SUPPLEMENTS TO MATHEMATICAL METHODS IN SCIENCE AND ENGINEERING
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CHAPTER 13: COMPLEX INTEGRALS and SERIES

I. Solutions or Hints to Selected Problems:

1. Let $f(z)$ be an analytic function within and on a simple closed curve C and let z_0 be a point not on C.

   If

$$I_1 = \oint_C \frac{f'(z)dz}{(z-z_0)} \tag{0.1}$$

   and

$$I_2 = \oint_C \frac{f(z)dz}{(z-z_0)^2}, \tag{0.2}$$

   then show that $I_1 = I_2$ and evaluate I_1 in terms of z_0.

   Solution:

   When z_0 is within C, using the Cauchy integral theorem [Eqs. (13.6) and (13.19)] we can write

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z-z_0)} \tag{0.3}$$

   and

$$\frac{2\pi i}{n!} f^{(n)}(z_0) = \oint_C \frac{f(z)dz}{(z-z_0)^{n+1}}, \tag{0.4}$$

   hence

$$I_1 = I_2 = 2\pi i f'(z_0). \tag{0.5}$$

When z_0 is outside of C and $\frac{f'(z)}{(z-z_0)}$ and $\frac{f'(z)}{(z-z_0)^2}$ are analytic within and on C, then

$$I_1 = I_2 = 0. \tag{0.6}$$

2. Evaluate the integral

$$I = \oint_C z^m z^{*n} dz, \;\; m, n \text{ are integers,} \tag{0.7}$$

over the unit circle.

Solution:

Over the unit circle we write

$$z = e^{i\theta}, \;\; dz = iz d\theta, \tag{0.8}$$

hence

$$I = i \int_0^{2\pi} e^{i(m+1-n)\theta} d\theta, \tag{0.9}$$

$$= \frac{i}{i(m+1-n)} \; e^{i(m+1-n)\theta}\Big|_0^{2\pi}, \tag{0.10}$$

$$= \begin{cases} 2\pi i, & m+1 = n \\ 0, & m+1 \neq n \end{cases}. \tag{0.11}$$

3. If a function has an isolated pole of order m :

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \frac{b_1}{(z-z_0)} + \cdots + \frac{b_m}{(z-z_0)^m}, \tag{0.12}$$

first show that its residue at z_0 can be given as

$$\mathrm{Res}[f(z_0)] = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} [(z-z_0)^m f(z)] \tag{0.13}$$

and then find the residues of

$$f(z) = \frac{z}{(z+1)^2(z-1)}. \tag{0.14}$$

Finally, evaluate the integral

$$I = \oint_C \frac{z \; dz}{(z+1)^2(z-1)} \tag{0.15}$$

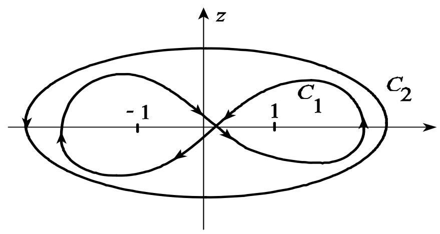

Fig. 0.1 Contours for Problem 0.3.

over the contours, C_1 and C_2, shown in Figure (0.1).

Solution:

If $f(z)$ has a pole of order m, then

$$g(z) = (z - z_0)^m f(z) \tag{0.16}$$

is analytic at z_0 :

$$g(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{n+m} + b_m + b_{m-1}(z - z_0) + \cdots + b_1 (z - z_0)^{m-1}, \tag{0.17}$$

hence

$$\lim_{z \to z_0} \frac{dg(z)}{dz} = b_{m-1}, \tag{0.18}$$

$$\lim_{z \to z_0} \frac{d^2 g(z)}{dz^2} = 2! b_{m-2}, \tag{0.19}$$

$$\vdots$$

$$\lim_{z \to z_0} \frac{d^{m-1} g(z)}{dz^{m-1}} = (m-1)! b_1. \tag{0.20}$$

Since

$$\mathrm{Res}[f(z_0)] = b_1, \tag{0.21}$$

we can write

$$\mathrm{Res}[f(z_0)] = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1} g(z)}{dz^{m-1}}, \tag{0.22}$$

thus proving the desired result.

