader can check, we can also generate these functions by acting on the normalized $d^2_{22}(\beta)$ with the normalized ladder operator $\pounds_-(m')$, which acts on m' and lowers it by one while keeping m fixed. Equation (0.30) gives only the first column of the 5×5 matrix, $d^2_{m'm}(\beta)$, where $m = 2$ and m' takes the values $2, 1, 0, -1, -2$. For the remaining columns we use the normalized ladder operator:

$$\pounds_-(m) = \frac{-\frac{d}{d\beta} - (m-\frac{1}{2})\cot\beta + \frac{m'}{\sin\beta}}{\sqrt{(l+m)(l-m+1)}}, \tag{0.31}$$

which keeps m' fixed and lowers m by one as

$$\pounds_-(m)\ y(\lambda, m', m, \beta) = y(\lambda, m', m-1, \beta). \tag{0.32}$$

Similarly, we write the other normalized ladder operator $\pounds_+(m)$.

We now use

$$\sqrt{\sin\beta}\, d^l_{m'm} = y(\lambda, m', m, \beta) \tag{0.33}$$

in Equation [0.32] to obtain

$$d^l_{m',m-1} = \frac{1}{\sqrt{(l+m)(l-m+1)}}\left[-\frac{d}{d\beta} - m\cot\beta + \frac{m'}{\sin\beta}\right] d^l_{m'm}. \tag{0.34}$$

In conjunction with the normalized eigenfunctions [Eq. (0.30)], each of which is the top of the ladder for the corresponding row, we use this formula to generate the remaining 20 elements of the $d^l_{m'm}$ matrix.

Note:

You can use the symmetry relation in Equation (0.20) to check your algebra. Also show the relation

$$d^l_{m'm}(\beta) = (-1)^{m'-m} d^l_{-m'-m}(\beta). \tag{0.35}$$

v) We start with the equation that $d^l_{m'm}(\beta)$ satisfies:

$$\left\{\frac{d^2}{d\beta^2} + \cot\beta\frac{d}{d\beta} + \left[l(l+1) - \frac{m^2 + m'^2 - 2mm'\cos\beta}{\sin^2\beta}\right]\right\} d^l_{m'm}(\beta) = 0, \tag{0.36}$$

and substitute

$$\left\{\begin{array}{l} z = \ln(\tan\beta/2), \\ d^l_{m'm}(\beta) = K^l_{m'm}(z), \end{array}\right\} \tag{0.37}$$

to obtain

$$\frac{d^2K^l_{m'm}}{dz^2} + \left[-(m^2 + m'^2) + \frac{l(l+1)}{\cosh^2 z} - 2mm'\tanh z\right] K^l_{m'm}(z) = 0. \tag{0.38}$$

This is in second canonical form. We now proceed as in the previous case to obtain the recursion relations for the normalized eigenfunctions:

$$d^{l-1}_{m'm}(\beta) = \left[\frac{l\sqrt{(2l-1)/(2l+1)}}{\sqrt{[l^2-m^2][l^2-m'^2]}}\right]\left[-\sin\beta\frac{d}{d\beta} + l\cos\beta - \frac{m'm}{l}\right] d^l_{m'm}(\beta) \tag{0.39}$$

and

$$d^{l+1}_{m'm}(\beta) = \left[\frac{(l+1)\sqrt{(2l+3)/(2l+1)}}{\sqrt{[(l+1)^2 - m^2][(l+1)^2 - m'^2]}} \right] \tag{0.40}$$

$$\times [\sin\beta \frac{d}{d\beta} + (l+1)\cos\beta - \frac{m'm}{(l+1)}] d^l_{m'm}(\beta) \tag{0.41}$$

vi) For the needed recursion relation, simply add the above expressions.

2. **(Problem 11.9)** Show that

i)

$$D^l_{m0}(\alpha, \beta, -) = \sqrt{\frac{4\pi}{(2l+1)}} Y^*_{lm}(\beta, \alpha) \tag{0.42}$$

and

ii)

$$D^l_{0m}(-, \beta, \gamma) = (-1)^m \sqrt{\frac{4\pi}{(2l+1)}} Y^*_{lm}(\beta, \gamma). \tag{0.43}$$

Hint. Use the invariant

$$\sum_{m=-l}^{m=l} Y^*_{lm}(\theta_1, \phi_1) Y_{lm}(\theta_2, \phi_2) \tag{0.44}$$

with $(\theta_1, \phi_1) = (\beta, \alpha)$ and $(\theta_2, \phi_2) = (\theta, \phi)$, $\theta_{12} = \theta'$, and

$$\left[D^l_{mm'}(\alpha, \beta, \gamma) \right]^{-1} = \left[D^l_{m'm}(\alpha, \beta, \gamma) \right]^* = D^l_{mm'}(-\gamma, -\beta, -\alpha). \tag{0.45}$$

