Supplements to Mathematical Methods in Science and Engineering
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CHAPTER 12: COMPLEX VARIABLES and FUNCTIONS

I. Solutions or Hints to Selected Problems:

1. Discuss the analyticity and the differentiability of

$$f(z) = \frac{x^2 y^2 (x + iy)}{x^2 + y^2}. \tag{0.1}$$

Solution:

We first write the u and v functions as

$$u(x, y) = \frac{x^3 y^2}{x^2 + y^2}, \tag{0.2}$$

$$v(x, y) = \frac{x^2 y^3}{x^2 + y^2} \tag{0.3}$$

and then evaluate the following partial derivatives:

$$\frac{\partial u}{\partial x} = \frac{x^2 y^2 (x^2 + 3y^2)}{(x^2 + y^2)^2}, \tag{0.4}$$

$$\frac{\partial u}{\partial y} = \frac{2x^5 y}{(x^2 + y^2)^2}, \tag{0.5}$$

$$\frac{\partial v}{\partial x} = \frac{2xy^5}{(x^2 + y^2)^2}, \tag{0.6}$$

$$\frac{\partial v}{\partial y} = \frac{x^2 y^2 (3x^2 + y^2)}{(x^2 + y^2)^2}. \tag{0.7}$$

Substituting these into the Cauchy-Riemann conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{0.8}$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}, \tag{0.9}$$

we obtain

$$\begin{aligned}\frac{x^2y^2(x^2+3y^2)}{(x^2+y^2)^2} &= \frac{x^2y^2(3x^2+y^2)}{(x^2+y^2)^2}\\ \frac{2xy^5}{(x^2+y^2)^2} &= -\frac{2x^5y}{(x^2+y^2)^2}.\end{aligned} \tag{0.10}$$

These two conditions can be satisfied simultaneously only at the origin. In conclusion, the derivative exists at the origin but the function is analytic nowhere. This is also apparent from the expression of $f(z)$ as

$$f(z) = -\frac{z}{16(zz^*)}(z+z^*)^2(z-z^*)^2, \tag{0.11}$$

which depends on z^* explicitly.

Important:

Cauchy-Riemann conditions are necessary for the derivative to exist at a given point z_0. It is only when the partial derivatives of u and v are continuous at z_0 that they become both necessary and sufficient. In this case one should check that the partial derivatives of u and v are indeed continuous at $z = 0$, hence the derivative of $f(z)$ [Eq. (0.1)] exists at $z = 0$ (Bayin, 2008).

2. Check the differentiability and the analyticity of the function

$$f(z) = \begin{cases} \dfrac{x^3-y^3}{x^2+y^2} + i\dfrac{x^3+y^3}{x^2+y^2}, & |z| \neq 0. \\ \\ 0, & z = 0. \end{cases} \tag{0.12}$$

Solution:

Follow the steps of the previous question and also check the continuity of the partial derivatives. Also see Example 7.2 in Bayin (2008).

3. **(Problem 12.8)** Find the Riemann surface on which

$$w = \sqrt[3]{(z-1)(z-2)(z-3)} \tag{0.13}$$

is a single valued function, analytic except at $z = 1, 2, 3$.

Solution:

We have discussed the square root function,

$$w = \sqrt{z}, \tag{0.14}$$

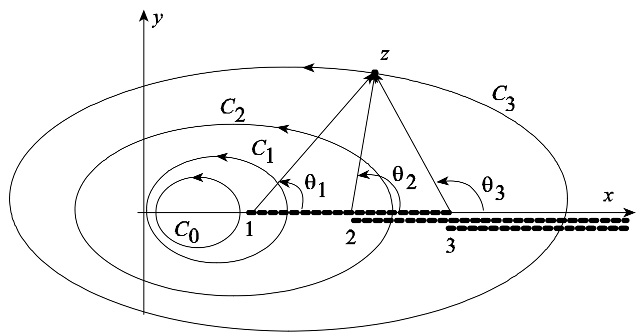

Fig. 0.1 Branch cuts for the branch points at $z = 1, 2, 3$.

in Chapter 12 in detail (Bayin, 2006), which has a branch point at $z = 0$ and two branch values

$$w = \sqrt{r}e^{i\left(\dfrac{\theta + 2\pi k}{2}\right)}, \tag{0.15}$$

where $0 \leq \theta < 2\pi$ and $k = 0, 1$. In general,

$$w = z^{1/n} \tag{0.16}$$

has a single branch point at $z = 0$, but n branch values given by

$$w = \sqrt[n]{r}e^{i\left(\dfrac{\theta + 2\pi k}{n}\right)}, \ 0 \leq \theta < 2\pi, \ k = 0, 1, \ldots, n-1. \tag{0.17}$$