The given function:

$$f(z) = \frac{z}{(z+1)^2 (z-1)} \tag{0.23}$$

has a second-order isolated pole at $z = -1$ and a first-order isolated pole $z = 1$, hence we can write its residues as

$$\mathrm{Res}[f(1)] = \frac{1}{0!}\lim_{z\to 1}\frac{d^{1-1}}{dz^{1-1}}\left[(z-1)\frac{z}{(z+1)^2(z-1)}\right] \tag{0.24}$$

$$= \frac{1}{4} \tag{0.25}$$

and

$$\mathrm{Res}[f(-1)] = \frac{1}{(2-1)!}\frac{d^{2-1}}{dz^{2-1}}\lim_{z\to -1}\left[(z+1)^2\frac{z}{(z+1)^2(z-1)}\right] \tag{0.26}$$

$$= -\frac{1}{4}. \tag{0.27}$$

We can now evaluate the integral for path C_1 as

$$I = \oint_{C_1}\frac{z\,dz}{(z+1)^2(z-1)} = 2\pi i\left[\frac{1}{4} - (-\frac{1}{4})\right] = \pi i \tag{0.28}$$

and for C_2 as

$$I = \oint_{C_2}\frac{z\,dz}{(z+1)^2(z-1)} = 2\pi i\left[\frac{1}{4} + (-\frac{1}{4})\right] = 0. \tag{0.29}$$

4. **(Problem 13.10)** The Jacobi polynomials $P_n^{(a,b)}(x)$, where $n =$ positive integer, $x = \cos\theta$ and a, b are arbitrary real numbers are defined by the Rodriguez formula

$$P_n^{(a,b)}(x) = \frac{(-1)^n}{2^n n!(1-x)^a(1+x)^b}\frac{d^n}{dx^n}\left[(1-x)^{n+a}(1+x)^{n+b}\right], \quad |x| < 1. \tag{0.30}$$

Find a contour integral representation for this polynomial valid for $|x| < 1$ and use this to show that the polynomial can be expanded as

$$P_n^{(a,b)}(\cos\theta) = \sum_{k=0}^{n} A(n,a,b,k)\left(\sin\frac{\theta}{2}\right)^{2n-2k}\left(\cos\frac{\theta}{2}\right)^{2k}. \tag{0.31}$$

Determine the coefficients $A(n,a,b,k)$ for the special case, where a and b are both integers.

Solution:

Using the Cauchy Theorem [Eq. (13.19)] we obtain

$$P_n^{(a,b)}(x) = \frac{(-1)^n}{2^n n!}\frac{1}{(1-x)^a(1+x)^b}\frac{n!}{2\pi i}\oint_C \frac{(1-z')^{n+a}(1+z')^{n+b}}{(z'-x)^{n+a}}dz', \tag{0.32}$$

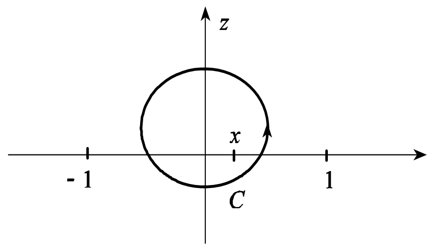

Fig. 0.2 Contour for $P_n^{(a,b)}(x)$.

where C is a closed contour as shown in Figure (0.1).

To evaluate the contour integral we need to write $(1-z')^{n+a}$ and $(1+z')^{n+b}$ in powers of $(z'-x)$. When a = integer, we use the binomial formula to write

$$(1-z')^{n+a} = (1-x)^{n+a}\left[1-\frac{(z'-x)}{(1-x)}\right]^{n+a} \tag{0.33}$$

$$= (1-x)^{n+a}\sum_{k=0}^{n+a}(-1)^k\frac{(n+a)!}{(n+a-k)!k!}\left[\frac{(z'-x)}{(1-x)}\right]^k, \tag{0.34}$$

$$= (1-x)^{n+a}\sum_{k=0}^{n+a}(-1)^k\frac{(n+a)_k}{k!}\left[\frac{(z'-x)}{(1-x)}\right]^k, \tag{0.35}$$

