SUPPLEMENTS TO MATHEMATICAL METHODS IN SCIENCE AND ENGINEERING
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CHAPTER 5: GEGENBAUER and CHEBYSHEV POLYNOMIALS

I. Solutions or Hints to Selected Problems:

1. Write the wave equation [Eq. (5.3)] for the massless conformal scalar field in a closed static Friedmann (Einstein) universe explicitly.

   Solution:

   The wave equation for the massless conformal scalar field in a closed static Friedmann universe is given as

$$\Box\Phi(t,\chi,\theta,\phi)+\frac{1}{R_0^2}\Phi(t,\chi,\theta,\phi)=0, \tag{0.1}$$

   where R_0 is the radius of the universe. D'Alembert (wave) operator, $\Box$, is defined as

$$\Box\equiv g_{\mu\nu}\nabla^{\mu}\nabla^{\nu},\ \ \mu,\nu=0,1,2,3, \tag{0.2}$$

   where ∇_μ stands for the covariant derivative, which is also shown as ∂_μ. The $\Box$ operator can also be written as [Eq. (10.225)]

$$\Box\equiv g^{1/2}\frac{\partial}{\partial x^{\mu}}\left[g^{1/2}g^{\mu\nu}\frac{\partial}{\partial x^{\nu}}\right], \tag{0.3}$$

   where g stands for the absolute value of the determinant of the metric tensor [Eq. (10.238)]. Note that we use the Einstein summation convention, that is, the repeated indices are summed over (Chapter 10). In a closed static Friedmann universe, the metric tensor is defined by the line element [Eq. (5.1)]

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^{\mu}dx^{\nu} \\ &= dt^2-R_0^2\left[d\chi^2+\sin^2\chi d\theta^2+\sin^2\chi\sin^2\theta d\phi^2\right], \end{aligned} \tag{0.4}$$

where R_0 is the constant radius of the universe and

$$g = R_0^6 \sin^4\chi \sin^2\theta.$$

Using the summation convention and the identification

$$x^0 = t,\ x^1 = \chi,\ x^2 = \theta,\ x^3 = \phi, \tag{0.5}$$

Equation (0.1) with Equation (0.3) can be written as

$$\left[\frac{1}{2g}\frac{\partial g}{\partial x^\mu} g^{\mu\nu}\partial_\nu + \frac{\partial g^{\mu\nu}}{\partial x^\mu}\partial_\nu + g^{\mu\nu}\partial_\mu\partial_\nu + \frac{1}{R_0^2}\right]\Phi = 0, \tag{0.6}$$

which eventually leads to

$$\begin{aligned}&\frac{\partial^2\Phi}{\partial t^2} - \frac{1}{R_0^2}\frac{\partial^2\Phi}{\partial\chi^2} - \frac{1}{R_0^2\sin^2\chi}\frac{\partial^2\Phi}{\partial\theta^2} - \frac{1}{R_0^2\sin^2\theta\sin^2\chi}\frac{\partial^2\Phi}{\partial\phi^2}\\ &-\frac{2\cos\chi}{R_0^2\sin\chi}\frac{\partial\Phi}{\partial\chi} - \frac{\cos\theta}{R_0^2\sin^2\chi\sin\theta}\frac{\partial\Phi}{\partial\theta} + \frac{3\dot{R}_0}{R_0}\frac{\partial\Phi}{\partial t} + \frac{1}{R_0^2}\Phi = 0.\end{aligned} \tag{0.7}$$

We now try the separation of variables method and substitute a solution of the form

$$\Phi(t,\chi,\theta,\phi) = T(t)X(\chi)Y(\theta,\phi), \tag{0.8}$$

which yields Equation (5.6) and eventually leads to Equations (5.7)−(5.9) in the book.