In the case of the square root function, there are two Riemann sheets connected along the branch cut (Fig. 12.9 in Bayin (2006). For both of the above cases [Eqs. (0.15) and (0.17)], branch cuts are chosen to be along the positive real axis. For $w = z^{1/n}$, there are n Riemann sheets connected along cut line (Bayin, 2006). For the function

$$w = \sqrt[n]{z - z_0}, \tag{0.18}$$

the situation is not very different. There are n Riemann sheets connected along a suitably chosen branch cut, which ends at the branch point z_0. For the function

$$w = \sqrt[3]{z - z_0}, \tag{0.19}$$

for a full revolution about z_0 in the z-plane, where θ goes from 0 to 2π, the corresponding point in the w-plane completes only 1/3 of a revolution, where ϕ changes from 0 to $2\pi/3$. In other words, for a single revolution in the w-plane, one has to complete three revolutions in the z-plane. In this case the three branch values are given as

$$w = \sqrt[3]{r} e^{i\left(\dfrac{\theta + 2\pi k}{3}\right)}, \ 0 \leq \theta < 2\pi, \ k = 0, 1, 2, \tag{0.20}$$

where $r = |z - z_0|$. To avoid multiple revolutions in the z-plane, we need three Riemann sheets.

For the function at hand:

$$w = \sqrt[3]{(z-1)(z-2)(z-3)}, \tag{0.21}$$

we have three branch points located at the points

$$z_1 = 1, \ z_2 = 1, \ z_3 = 1. \tag{0.22}$$

We choose the branch cuts to be along the real axis and to the right of the corresponding branch point as shown in Figure (0.1). We can now write

$$z - 1 = r_1 e^{i\theta_1}, \tag{0.23}$$
$$z - 2 = r_2 e^{i\theta_2}, \tag{0.24}$$
$$z - 3 = r_3 e^{i\theta_3}, \tag{0.25}$$

where $0 \leq \theta_1, \theta_2, \theta_3 < 2\pi$. The corresponding branch values for the cube root function are now given as

$$\sqrt[3]{z-1} = \sqrt[3]{r_1}\ e^{i\left(\dfrac{\theta_1 + 2\pi k}{3}\right)}, \ 0 \leq \theta_1 < 2\pi, \ k = 0, 1, 2, \tag{0.26}$$
$$\sqrt[3]{z-2} = \sqrt[3]{r_2}\ e^{i\left(\dfrac{\theta_2 + 2\pi l}{3}\right)}, \ 0 \leq \theta_2 < 2\pi, \ l = 0, 1, 2, \tag{0.27}$$
$$\sqrt[3]{z-3} = \sqrt[3]{r_3}\ e^{i\left(\dfrac{\theta_3 + 2\pi m}{3}\right)}, \ 0 \leq \theta_3 < 2\pi, \ m = 0, 1, 2. \tag{0.28}$$

Hence, for

$$w = \sqrt[3]{(z-1)(z-2)(z-3)} = \rho e^{i\phi}, \tag{0.29}$$

where

$$\rho = \sqrt[3]{r_1 r_2 r_3}, \ \phi = (\theta_1 + \theta_2 + \theta_3)/3, \tag{0.30}$$

the branch values are given as

$$w = \sqrt[3]{r_1 r_2 r_3}\ e^{i\left(\dfrac{\theta + 2\pi(k + l + m)}{3}\right)}, \tag{0.31}$$

where k, l, m take the values

$$k = 0, 1, 2, \tag{0.32}$$
$$l = 0, 1, 2, \tag{0.33}$$
$$m = 0, 1, 2. \tag{0.34}$$

For points on a closed path, C_0, that does not include any of the branch points, the function [Eq. 0.21] is single valued and takes its first branch value, that is, $k = l = m = 0$. For the path C_1 only one of the branch points, $z = 1$, is within the path, hence there are three branch values corresponding to the (k, l, m) values

$$(k, l, m) = \begin{cases} (0, 0, 0) \\ (1, 0, 0) \\ (2, 0, 0) \end{cases} . \tag{0.35}$$