where

$$(n+a)_k = (n+a)(n+a-1)(n+a-2)\cdots(n+a+1-k). \tag{0.36}$$

When $a \neq$ integer, we have an infinite sum:

$$(1-z')^{n+a} = (1-x)^{n+a}\sum_{k=0}^{\infty}(-1)^k\frac{(n+a)_k}{k!}\left[\frac{(z'-x)}{(1-x)}\right]^k, \tag{0.37}$$

which can also be written as

$$(1-z')^{n+a} = (1-x)^{n+a}\sum_{k=0}^{\infty}\frac{[-(n+a)]_k}{k!}\left[\frac{(z'-x)}{(1-x)}\right]^k, \tag{0.38}$$

where

$$[-(n+a)]_k = [-(n+a)][-(n+a-1)][-(n+a-2)]\cdots[-(n+a-k+1)]. \tag{0.39}$$

Note that when a is an integer, for $k > (n+a)$ we have

$$[-(n+a)]_k = 0, \tag{0.40}$$

hence we can use Equation (0.38) for all a. Similarly, we write

$$(1+z')^{n+b} = (1+x)^{n+b} \sum_{j=0}^{\infty} \frac{[-(n+b)]_j}{j!} \left[\frac{(z'-x)}{(1+x)}\right]^j (-1)^j. \qquad (0.41)$$

Using the result

$$\oint_C \frac{(z'-x)^{k+j}}{(z'-x)^{n+1}} dz' = 2\pi i \ \delta_{k+1,n}, \qquad (0.42)$$

which the reader should show, and for a, b integers, we can now write

$$P_n^{(a,b)}(x) = \frac{(-1)^n}{2^n}(1-x)^n(1+x)^n \sum_{k=0}^{n+a}\sum_{j=0}^{n+b} \frac{(-1)^{2k}(n+a)!}{(n+a-k)!k!}\frac{(-1)^j(n+b)!}{(n+b-j)!j!} \qquad (0.43)$$
$$\times \frac{1}{(1-x)^k(1+x)^j}\delta_{k+j,n}.$$

Simplification of the above expression yields

$$P_n^{(a,b)}(x) = \sum_{k=0}^{n} \frac{(1-x)^{n-k}(1+x)^k(-1)^{n+k}}{2^n} \frac{(n+a)!(n+b)!}{k!(n+a-k)!(b+k)!(n-k)!} \qquad (0.44)$$

or

$$P_n^{(a,b)}(\theta) = \sum_{k=0}^{n} \left(\sin\frac{\theta}{2}\right)^{2n-k} \left(\cos\frac{\theta}{2}\right)^{2k} \frac{(-1)^{n+k}(n+a)!(n+b)!}{k!(n+a-k)!(b+k)!(n-k)!}. \qquad (0.45)$$

Note that $j = n-k$ but $j+k \leq n$, where $j \geq 0$, hence the upper limit of the sum in Equation (0.44) is n.

An important application of this result is to the reduced rotation matrix [Eq. (11.276)] as

$$d^j_{m'm}(\beta) = \left[\frac{(j+m')!(j-m')!}{(j+m)!(j-m)!}\right]^{1/2}$$
$$\times \left(\sin\frac{\beta}{2}\right)^{m'-m} \left(\cos\frac{\beta}{2}\right)^{m'+m} P_{(j-m')}^{(m'-m,m+m')}(\beta). \qquad (0.46)$$

5. **(Problem 13.16** and **Problem 9.11)** First use the factorization method to show that the spherical Hankel functions of the first kind:

$$h_l^{(1)} = j_l + in_l, \qquad (0.47)$$

can be expressed as

$$h_l^{(1)}(x) = (-1)^l x^l \left[\frac{1}{x}\frac{d}{dx}\right]^l h_0^{(1)}(x) \tag{0.48}$$

$$= (-1)^l x^l \left[\frac{1}{x}\frac{d}{dx}\right]^l \left(\frac{-ie^{ix}}{x}\right). \tag{0.49}$$

and then use the above result to define $h_l^{(1)}(x)$ by a contour integral in the j'-plane, $j' = t' + is'$, where

$$\frac{d}{dt} = \frac{1}{x}\frac{d}{dx}. \tag{0.50}$$

Indicate your contour by carefully showing the singularities to be avoided.