2. Show that the function

$${}_1\Pi_l^N = \sin^l\chi\frac{d^{l+1}(\cos N\chi)}{d(\cos\chi)^{l+1}} \tag{0.9}$$

also satisfies the differential equation for $X(\chi)$ [Eq. (5.9)]:

$$\sin^2\chi\frac{d^2X}{d\chi^2} + 2\sin\chi\cos\chi\frac{dX}{d\chi} + \left[(\omega^2 - \frac{1}{R_0^2})R_0^2\sin^2\chi - l(l+1)\right]X(\chi) = 0 \tag{0.10}$$

with

$$\omega = \frac{N}{R_0}. \tag{0.11}$$

Solution:

We first write Equation (0.10) as

$$\frac{d}{d\chi}\left[\sin^2\chi\frac{dX}{d\chi}\right] + \left[(\omega^2 - \frac{1}{R_0^2})R_0^2\sin^2\chi - l(l+1)\right]X(\chi) = 0 \tag{0.12}$$

and then make the transformation

$$x = \cos\chi \tag{0.13}$$

to obtain

$$(1-x^2)^2\frac{d^2X}{dx^2} - 3x(1-x^2)\frac{dX}{dx} + \left[(\omega^2R_0^2-1)(1-x^2) - l(l+1)\right]X = 0. \tag{0.14}$$

We now substitute

$${}_1\Pi_l^N = (1-x^2)^{l/2}\frac{d^{l+1}(\cos N\chi)}{dx^{l+1}} \tag{0.15}$$

into the above differential equation to get

$$\begin{aligned}(1-x^2)\frac{d^{l+3}(\cos Nx)}{dx^{l+3}} &- (2l+3)x\frac{d^{l+2}(\cos Nx)}{dx^{l+2}}\\ +[-l^2-2l-1+\omega^2R_0^2]&\frac{d^{l+1}(\cos Nx)}{dx^{l+1}} = 0.\end{aligned} \tag{0.16}$$

Now, the problem boils down to showing that the function

$$y(x) = \frac{d^{l+1}(\cos Nx)}{dx^{l+1}} \tag{0.17}$$

satisfies the following second-order differential equation:

$$\begin{aligned}(1-x^2)\frac{d^2}{dx^2}\left[\frac{d^{l+1}(\cos Nx)}{dx^{l+1}}\right] &- (2l+3)x\frac{d}{dx}\left[\frac{d^{l+1}(\cos Nx)}{dx^{l+1}}\right]\\ +[-(l+1)^2+\omega^2R_0^2]&\left[\frac{d^{l+1}(\cos Nx)}{dx^{l+1}}\right] = 0,\end{aligned} \tag{0.18}$$

for $\omega = \frac{N}{R_0}$. We first show that the above equation is true for $l = 0$:

$$\begin{aligned}(1-x^2)\frac{d^3(\cos Nx)}{dx^3} &- 3x\frac{d^2(\cos Nx)}{dx^2}\\ +[-1+\omega^2R_0^2]&\frac{d(\cos Nx)}{dx} = 0.\end{aligned} \tag{0.19}$$

Evaluating the derivatives explicitly gives

$$\omega = \frac{N}{R_0}. \tag{0.20}$$

Finally, differentiating Equation (0.19) l times and using the Leibnitz rule:

$$\frac{d^l(uv)}{dx^l} = \sum_{r=0}^{l}\binom{l}{r}\frac{d^ru}{dx^r}\frac{d^{l-r}v}{dx^{l-r}}, \tag{0.21}$$

we get the desired result. Note that N is not quantized yet. However, for finite solutions everywhere in the interval $x \in [-1, 1]$, we restrict N to integer values:

$$N = 1, 2, \ldots .$$

3. Show that a second and linearly independent solution of Equation (0.10) can be written as

$$ {}_2\Pi_l^N(x) = (1 - x^2)^l \frac{d^{l+1}(\sin N\chi)}{dx^{l+1}}. \tag{0.22}$$

Discuss the boundary conditions for the general solution. Establish the connection between ${}_1\Pi_l^N(x)$ and the solution given in the book in terms of Gegenbauer polynomials [Eq.(5.31)].