For the path C_2, both $z = 1$ and $z = 2$ are within the path, hence when we complete a full circuit, we cross over both of the branch cuts. In this case, the three branch values are given by

$$(k, l, m) = \begin{cases} (0, 0, 0) \\ (1, 1, 0) \\ (2, 2, 0) \end{cases} . \tag{0.36}$$

For the third circuit, C_3, all three of the branch points are within the path, hence to complete a full circuit, one has to cross over all three of the branch cuts. In this case, the function is single valued and (k, l, m) take the values

$$(k, l, m) = \begin{cases} (0, 0, 0) \\ (1, 1, 1) \\ (2, 2, 2) \end{cases} . \tag{0.37}$$

In other words, for the points to the right of $z = 3$, the three branch cuts combine to cancel each others effect, thus producing a single valued function (Fig. 0.2). To see the situation along the real axis, where the branch cuts overlap, we construct the following table, where the points

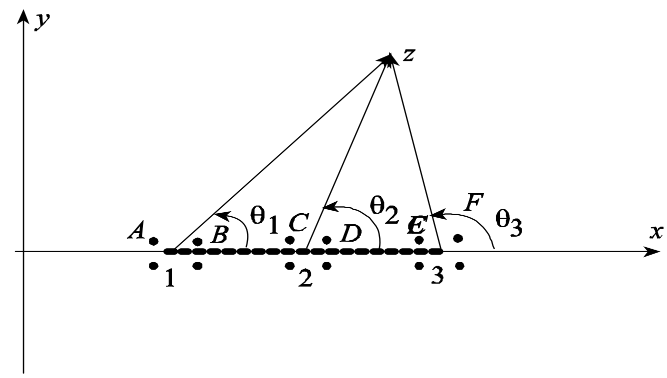

Fig. 0.2 Points below the real axis, which are symmetric to A, B, C, D, E, F, are A', B', C', D', E', F', respectively.

are defined as in Figure (0.2):

$$\begin{array}{lllll}
\text{point}\backslash\text{angle} & \theta_1 & \theta_2 & \theta_3 & \phi \\
A & \pi & \pi & \pi & \pi \\
B & 0 & \pi & \pi & 2\pi/3 \\
C & 0 & \pi & \pi & 2\pi/3 \\
D & 0 & 0 & \pi & \pi/3 \\
E & 0 & 0 & \pi & \pi/3 \\
F & 0 & 0 & 0 & 0 \\
A' & \pi & \pi & \pi & \pi \\
B' & 2\pi & \pi & \pi & 4\pi/3 \\
C' & 2\pi & \pi & \pi & 4\pi/3 \\
D' & 2\pi & 2\pi & \pi & 5\pi/3 \\
E' & 2\pi & 2\pi & \pi & 5\pi/3 \\
F' & 2\pi & 2\pi & 2\pi & 6\pi/3
\end{array}, \tag{0.38}$$

which gives

$$\begin{array}{ll}
A, A' & \text{Same pt. in the } w\text{-plane} \\
B, B' & \text{Not single valued} \\
C, C' & \text{Not single valued} \\
D, D' & \text{Not single valued} \\
E, E' & \text{Not single valued} \\
F, F' & \text{Same pt. in the } w\text{-plane}
\end{array} \tag{0.39}$$

From this table we see that the 3 Riemann sheets are sawn together along the dotted lines between the points $z = 1$ and $z = 3$ as shown in Figure (0.2).

4. (i) Show that the transformation

$$w = \frac{z+1}{1-z}, \tag{0.40}$$

maps the following region:

$$x \leq 0 \text{ and } -\infty < y < \infty, \tag{0.41}$$

onto the unit disc in the w-plane.

(ii) Find the image of the unit disc centered at the origin in the z-plane, under the transformation

$$z = \frac{i-w}{i+w}. \tag{0.42}$$

Solution:

(i) We first use the general expression

$$w = \frac{az+b}{cz+d} \tag{0.43}$$

and its inverse

$$z = \frac{dw-b}{-cw+a} \tag{0.44}$$

to write

$$z = \frac{w-1}{w+1}. \tag{0.45}$$

Using Equation (0.42) we write

$$\begin{aligned} x+iy &= \frac{(u-1)+iv}{(u+1)+iv}.\frac{(u+1)-iv}{(u+1)-iv} \\ &= \frac{(u^2+v^2-1)+i(2v)}{(u+1)^2+v^2} \end{aligned} \tag{0.46}$$

and obtain the relations

$$x = \frac{(u^2+v^2-1)}{(u+1)^2+v^2}, \tag{0.47}$$

$$y = \frac{2v}{(u+1)^2+v^2}. \tag{0.48}$$

For $x \leq 0$ these imply

$$u^2+v^2 \leq 1, \tag{0.49}$$

which is the unit disc with its center located at the origin.