Solution:

Using the substitution

$$y_l(x) = xh_l^{(1)}(x) \tag{0.51}$$

in Equation (6.43), it is easy to show that $y_l(x)$ satisfies the differential equation

$$y_l'' + \left[1 - \frac{l(l+1)}{x^2}\right] y_l(x) = 0. \tag{0.52}$$

We now solve the above differential equation via the factorization method (Chapter 9). Since

$$r(x,l) = -\frac{l(l+1)}{x^2}, \tag{0.53}$$

$$\lambda = 1, \tag{0.54}$$

using the table in Infeld and Hull (Bayin, 2006) for type C, we find

$$K(x,l) = \frac{l}{x}, \tag{0.55}$$

$$\mu(l) = 0. \tag{0.56}$$

This gives the ladder operators as

$$O_\pm = \pm\frac{d}{dx} - \frac{l}{x}. \tag{0.57}$$

The normalized ladder operators yield

$$y_{l+1} = \frac{1}{\sqrt{\lambda - \mu(l+1)}} \left[\frac{d}{dx} - \frac{l+1}{x}\right] y_l, \tag{0.58}$$

$$y_{l-1} = \frac{1}{\sqrt{\lambda - \mu(l)}} \left[-\frac{d}{dx} - \frac{l}{x}\right] y_l. \tag{0.59}$$

Using Equations (0.55) and (0.56) we write Equation (0.58) as

$$y_{l+1} = \frac{-1}{\sqrt{1-0}} \left[\frac{d}{dx} - \frac{l+1}{x} \right] y_l, \tag{0.60}$$

where an extra minus sign is introduced for convention. We now substitute

$$u_l(x) = \frac{y_l(x)}{x^{l+1}} \tag{0.61}$$

to write

$$x^{l+2} u_{l+1} = - \left\{ \frac{d}{dx} - \frac{l+1}{x} \right\} u_l x^{l+1}, \tag{0.62}$$

$$= - \left\{ \frac{du_l}{dx} x^{l+1} + (l+1) x^l u_l - \frac{(l+1)}{x} u_l x^{l+1} \right\}, \tag{0.63}$$

$$= -x^{l+1} \frac{du_l}{dx}. \tag{0.64}$$

Iterating the above result:

$$u_{l+1} = -\frac{1}{x} \frac{du_l}{dx}, \tag{0.65}$$

as

$$u_{l+1} = (-1)^2 \frac{1}{x} \frac{d}{dx} \left(\frac{1}{x} \frac{d}{dx} u_{l-1} \right) = (-1)^3 \frac{1}{x} \frac{d}{dx} \left(\frac{1}{x} \frac{d}{dx} \right) \left(\frac{1}{x} \frac{d}{dx} u_{l-2} \right) \cdots . \tag{0.66}$$

we eventually reach

$$u_{l+1} = (-1)^{l+1} \left(\frac{1}{x} \frac{d}{dx} \right)^{l+1} u_0. \tag{0.67}$$

Since

$$y_0 = x h_0^{(1)} = -i e^{ix}, \tag{0.68}$$

which comes from the solution of Equation (0.52), we can also write

$$\frac{y_0}{x} = \frac{-i e^{ix}}{x}, \tag{0.69}$$

$$y_1 = x h_1^{(1)} = -\left(\frac{d}{dx} - \frac{1}{x} \right) y_0, \tag{0.70}$$

$$y_2 = x h_2^{(1)} = (-1)^2 \left(\frac{d}{dx} - \frac{2}{x} \right) \left(\frac{d}{dx} - \frac{1}{x} \right) y_0, \tag{0.71}$$

$$\vdots$$

$$y_l = x h_l^{(1)}. \tag{0.72}$$

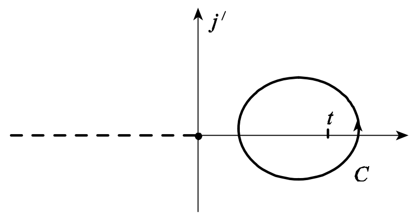

Fig. 0.3 The dotted line indicates the branch cut and t is any point on the real axis of the complex j'-plane within C.