Solution:

To prove that the above function is indeed a solution of Equation (0.10), we use the same method used in the previous problem for ${}_1\Pi_l^N(x)$. For their linear independence, check their Wronskian [Eq. (6.84)]. The general solution can now be given as the linear combination:

$$X(x) = c_0\ {}_1\Pi_l^N(x) + c_1\ {}_2\Pi_l^N(x), \tag{0.23}$$

where c_0 and c_1 are two integration constants. We now impose the boundary condition that the solution be finite everywhere in the interval

$$\chi \in [0, \pi], \text{ or } x \in [-1, 1]. \tag{0.24}$$

The second solution diverges for $\chi = 0$, hence we set its coefficient to zero.

To establish the connection with the Gegenbauer polynomials, we make use of the trigonometric expansion

$$\cos N\chi = \sum_{j=0}^{[N/2]} \frac{N(N-j-1)!(-1)^j 2^{N-(2j+1)}}{(N-2j)!j!} (\cos\chi)^{N-2j} \tag{0.25}$$

$$= \sum_{j=0}^{[N/2]} \frac{N(N-j-1)!(-1)^j 2^{N-(2j+1)}}{(N-2j)!j!} x^{N-2j}, \tag{0.26}$$

which terminates when a coefficient is zero. We can now write

$$\frac{d^{l+1}(\cos Nx)}{dx^{l+1}} = N \sum_{j=0}^{[(N-l-1)/2]} (-1)^j \frac{2^{N-(2j+1)}(N-j-1)!}{(N-2j-l-1)!j!} x^{N-2j-l-1}, \tag{0.27}$$

where $N = 1, 2, \ldots$ and we have used the formula

$$\frac{d^m x^n}{dx^m} = \frac{n!}{(m-n)!} x^{m-n}. \tag{0.28}$$

Comparing this with the Gegenbauer polynomials [Eq. (5.28)]:

$$C_n^\lambda(x) = \sum_{r=0}^{[n/2]} (-1)^r \frac{\Gamma(n-r+\lambda)}{\Gamma(\lambda) r! (n-2r)!} (2x)^{n-2r}, \tag{0.29}$$

which satisfies the differential equation [Eq. (5.27)]

$$(1-x^2)\frac{d^2 C_n^\lambda(x)}{dx^2} - (2\lambda+1)x\frac{dC_n^\lambda(x)}{dx} + n[n+2\lambda]C_n^\lambda(x) = 0, \tag{0.30}$$

we see that the function $\left[\frac{d^{l+1}(\cos Nx)}{dx^{l+1}}\right]$ is proportional to $C_{N-l-1}^{l+1}(x)$, $N = 1, 2, \ldots$. That is,

$$\begin{aligned}
C_{N-l-1}^{l+1}(x) &= \sum_{j=0}^{[(N-l-1)/2]} (-1)^j \frac{\Gamma(N-j) 2^{N-2j-1-l}}{j!\Gamma(l+1)(N-2j-l-1)!} x^{N-2j-l-1} \\
&= \sum_{j=0}^{[(N-l-1)/2]} (-1)^j \frac{(N-j-1)! 2^{-l} 2^{N-2j-1}}{j! l! (N-2j-l-1)!} x^{N-2j-l-1} \\
&= \left(\frac{2^{-l}}{l!}\right) \sum_{j=0}^{[(N-l-1)/2]} (-1)^j \frac{2^{N-2j-1}(N-j-1)!}{j!(N-2j-l-1)!} x^{N-2j-l-1}, \\
&= \left(\frac{2^{-l}}{l!}\right) \frac{1}{N} \left[\frac{d^{l+1}(\cos Nx)}{dx^{l+1}}\right].
\end{aligned} \tag{0.31}$$