Solution:

We demonstrate the solution of the first part. First write the addition theorem [Eq. (11.325)]:

$$\sum_{m=-l}^{m=l} Y^*_{lm}(\theta_1, \phi_1) Y_{lm}(\theta_2, \phi_2) = \frac{2l+1}{4\pi} P_l(\cos\theta_{12}), \tag{0.46}$$

and then make the substitutions

$$\begin{aligned} \theta_1 &= \beta, \\ \phi_1 &= \alpha, \\ \theta_2 &= \theta, \\ \phi_2 &= \phi, \\ \theta_{12} &= \theta' \end{aligned} \tag{0.47}$$

to write

$$\sum_{m=-l}^{m=l} Y_{lm}^*(\beta,\alpha) Y_{lm}(\theta,\phi) = \frac{2l+1}{4\pi} P_l(\cos\theta'). \tag{0.48}$$

We now multiply with $Y_{l'm'}^*(\theta,\phi)$ and then integrate to obtain

$$\frac{4\pi}{2l+1} \int \sum_{m=-l}^{m=l} Y_{lm}^*(\beta,\alpha) Y_{lm}(\theta,\phi) Y_{l'm'}^*(\theta,\phi) d\Omega = \int P_l(\cos\theta') Y_{l'm'}^*(\theta,\phi) d\Omega. \tag{0.49}$$

Use the definition of $Y_{lm}(\theta,\phi)$ and its orthogonality relation [Eqs. (2.177) and (2.179)] to write this as

$$\frac{4\pi}{2l+1} \sum_{m=-l}^{m=l} Y_{lm}^*(\beta,\alpha) \delta_{ll'} \delta_{mm'} = \sqrt{\frac{4\pi}{2l+1}} \int Y_{l0}(\theta') Y_{l'm'}^*(\theta,\phi) d\Omega, \tag{0.50}$$

$$\sqrt{\frac{4\pi}{2l'+1}} Y_{l'm'}^*(\beta,\alpha) = \int Y_{l'0}(\theta') Y_{l'm'}^*(\theta,\phi) d\Omega. \tag{0.51}$$

From the definition of θ_{12} :

$$\theta_1 + \theta_2 = \beta + \theta = \theta', \tag{0.52}$$

write

$$Y_{l0}(\beta+\theta) = R(\alpha,\beta,\gamma) Y_{l0}(\theta,\phi) \tag{0.53}$$

and

$$\sqrt{\frac{4\pi}{2l'+1}} Y_{l'm'}^*(\beta,\alpha) = \int Y_{l'm'}^*(\theta,\phi) R(\alpha,\beta,\gamma) Y_{l'0}(\theta,\phi) d\Omega. \tag{0.54}$$

Finally, use the definition [Eq. (11.273)] of $D_{mm'}^{l'}(\alpha,\beta,\gamma)$ to obtain the desired result.

A different approach:

We are now going to approach this problem from a different direction. We have obtained the differential equation that $d_{m'm}^l(\beta)$ satisfies as

$$\left\{ \frac{d^2}{d\beta^2} + \cot\beta \frac{d}{d\beta} + \left[l(l+1) - \frac{m^2 + m'^2 - 2mm'\cos\beta}{\sin^2\beta} \right] \right\} d_{m'm}^l(\beta) = 0. \tag{0.55}$$

Given the solution of the above equation:

$$d^l_{m'm}(\beta) = (-1)^{m'+m} \left[\frac{(l+m')!(l-m')!}{(l+m)!(l-m)!}\right]^{1/2} \sum_k \binom{l+m}{l-m'-k}\binom{l-m}{k}$$
$$\times (-1)^{l-m'-k} \left(\cos\frac{\beta}{2}\right)^{2k+m'+m} \left(\sin\frac{\beta}{2}\right)^{2l-2k-m'-m}, \quad (0.56)$$

we now use the Jacobi polynomials:

$$P_n^{(a,b)}(x) = 2^{-n} \sum_{k=0}^{n} \binom{n+a}{k}\binom{n+b}{n-k}(x-1)^{n-k}(x+1)^k, \qquad (0.57)$$

which satisfy the differential equation

$$(1-x^2)\frac{dy^2}{dx^2} + [b-a-(a+b+2)x]\frac{dy}{dx} + n(n+a+b+1)y(x) = 0, \qquad (0.58)$$

to express $d^l_{m'm}(\beta)$ as

$$d^l_{m'm}(\beta) = (-1)^{m'+m} \left[\frac{(l+m')!(l-m')!}{(l+m)!(l-m)!}\right]^{1/2} \left(\cos\frac{\beta}{2}\right)^{m'+m} \left(\sin\frac{\beta}{2}\right)^{m'-m}$$
$$\times P_{l-m'}^{(m'-m,m'+m)}(\cos\beta). \qquad (0.59)$$