Using Equations (0.61) and (0.72) in Equation (0.67), we obtain the desired result as

$$h_l^{(1)}(x) = (-1)^l x^l \left(\frac{1}{x}\frac{d}{dx}\right)^l \left(\frac{-e^{ix}}{x}\right). \quad (0.73)$$

To find the contour integral representation we let

$$\frac{1}{x}\frac{d}{dx} = \frac{d}{dt}, \quad (0.74)$$

$$dt = xdx, \quad (0.75)$$

hence

$$t = \frac{x^2}{2}, \quad (0.76)$$

$$x = \sqrt{2t}, \quad (0.77)$$

to write Equation (0.67) as

$$u_l = -(-1)^l i \frac{d^l}{dt^l}\left(\frac{e^{i\sqrt{2t}}}{\sqrt{2t}}\right) \quad (0.78)$$

$$= \frac{-i(-1)^l l!}{2\pi i} \oint_C \frac{e^{i\sqrt{2j'}}dj'}{\sqrt{2j'}(j'-t)^l}. \quad (0.79)$$

That is,

$$h_l^{(1)}(x) = \frac{-i(-1)^l l!}{2\pi i} x^l \oint_C \frac{e^{i\sqrt{2j'}}dj'}{\sqrt{2j'}(j'-t)^l}, \quad (0.80)$$

where C is as shown in Figure (0.3).

6. (**Extra problem**) Using the integral definition of $h_l^{(1)}(x)$ found in the previous problem and the transformation

$$z'' = -\frac{[2j']^{1/2}}{x}, \quad (0.81)$$

show that an even more useful integral definition can be obtained as

$$h_l^{(1)}(x) = \frac{(-1)^l 2^l l!}{\pi x^{l+1}} \oint_{C_{z''}} \frac{e^{-ixz} dz''}{[(z''-1)(z''+1)]^{l+1}}.$$

Compare the two contours, C and $C_{z''}$.

7. **(Problem 13.9)** Using contour integral techniques evaluate

$$\int_{-\infty}^{+\infty} \frac{\sin^2 x \, dx}{x^2(1+x^2)}. \tag{0.82}$$

Solution:

First note that $z = 0$ is not a pole and then show that

$$I = \left\{ \frac{\pi}{4}(e^{-2}-1) + \frac{\pi}{4}(e^{-2}-1) + \pi \right\}, \tag{0.83}$$

$$= \frac{\pi}{2}[e^{-2}+1]. \tag{0.84}$$

II. Integral Representation of Bessel Functions

Using the generating function definition of $J_n(x)$, which is derived in Bayin (2008):

$$\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right] = \sum_{n=0}^{\infty} t^n J_n(x) \tag{0.85}$$

we can write the integral definition

$$J_n(x) = \frac{1}{2\pi i} \oint_C \frac{\exp\left[\frac{x}{2}\left(t - \frac{1}{t}\right)\right]}{t^{n+1}} dt, \tag{0.86}$$

where t is now a point on the complex t-plane and C is a closed contour enclosing the origin. We can extend this definition to the complex z-plane as

$$J_n(z) = \frac{1}{2\pi i} \oint_C \frac{\exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right]}{t^{n+1}} dt, \quad |\arg z| < \frac{\pi}{2}, \tag{0.87}$$

where $J_n(z)$ satisfies the differential equation [Eq. (6.21)]

$$\left[z^2 \frac{d^2}{dz^2} + z\frac{d}{dz} + (z^2 - n^2)\right] J_n(z) = 0. \tag{0.88}$$

One can check this by substituting Equation (0.87) into (0.88). To extend this definition to the noninteger values of n, we write the integral

$$g_n(z) = \frac{1}{2\pi i} \int_{C'} \frac{\exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right]}{t^{n+1}} dt, \quad |\arg z| < \frac{\pi}{2}, \tag{0.89}$$

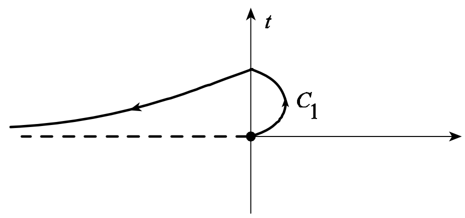

Fig. 0.4 Path for $\frac{1}{2}H_n^{(1)}(z)$.