Hence, they both satisfy the same differential equation [Eq. (0.18)]. We can now write the solution of Equation (5.9) as

$$X(\chi) = c_0 \, {}_1\Pi_l^N(\chi), \tag{0.32}$$

$$= c_0 \sin^l \chi \frac{d^{l+1}(\cos N\chi)}{d(\cos \chi)^{l+1}}, \tag{0.33}$$

or

$$X(x) = C_0 (1-x^2)^{l/2} C_{N-l-1}^{l+1}(x), \tag{0.34}$$

where $x = \cos \chi$. For finite solutions everywhere in the interval $x \in [-1, 1]$, we also restrict N to integers:

$$N = 1, 2, \ldots \ . \tag{0.35}$$

4. Evaluate the normalization constants in $X(\chi)$ and $X(x)$ [Eqs. (0.33) and (0.34)].

   Solution:

   Using the normalization condition of the Gegenbauer polynomials [Eq. (5.29)], we can write

$$\int_{-1}^{+1}(1-x^2)^{l+1/2}C_{N-l-1}^{l+1}dx = \frac{\pi}{2^{l+1}N(l!)^2}\frac{(N+l)!}{(N-l-1)!}. \qquad (0.36)$$

   We can also write the ratio

$$\begin{aligned}\frac{(N+l)!}{(N-l-1)!} &= \frac{(N+l)\cdots N\cdots(N-l)(N-l-1)!}{(N-l-1)!} \\ &= (N^2-l^2)\cdots(N^2-1)N, \end{aligned} \qquad (0.37)$$

   which yields

$$\int_{-1}^{+1}(1-x^2)^{l+1/2}C_{N-l-1}^{l+1}dx = \frac{\pi(N^2-l^2)\cdots(N^2-1)}{2^{l+1}(l!)^2}. \qquad (0.38)$$

   This gives C_0 as

$$C_0 = \frac{2^{(l+1)/2}l!}{\sqrt{\pi}}\left[(N^2-l^2)\cdots(N^2-1)\right]^{-1/2}. \qquad (0.39)$$

   We now use Equations (0.27) and (0.31) to establish the relation

$$C_{N-l-1}^{l+1} = l!2^l N\left[\frac{d^{l+1}(\cos N\chi)}{d(\cos\chi)^{l+1}}\right] \qquad (0.40)$$

   to obtain

$$c_0 = \left[\frac{\pi}{2}(N^2-l^2)\cdots(N^2-1)N^2\right]^{-1/2}. \qquad (0.41)$$

5. The wave equation for the massless conformal scalar field in an open static Friedmann universe is written as

$$\Box\Phi(t,\chi,\theta,\phi) - \frac{1}{R_0^2}\Phi(t,\chi,\theta,\phi) = 0, \qquad (0.42)$$

   where the line element is given as

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu \\ &= dt^2 - R_0^2\left[d\chi^2 + \sinh^2\chi d\theta^2 + \sinh^2\chi\sin^2\theta d\phi^2\right] \end{aligned} \qquad (0.43)$$

   and the range of the coordinates are:

$$\chi\in[0,\infty],\ \theta\in[0,\pi],\ \phi\in[0,2\pi]. \qquad (0.44)$$

Following similar steps as in the previous problems, find separable solutions of the wave equation and show that the solution you have found can be obtained from the closed universe case by the transformation

$$N \to iN, \ \chi \to i\chi. \tag{0.45}$$

What are the allowed frequencies in this case?

Note:

These results have found great use in the study of quantum fields in Friedmann universes. For details and other references we refer to *Quantum Fields in Curved Space* by Birrell and Davies (Cambridge, 1984). In particular, see their Chapter 5.

II. Useful Sites

More references and other useful information about Gegenbauer and Chebyshev polynomials can be found in the following sites:

`http://en.wikipedia.org/wiki/Gegenbauer_polynomials`,

`http://mathworld.wolfram.com/GegenbauerPolynomial.html`,

`http://en.wikipedia.org/wiki/Chebyshev_polynomials`,

`http://mathworld.wolfram.com/ChebyshevPolynomialoftheFirstKind.html`.

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