Notes:

i) The normalization constant of $d^l_{m'm}(\beta)$ can be evaluated via the integral

$$\int_{-1}^{1} (1-x)^a(1+x)^b P_n^{(a,b)}(x)P_m^{(a,b)}(x)dx$$
$$= \frac{2^{a+b+1}}{2n+a+b+1}\frac{\Gamma(n+a+1)\Gamma(n+b+1)}{\Gamma(n+1)\Gamma(n+a+b+1)}\delta_{nm}. \qquad (0.60)$$

Also note that the Jacobi polynomials are normalized so that

$$P_n^{(a,b)}(1) = \binom{n+a}{n}. \qquad (0.61)$$

ii) You can use Equation (0.59) to check the matrix elements found in Problem 2 [Eq. (0.30)] and in Equation (11.281). You can use Mathematica® to obtain the needed Jacobi polynomials via the command " JacobiP[a,b,n,x] ".

iii) To see that $d^l_{m'm}(\beta)$ given in Equation (0.59) is indeed a solution to Equation (0.55), substitute

$$d^l_{m'm}(\beta) = C\left(\cos\frac{\beta}{2}\right)^{m'+m}\left(\sin\frac{\beta}{2}\right)^{m'-m} f(\cos\beta), \tag{0.62}$$

where C is an appropriate normalization constant, into Equation (0.55) and then show that $f(\cos\beta)$ satisfies the Jacobi equation [Eq. (0.57)] with an appropriate choice of the parameters.

For the first part of Problem 2, we need the value of $d^l_{m0}(\beta)$, which from Equation (0.59) can be written as

$$d^l_{m0}(\beta) = (-1)^m\left[\frac{(l+m)!(l-m)!}{(l!)^2}\right]^{1/2}\frac{1}{2^m}(\sin^m\beta)P^{(m,m)}_{l-m}(x). \tag{0.63}$$

We now use the relation

$$P^{(m,m)}_{l-m}(x) = (-2)^m\frac{l!}{(l-m)!}(1-x^2)^{-m/2}P_l^{-m}(x), \tag{0.64}$$

to write

$$d^l_{m0}(\beta) = \left[\frac{(l+m)!}{(l-m)!}\right]^{1/2} P_l^{-m}(\cos\beta) \tag{0.65}$$

$$= (-1)^m\left[\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_l^{m}(\cos\beta). \tag{0.66}$$

Using the definition [Eq. (11.277)]

$$D^l_{m'm}(\alpha,\beta,\gamma) = e^{-i\alpha m'}d^l_{m'm}(\beta)e^{-i\gamma m}, \tag{0.67}$$

we write

$$D^l_{m0}(\alpha,\beta,\gamma) = e^{-i\alpha m}d^l_{m0}(\beta). \tag{0.68}$$

Since [Eq. (2.177)]

$$Y_l^m(\theta,\phi) = (-1)^m\sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}e^{im\phi}P_l^m(\cos\theta), \tag{0.69}$$

Equations (0.64) and (0.66) yield the desired result:

$$D^l_{m0}(\alpha,\beta) = \sqrt{\frac{4\pi}{2l+1}}Y^*_{lm}(\beta,\alpha). \tag{0.70}$$

For the second part, first show and then use the symmetry property:

$$d^l_{m'm}(-\beta) = d^l_{mm'}(\beta). \tag{0.71}$$

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3. **(Problem 11.10)** For $l = 2$ construct the matrices
$$\mathbf{L}_y^k = \left(L_y^k\right)_{mm'} \tag{0.72}$$
for $k = 0, 1, 2, 3, 4, ...$ and show that the matrices with $k \geq 5$ can be expressed as linear combinations of these. Use this result to check the result of Problem 11.8.4.

Solution:

Use [Eq. (11.276)]
$$d^l_{m'm}(\beta) = \int\int d\Omega Y^*_{lm'}(\theta,\phi)e^{-i\beta L_y}Y_{lm}(\theta,\phi), \tag{0.73}$$
which for $l = 2$ becomes
$$d^2_{m'm}(\beta) = \int\int d\Omega Y^*_{2m'}(\theta,\phi)e^{-i\beta L_y}Y_{2m}(\theta,\phi). \tag{0.74}$$
Matrix elements, $(y_{lm'}, L_y Y_{lm})$, of L_y are obtained from [Eq. (11.263)]:
$$\begin{aligned}(L_y)_{m'm} &= -\frac{i}{2}\sqrt{(l-m)(l+m+1)}\delta_{m',m+1} \\ &\quad + \frac{i}{2}\sqrt{(l+m)(l-m+1)}\delta_{m',m-1},\end{aligned} \tag{0.75}$$
which gives
$$(L_y)_{m'm} = \begin{pmatrix} 0 & -i & 0 & 0 & 0 \\ i & 0 & -i\sqrt{3/2} & 0 & 0 \\ 0 & i\sqrt{3/2} & 0 & -i\sqrt{3/2} & 0 \\ 0 & 0 & i\sqrt{3/2} & 0 & -i \\ 0 & 0 & 0 & i & 0 \end{pmatrix}, \tag{0.76}$$
where the rows correspond to the m' values as $m' = 2, 1, 0, -1, 2$ and the columns correspond to the m values $m = 2, 1, 0, -1, 2$, respectively. We now evaluate the higher powers:
$$\left(L_y^2\right)_{m'm} = \begin{pmatrix} 1 & 0 & -\sqrt{3/2} & 0 & 0 \\ 0 & 5/2 & 0 & -3/2 & 0 \\ -\sqrt{3/2} & 0 & 3 & 0 & -\sqrt{3/2} \\ 0 & -3/2 & 0 & 5/2 & 0 \\ 0 & 0 & -\sqrt{3/2} & 0 & 1 \end{pmatrix}, \tag{0.77}$$
note the symmetry with respect to the two diagonals,
$$\left(L_y^4\right)_{m'm} = \begin{pmatrix} 5/2 & 0 & -2\sqrt{6} & 0 & 3/2 \\ 0 & 17/2 & 0 & -15/2 & 0 \\ -2\sqrt{6} & 0 & 12 & 0 & -2\sqrt{6} \\ 0 & -15/2 & 0 & 17/2 & 0 \\ 3/2 & 0 & -2\sqrt{6} & 0 & 5/2 \end{pmatrix}, \tag{0.78}$$

$$\left(L_y^3\right)_{m'm} = \begin{pmatrix} 0 & -5i/2 & 0 & 3i/2 & 0 \\ & 0 & -2i\sqrt{6} & 0 & \\ & & 0 & & \\ & & & & \\ & & & & \end{pmatrix}, \tag{0.79}$$

$$\left(L_y^5\right)_{m'm} = \begin{pmatrix} 0 & -17i/2 & 0 & 15i/2 & 0 \\ & 0 & -i8\sqrt{6} & 0 & \\ & & 0 & & \\ & & & & \\ & & & & \end{pmatrix}, \tag{0.80}$$

$$\left(L_y^6\right)_{m'm} = \begin{pmatrix} 17/2 & 0 & -8\sqrt{6} & 0 & 15/2 \\ & 65/2 & 0 & -65/2 & \\ & & 48 & & \\ & & & & \\ & & & & \end{pmatrix}, \tag{0.81}$$

$$\left(L_y^5\right)_{m'm} = -4\left(L_y\right)_{m'm} + 5\left(L_y^3\right)_{m'm}, \tag{0.82}$$

$$\left(L_y^7\right)_{m'm} = -4\left(L_y^3\right)_{m'm} + 5\left(L_y^5\right)_{m'm} = -20\left(L_y\right)_{m'm} + 21\left(L_y^3\right)_{m'm}, \tag{0.83}$$

$$\left(L_y^9\right)_{m'm} = -20\left(L_y^3\right)_{m'm} + 21\left(L_y^5\right)_{m'm} = -84\left(L_y\right)_{m'm} + 85\left(L_y^3\right)_{m'm}, \tag{0.84}$$

$$\left(L_y^{11}\right)_{m'm} = -340\left(L_y\right)_{m'm} + 341\left(L_y^3\right)_{m'm}, \tag{0.85}$$

$$\left(L_y^6\right)_{m'm} = -4\left(L_y^2\right)_{m'm} + 5\left(L_y^4\right)_{m'm} \tag{0.86}$$

$$\vdots$$

We now write $d^2_{2,-2}(\beta)$ as the series

$$\begin{aligned} d^2_{2,-2}(\beta) &= I - i\beta L_y - \frac{\beta^3}{3!}L_y^3 + \frac{\beta^5}{5!}L_y^5 - \cdots \\ &\quad + \left(-\frac{\beta^2}{2}L_y^2 + \frac{\beta^4}{4!}L_y^4 - \frac{\beta^6}{6!}L_y^6 + \cdots\right) \end{aligned} \tag{0.87}$$

and proceed to show

$$d^2_{2,-2}(\beta) = \frac{(1-\cos\beta)^2}{4}. \tag{0.88}$$

II. Useful Sites

Additional references and other useful information about the Jacobi polynomials can be found in the following sites:

`http://mathworld.wolfram.com/JacobiPolynomial.html`,

`http://en.wikipedia.org/wiki/Jacobi_polynomials`.

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