(ii) Follow similar steps to find the solution.

5. Show that the transformation

$$w = i\frac{1-z}{1+z}, \tag{0.50}$$

maps the upper half of the unit disc,

$$y \geq 0 \text{ and } x^2 + y^2 \leq 1, \tag{0.51}$$

onto the first quadrant, $u \geq 0$, $v \geq 0$, of the w-plane.

Solution:

We first write the mapping [Eq. (0.50)] as

$$\begin{aligned} u + iv &= i\frac{1-x-iy}{1+x+iy} \\ &= \frac{2y + i(1-x^2-y^2)}{(1+x)^2+y^2}, \end{aligned} \tag{0.52}$$

which for $y \geq 0$ implies $u \geq 0$ and for $(1-x^2-y^2) \geq 1$ gives $v \geq 0$.

We now show that the diameter of the unit circle, $-1 < x < 1$, $y = 0$, is mapped onto the positive v-axis with $v < 1$. Using Equation (0.52) for $y = 0$, we write

$$u + iv = \frac{0 + i(1-x^2)}{(1+x^2)}, \tag{0.53}$$

which implies

$$u = 0 \tag{0.54}$$

and

$$v > 0. \tag{0.55}$$

Since $1 - x^2 < 1 + x^2$, we also have $v < 1$.

6. Determine the image of the horizontal strip

$$-\pi/2 < \operatorname{Im} z < \pi/2 \tag{0.56}$$

under the transformation

$$w = \frac{e^z - 1}{e^z + 1}. \tag{0.57}$$

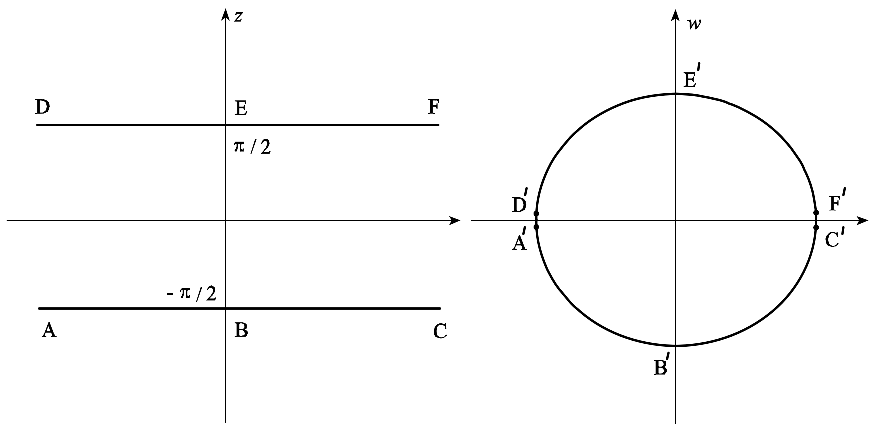

Fig. 0.3 Mappings.

Solution:

We first write the inverse of above mapping:

$$e^z = \frac{w+1}{1-w}, \tag{0.58}$$

as

$$e^x e^{iy} = \frac{(u+1)+iv}{(1-u)-iv}, \tag{0.59}$$

$$e^x(\cos y + i \sin y) = \frac{(1-u^2-v^2)+2iv}{(1-u)^2+v^2}. \tag{0.60}$$

For $y = \pm\pi/2$ this gives

$$\pm i e^x = \frac{(1-u^2-v^2)+2iv}{(1-u)^2+v^2}, \tag{0.61}$$

which implies the unit circle:

$$1 = u^2 + v^2. \tag{0.62}$$

Similarly, we can find the images of the points A, B, C, D, E, F as A', B', C', D', E', F', respectively (Fig. 0.3).

