SUPPLEMENTS TO MATHEMATICAL METHODS IN SCIENCE AND ENGINEERING
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CHAPTER 2: LEGENDRE POLYNOMIALS

I. Solutions or Hints to Selected Problems:

1. **(Problem 2.11)** Show the integral

$$\int_{-1}^{1} dx x^l P_n(x) = \frac{2^{n+1} l! \left(\frac{l+n}{2}\right)!}{(l+n+1)! \left(\frac{l-n}{2}\right)!}, \tag{0.1}$$

where

$$(l-n) = |\text{even integer}|. \tag{0.2}$$

Solution:

We show the solution for the special case where $n = l$:

$$I_{nn} = \int_{-1}^{1} dx x^n P_n(x). \tag{0.3}$$

Using the Rodriguez formula [Eq. (2.60)] we write the integral

$$I_{nn} = \frac{1}{2^n n!} \int_{-1}^{1} x^n \frac{d^n}{dx^n} (x^2-1)^n, \tag{0.4}$$

which after $n-$fold integration by parts gives

$$I_{nn} = \frac{1}{2^n} \int_{-1}^{1} (1-x^2)^n dx. \tag{0.5}$$

Comparing with the beta function [Eq. (13.151)]:

$$B(r,s) = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} = \int_0^1 t^{r-1}(1-t)^{s-1} dt, \tag{0.6}$$

we finally obtain the desired result as

$$I_{nn} = \frac{2^{n+1}(n!)^2}{(2n+1)!}. \tag{0.7}$$

For the general case, follow the same procedure and use the properties of gamma functions [Eqs. (13.136) and (13.155)].

2. Using the Cauchy integral formula:

$$\frac{d^n f(z_0)}{dz_0^n} = \frac{n!}{2\pi i} \oint_C \frac{f(z)dz}{(z-z_0)^{n+1}}, \tag{0.8}$$

where $f(z)$ is analytic on and within the closed contour C, and z_0 is a point within C, obtain an integral representation of $P_l(x)$ and $P_l^m(x)$.

Solution:

Using any closed contour C enclosing the point $z_0 = x$ on the real axis and the Rodriguez formula for $P_l(x)$ [Eq (2.60)]:

$$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l}(x^2-1)^l, \tag{0.9}$$

we can write

$$P_l(x) = \frac{2^{-l}}{2\pi i} \oint_C \frac{(z^2-1)^l}{(z-x)^{l+1}} dz. \tag{0.10}$$

Using the definition [Eq. (2.162)]:

$$P_l^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x), \tag{0.11}$$

we also obtain

$$P_l^m(x) = \frac{1}{2^l 2\pi i} \frac{(l+m)!}{l!} (1-x^2)^{m/2} \oint_C \frac{(z^2-1)^l}{(z-x)^{l+m+1}} dz. \tag{0.12}$$

3. In Equation (0.8) C is any closed contour enclosing the point x. Let C be a unit circle centered at x with the parametrization

$$z = \cos\theta + i \sin\theta e^{i\phi}. \tag{0.13}$$

Using ϕ as the new integration variable, show the following integral representation:

$$p_l^m(\cos\theta) = \frac{(-1)^m i^m}{2\pi} \frac{(l+m)!}{m!} \int_{-\pi}^{\pi} [\cos\theta + i \sin\theta \cos\phi]^l e^{-im\phi} d\phi. \tag{0.14}$$

The advantage of this representation is that the definite integral is taken over the real domain.

Solution:

Using Equation (0.13) we first write the following relations:

$$\begin{aligned} (z-\cos\theta)^{l+m+1} &= i^{l+m+1}\sin^{l+m+1}\theta e^{i(l+m+1)\phi}, \\ (z^2-1) &= 2i\sin\theta e^{i\phi}[\cos\theta + i\sin\theta\cos\phi], \\ dz &= -\sin\theta e^{i\phi}d\phi, \end{aligned} \tag{0.15}$$

which when substituted into Equation (0.12) gives the desired result [Eq. (0.14)]. Note that $x = \cos\theta$.

4. Show that the function

$$V(x,y,z) = [z + ix\cos u + i\sin u]^l, \tag{0.16}$$

where (x,y,z) are the Cartesian coordinates of a point and u is a real parameter, is a solution of the Laplace equation. Next show that an integral representation of $P_l^m(\cos\theta)$ given in terms of the angles, θ and ϕ, of the spherical polar coordinates also yields Equation (0.14) up to a proportionality constant.