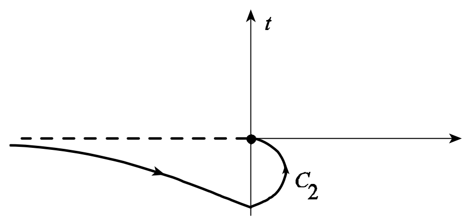

Fig. 0.5 Path for $\frac{1}{2}H_n^{(2)}(z)$.

where C' is a path in the complex t-plane. We operate on $g_n(z)$ with the Bessel's differential operator to get

$$\left[z^2\frac{d^2}{dz^2}+z\frac{d}{dz}+(z^2-n^2)\right]g_n(z) \tag{0.90}$$

$$=\frac{1}{2\pi i}\int_{C'}\frac{dt}{t^{n+1}}\exp\left[\frac{z}{2}\left(t-\frac{1}{t}\right)\right]\left\{\frac{z^2}{4}\left(t-\frac{1}{t}\right)^2+\frac{z}{2}\left(t-\frac{1}{t}\right)+z^2-n^2\right\} \tag{0.91}$$

$$=\frac{1}{2\pi i}\int_{C'}dt\frac{d}{dt}\left\{\frac{\exp\left[\frac{z}{2}\left(t-\frac{1}{t}\right)\right]}{t^n}\left[\frac{z}{2}\left(t+\frac{1}{t}\right)+n\right]\right\} \tag{0.92}$$

$$=\frac{1}{2\pi i}\left[G_n(z,t_2)-G_n(z,t_1)\right], \tag{0.93}$$

where

$$G_n(z,t)=\frac{\exp\left[\frac{z}{2}\left(t-\frac{1}{t}\right)\right]}{t^n}\left[\frac{z}{2}\left(t+\frac{1}{t}\right)+n\right] \tag{0.94}$$

and t_1 and t_2 are the end points of the path C'. Obviously, for a path that makes the difference in Equation (0.93) zero, we have a solution of the Bessel's equation. For the integer values of n, choosing C' as a closed path that encloses the origin does the job, which reduces to the Schläfli definition [Eq. (0.87)]. For the noninteger values of n, we have a branch cut, which we choose to be

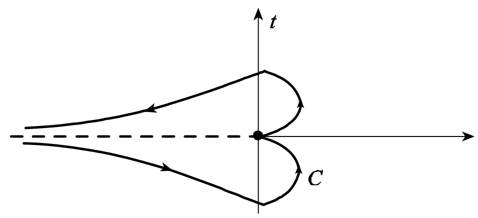

Fig. 0.6 Contour for $J_n(z) = \frac{1}{2}[H_n^{(1)}(z) + H_n^{(2)}(z)]$.

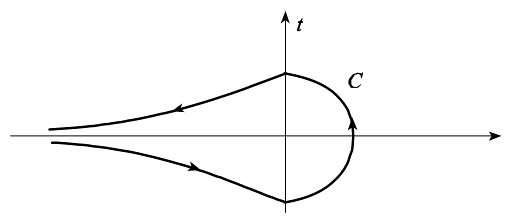

Fig. 0.7 For the integer values n there is no need for the branch cut, hence the path for the integral definition of $J_n(z)$ can be deformed into C.

along the negative real axis. Along the real axis, $G_n(z,t)$ has the limits

$$G_n(z,t) \to 0 \text{ as } t \to 0^+ \text{ and } t \to -\infty. \tag{0.95}$$

Hence, the two paths, C_1 and C_2, shown in Figures (0.4) and (0.5) give two linearly independent solutions corresponding to $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$, respectively. Their sum gives

$$\frac{1}{2}[H_n^{(1)}(z) + H_n^{(2)}(z)] = J_n(z). \tag{0.96}$$

We can now write $J_n(z)$ for general n as

$$J_n(z) = \frac{1}{2\pi i}\int_C \frac{\exp\left[\frac{z}{2}\left(t - \frac{1}{t}\right)\right]}{t^{n+1}} dt, \quad |\arg z| < \frac{\pi}{2}, \tag{0.97}$$

where the contour is given in Figure (0.6). For the integer values of n there is no need for a branch cut, hence the contour can be deformed into C as shown in Figure (0.7). Furthermore, since the integrand is now single valued, we can also collapse the contour to one enclosing the origin (Fig. 0.8).