7. Find the Schwarz-Christoffel transformation that maps the semi-infinite strip: $-\pi/2 < x < \pi/2$, onto the upper half w-plane, $v > 0$. Use this result to solve the Laplace equation within the given strip satisfying the boundary conditions

$$V(x, 0) = 1 \text{ and } V(-\pi/2, y) = V(\pi/2, y) = 0. \tag{0.63}$$

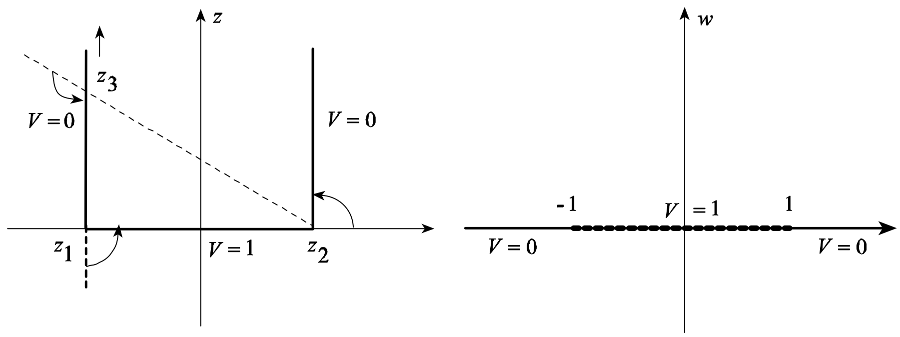

Fig. 0.4 Schwarz-Christoffel transformations.

Solution:

We map the points $(\pm\pi/2, 0)$ in the z-plane to $(\pm 1, 0)$ in the w-plane, respectively (Fig. 0.4). We also map the point z_3 to ∞. Schwarz-Christoffel transformation can now be written as

$$\frac{dz}{dw} = A(w+1)^{-k_1}(w-1)^{-k_2}(w-\infty)^{-k_3}, \tag{0.64}$$

where

$$k_1 = k_2 = 1/2,\ k_3 = 1. \tag{0.65}$$

We again absorb ∞ into the arbitrary constant A and define a new constant C_0 to write

$$\frac{dz}{dw} = C_0(w^2-1)^{-1/2}, \tag{0.66}$$

which upon integration yields

$$z = C_0 \cosh^{-1} w + C_1. \tag{0.67}$$

Since

$$\begin{array}{lll} z_1 = (-\pi/2, 0) & \rightarrow & w_1 = (-1, 0), \\ z_2 = (\pi/2, 0) & \rightarrow & w_2 = (1, 0), \end{array} \tag{0.68}$$

we determine C_0 and C_1 as

$$C_0 = i, \tag{0.69}$$

$$C_1 = \pi/2 \tag{0.70}$$

and write

$$z = i \cosh^{-1} w + \pi/2. \tag{0.71}$$

From the electromagnetic theory, solution of the Laplace equation in the w-plane is given as

$$V(u, v) = \frac{v}{\pi} \int_{-\infty}^{+\infty} \frac{V(\xi, 0)d\xi}{(u - \xi)^2 + v^2}, \tag{0.72}$$

which can be integrated to yield

$$V(u, v) = \frac{1}{\pi} \tan^{-1} \left[\frac{2v}{u^2 + v^2 - 1} \right]. \tag{0.73}$$

One should check that the above $V(u, v)$ does indeed satisfies the Laplace equation in the w-plane with the following boundary conditions:

$$V(u, 0) = 1 \quad \text{for} \quad -1 < u < 1, \tag{0.74}$$

$$V(u, v) = 0 \quad \text{elsewhere}. \tag{0.75}$$

For the solution in the z-plane, we need the transformation equations between (u, v) and (x, y). Using Equation (0.71) we write

$$\begin{aligned} w &= \cosh\left(\frac{z - \pi/2}{i}\right), \\ &= \sin x \cosh y + i \sinh y \cos x, \end{aligned} \tag{0.76}$$

thus obtaining the needed relations as

$$u = \sin x \cosh y, \tag{0.77}$$

$$v = \sinh y \cos x. \tag{0.78}$$

The solution in the z-plane can now be written as

$$V(x, y) = \frac{1}{\pi} \tan^{-1} \left[\frac{2 \sinh y \cos x}{\sin^2 x \cosh^2 y + \sinh^2 y \cos^2 x - 1} \right]. \tag{0.79}$$

Note:

The transformation we obtained [Eq. (0.71)]:

$$z = i[\cosh^{-1} w + \pi/2] \tag{0.80}$$

can in general be written as

$$z = i[A \cosh^{-1} w + B], \tag{0.81}$$

where the constants depend on the orientation of the strip in the z-plane. If you compare this with the horizontal strip used in Example 12.13, the factor i is essentially rotating the domain by $\pi/2$.

8. Find the Schwarz-Christoffel mapping that transforms the bent line, A, B, C, D, in the w-plane with the points

$$(-2,1),\ (-1,0),\ (1,0),\ (2,-1), \tag{0.82}$$

respectively, into a straight line along the real axis in the z-plane.