In Equation (0.97) we now make the transformation

$$t = \frac{2s}{z} \tag{0.98}$$

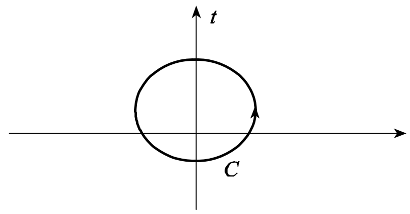

Fig. 0.8 Path for $J_n(z)$, where n takes integer values can be taken as any closed path enclosing the origin.

to write

$$J_n(z) = \frac{1}{2\pi i}\left(\frac{z}{2}\right)^n \int_C ds \frac{\exp\left[s - \frac{z^2}{4s}\right]}{s^{n+1}}. \tag{0.99}$$

Expanding $e^{-z^2/4s}$:

$$e^{-z^2/4s} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \frac{z^{2r}}{2^{2r} s^r}, \tag{0.100}$$

we write

$$J_n(z) = \left(\frac{z}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \quad \left(\frac{z}{2}\right)^{2r} \frac{1}{2\pi i} \int_C ds\ e^s s^{-n-r-1}. \tag{0.101}$$

The integral is nothing but one of the integral representations of the gamma function (see the next section of this supplement):

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_C dt\ e^t t^{-z}, \tag{0.102}$$

which the reader can show to lead to the series expression

$$J_n(z) = \left(\frac{z}{2}\right)^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)} \left(\frac{z}{2}\right)^{2r}. \tag{0.103}$$

An other useful formula can be obtained by using the contour integral representation in Equation (0.87) and the substitution

$$t = e^{i\theta}, \tag{0.104}$$

which allows us to write

$$J_n(z) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{iz\sin\theta}}{e^{(n+1)i\theta}} ie^{i\theta} d\theta \tag{0.105}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} e^{i(z\sin\theta - n\theta)} d\theta. \tag{0.106}$$

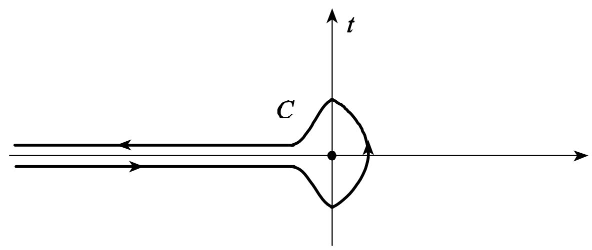

Fig. 0.9 Contour for the Hankel definition of $\Gamma(z)$.

This yields the **Bessel's integral** [Eq. (6.48)] as

$$J_n(z) = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta - z\sin\theta) d\theta. \tag{0.107}$$

III. Analytic Continuation of the Gamma Function

We have seen that the gamma function with real argument is defined as [Eq. (13.133)]

$$\Gamma(x) = \int_0^{\infty} dt\ e^{-t} t^{x-1},\ x > 0. \tag{0.108}$$

This formula can be analytically continued to the right-hand side of the z-plane easily as

$$\Gamma(z) = \int_0^{\infty} dt\ e^{-t} t^{z-1},\ \operatorname{Re} z > 0. \tag{0.109}$$

The above integral is convergent only for $\operatorname{Re} z > 0$. A definition valid in the entire z-plane exists and has been given by Hankel as

$$\Gamma(z) = \frac{1}{2i \sin \pi z} \int_C dt\ e^{-t} t^{z-1}, \tag{0.110}$$

where the integral is now taken in the complex t-plane over the contour shown in Figure (0.9). In this definition, the branch cut of t^{z-1} is located along the negative real axis as

$$t^{z-1} = e^{(z-1)\ln t} = e^{(z-1)(\ln|t| + i\theta)},\ \ -\pi \le \theta < \pi. \tag{0.111}$$