Solution:

We map point B to $(-1,0)$ and C to $(1,0)$ as shown in Figure (0.5). Differential form of the transform is written as

$$\frac{dw}{dz} = A(z-z_1)^{-k_1}(z-z_2)^{-k_2}, \tag{0.83}$$

where k_1 and k_2 are determined as

$$k_1\pi = \pi/4 \to k_1 = 1/4, \tag{0.84}$$
$$k_2\pi = -\pi/4 \to k_2 = -1/4. \tag{0.85}$$

Equation (0.83) now becomes

$$\begin{aligned}\frac{dw}{dz} &= A(z+1)^{1/4}(z-1)^{-1/4} \\ &= A\left(\frac{z-1}{z+1}\right)^{1/4},\end{aligned} \tag{0.86}$$

which upon integration yields

$$\begin{aligned}w &= A\int\left(\frac{z-1}{z+1}\right)^{1/4} dz + B \\ &= A\left[\frac{-2u}{u^4-1} + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| - \tan^{-1}u\right] + B,\end{aligned} \tag{0.87}$$

where

$$u = \left(\frac{z-1}{z+1}\right)^{1/4}. \tag{0.88}$$

Using the fact that

$$z = 1 \to w = 1, \tag{0.89}$$
$$z = -1 \to w = -1 \tag{0.90}$$

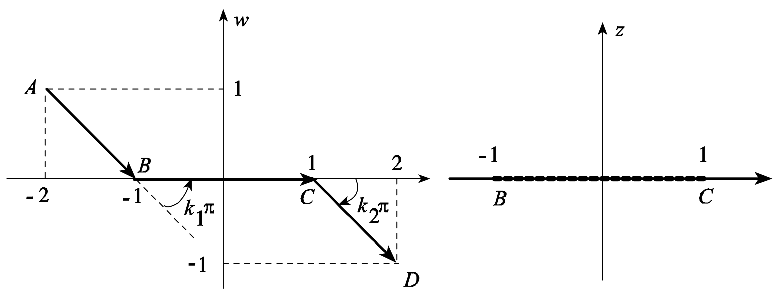

Fig. 0.5 Schwarz-Christoffel transfomation

we obtain two equations

$$1 = B + A\pi/2, \tag{0.91}$$
$$-1 = B + A\pi/2, \tag{0.92}$$

hence determine the integration constants A and B as

$$A = \frac{4}{\pi(i-1)}, \quad B = \frac{1+i}{1-i}. \tag{0.93}$$

9. **(Problem 12.10)** Show that the transformation

$$\frac{w}{2} = \tan^{-1}\frac{iz}{a} \tag{0.94}$$

or

$$w = -i\ln\frac{1+\frac{z}{a}}{1-\frac{z}{a}} \tag{0.95}$$

maps the v =constant lines into circles in the z-plane.

Solution:

Use Equation (0.94) to show that

$$x = a\frac{\sinh v}{\cosh v + \cos u}, \tag{0.96}$$
$$y = -a\frac{\sin u}{\cosh v + \cos u} \tag{0.97}$$

and than use these to show that v =constant lines are mapped to circles with radius

$$r = a\csc h(v) \tag{0.98}$$

and centered at

$$x = a \coth v, \tag{0.99}$$
$$y = 0. \tag{0.100}$$

II. Interesting Sites

In 1956 a Dutch artist Maurits Cornelis Escher (1898-1972) produced a lithograph that intrigued mathematicians. In 2003 Lenstra and his group at Leiden University succeeded in deciphering the mathematical secrets of this lithograph in terms of complex mappings. This article can be found in their web site:

`http://escherdroste.math.leidenuniv.nl/`,

which includes many other images and animations. Mathematical accuracy of the Escher's image was uncanny, since he was not trained in mathematics. A step by step description of the Lenstra's method and its applications to other images is given by Leys in the site

`http://www.josleys.com/articles/printgallery.htm`.

A gallery of other images produced by Escher along with other interesting material can be found in Leys's home page:

`http://www.josleys.com/`.

An overview of the Esher Droste effect can be found in Wikipedia:

`http://en.wikipedia.org/wiki/Droste_effect`.

If you are comfortable with Mathmap-For Windows and Photoshop, you can produce such images yourself by using the tutorial in the web site

`http://www.flickr.com/photos/joshsommers/sets/72157594515046947/`

and then submit them to Escher's Droste Print Galley group.

Selçuk Bayin (December, 2008)