As we deform the contour without touching either the branch point or crossing over the branch cut, the integral in Equation (0.110) reduces to two integrals over straight paths; one just over the branch cut and the other just below:

$$\int_C dt\ e^{-t} t^{z-1} = \int_C dt\ e^{-t} e^{(z-1)(\ln|t|+i\theta)} \tag{0.112}$$

$$= \int_0^{-\infty} dt\ e^{-t} e^{(z-1)(\ln|t|+i\pi)} - \int_{-\infty}^0 dt\ e^{-t} e^{(z-1)(\ln|t|-i\pi)} \tag{0.113}$$

$$= \int_0^{\infty} dt\ e^{-t} e^{-(z-1)\ln|t|} \left[e^{(z-1)i\pi} - e^{-(z-1)i\pi} \right] \tag{0.114}$$

$$= 2i \sin \pi z \int_0^{\infty} dt\ e^{-t} e^{-(z-1)\ln|t|}. \tag{0.115}$$

Substituting this into Equation (0.110) gives Equation (0.109), thus proving their equivalence.

Equation (0.110) tells us that $\Gamma(z)$ has simple poles located at

$$z = -n,\ n = 0, 1, 2, \dots\ . \tag{0.116}$$

We now write

$$\sin \pi z = (-1)^n \sin \pi(z+n) \tag{0.117}$$

$$\simeq (-1)^n \pi(z+n), \tag{0.118}$$

hence at $z = -n$, we can collapse the contour in Equation (0.110):

$$\int_C dt\ e^{-t} t^{z-1}, \tag{0.119}$$

to a closed contour about the origin. Considering that near the origin t^{-n-1} is single valued, thus we obtain the integral

$$\int_C dt\ e^{-t} t^{-n-1} = \frac{1}{n!} 2\pi i. \tag{0.120}$$

In other words,

$$\Gamma(z) \simeq \frac{(-1)^n}{n!} \frac{1}{z+n}, \tag{0.121}$$

that is, the residue of $\Gamma(z)$ at $z = -n$ is

$$\frac{(-1)^n}{n!}. \tag{0.122}$$

Finally, we use Equation (13.146):

$$B(u, v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \tag{0.123}$$

and the definition of the beta function [Eq. (13.149)]:

$$B(u,v) = \int_0^\infty dt \frac{t^{u-1}}{(1+t)^{u+v}}, \tag{0.124}$$

in conjunction with the integral representation we obtained [Eq. (0.110)] with the identifications

$$u = z, \tag{0.125}$$

$$v = 1 - z, \tag{0.126}$$

we obtain the following useful property of the gamma functions [Eq. (13.161)]:

$$\begin{aligned} \Gamma(z)\Gamma(1-z) &= \Gamma(1)B(z, 1-z) && (0.127) \\ &= \int_0^\infty dt \frac{t^{z-1}}{1+z} && (0.128) \\ &= \frac{\pi}{\sin \pi z}. && (0.129) \end{aligned}$$

Writing the above result as

$$\frac{1}{\Gamma(z)} = \frac{\sin \pi z}{\pi} \Gamma(1-z) \tag{0.130}$$

and substituting Equation (0.110) for $\Gamma(1-z)$, one obtains Equation (0.102):

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_C dt\ e^t t^{-z}, \tag{0.131}$$

used to derive Equation (0.103).

IV. Useful Sites and Sources

For the introductory topics of complex analysis and additional examples, we recommend *Essentials Mathematical Methods in Science and Engineering*:

`http://www.wiley.com/WileyCDA/WileyTitle/productCd-0470343796.html`.

For the biography of Cauchy we refer to

`http://en.wikipedia.org/wiki/Cauchy`

and

`http://scienceworld.wolfram.com/biography/Cauchy.html`.

On complex analysis we recommend the websites

`http://en.wikipedia.org/wiki/Complex_analysis` and

`http://mathworld.wolfram.com/Cauchy-RiemannEquations.html`.

In particular, we recommend the website by Prof. J.A. Mathews, where many interesting examples with computer graphics and applications with Mathematica and Maple, along with links to other useful internet sources can be found:

`http://mathews.ecs.fullerton.edu/c2000/`,

Selçuk Bayin (December, 2008)