Solution:

First evaluate the derivatives V_{xx}, V_{yy}, and V_{yy} to show that

$$\overrightarrow{\nabla}^2 V = V_{xx} + V_{yy} + V_{zz} = 0. \tag{0.17}$$

Since u is just a real parameter,

$$\int_{-\pi}^{\pi}[z + ix\cos u + i\sin u]^n e^{imu}du \tag{0.18}$$

is also a solution of the Laplace equation. We now transform x, y, z to spherical coordinates and let $\phi - u = \psi$, to obtain

$$r^l e^{im\phi}\int_{-\pi}^{+\pi}[\cos\theta + i\sin\theta\cos\psi]^l e^{-im\psi}d\psi. \tag{0.19}$$

Comparing with the solution of the Laplace equation: $r^l e^{im\phi}P_l^m(\cos\theta)$, we see that the integral

$$\int_{-\pi}^{+\pi}[\cos\theta + i\sin\theta\cos\psi]^l e^{-im\psi}d\psi, \tag{0.20}$$

must be proportional to $P_l^m(\cos\theta)$. Inserting the proportionality constant gives

$$P_l^m(\cos\theta) = \frac{(-1)^m i^m}{2\pi}\frac{(l+m)!}{l!}\int_{-\pi}^{+\pi}[\cos\theta + i\sin\theta\cos\psi]^l e^{-im\psi}d\psi. \tag{0.21}$$

If we write $e^{-im\psi} = \cos m\psi - im\sin\psi$, from symmetry the integral corresponding to $-im\sin\psi$ vanishes, thus allowing us to write

$$P_l^m(\cos\theta) = \frac{(-1)^m i^m}{2\pi}\frac{(l+m)!}{l!}\int_{-\pi}^{+\pi}[\cos\theta + i\sin\theta\cos\psi]^l \cos m\psi d\psi. \tag{0.22}$$

5. **(Problem 2.12)** Using the above expression for $P_l^m(\cos\theta)$, find $P_l^{-m}(\cos\theta)$.

   Solution:

   The differential equation that $P_l^m(x)$ satisfies [Eq. (2.21)], where $\lambda = l(l+1)$, depends on l as $l(l+1)$, which is unchanged when we let $l \to -l-1$. In other words, if we replace l with $-l-1$ in the right-hand side of Equation (0.22) we should get the same solution. Under the same replacement

$$\frac{(l+m)!}{l!} = (l+m)(l+m-1)\cdots(l+1) \tag{0.23}$$

   becomes $(-l-1+m)(-l-1+m-1)\cdots(-l) = (-1)^m\frac{l!}{(l-m)!}$, hence we can write

$$P_l^m(x) = \frac{(-1)^m(-i)^m l!}{2\pi(l-m)!}\int_{-\pi}^{+\pi}\frac{\cos m\psi}{[\cos\theta + i\sin\theta\cos\psi]^{l+1}}. \tag{0.24}$$

   Since m appears in the differential equation [Eq. (2.21)] as m^2, we can also replace m by $-m$ in Equation (0.22), thus allowing us to write

$$P_l^{-m}(x) = \frac{(-1)^m i^{-m}}{2\pi}\frac{(l-m)!}{l!}\int_{-\pi}^{+\pi}[\cos\theta + i\sin\theta\cos\psi]^l \cos m\psi d\psi \tag{0.25}$$

$$= \frac{(-1)^m(i)^m l!}{2\pi(l+m)!}\int_{-\pi}^{+\pi}\frac{\cos m\psi}{[\cos\theta + i\sin\theta\cos\psi]^{l+1}}. \tag{0.26}$$

   Comparing Equation (0.26) with Equation (0.24) we obtain

$$P_l^{-m}(x) = (-1)^m\frac{(l-m)!}{(l+m)!}P_l^m(x). \tag{0.27}$$

6. **(Problem 2.20)** Find solutions of the differential equation

$$2x(x-1)\frac{d^2y}{dx^2} + (10x-3)\frac{dy}{dx} + \left[8 + \frac{1}{x} - 2\lambda\right]y(x) = 0, \tag{0.28}$$

   satisfying the condition

$$y(x) = \text{finite} \tag{0.29}$$

in the entire interval $x \in [0,1]$. Write the solution explicitly for the third lowest value of λ.

Hint: First check the recursion relation and observe that it is a three-term recursion relation, then find a transformation that reduces the differential equation into an equation with a two-term recursion relation. Next, find a series solution and impose the boundary conditions, which will give you the allowed values of λ.

II. Additional Discussions

i) Other Recursion Relations For $P_l^m(x)$

Operating on the recursion relation [Prob. (2.9b)]:

$$(l+1)P_{l+1}(x) - (2l+1)xP_l(x) + lP_{l-1}(x) = 0 \tag{0.30}$$

with

$$(1-x^2)^{m/2}\frac{d^m}{dx^m} \tag{0.31}$$

and using the relation

$$(1-x^2)^{m/2}\frac{d^m P_l}{dx^m} = P_l^m, \tag{0.32}$$

we obtain another recursion relation for P_l^m as

$$\begin{aligned}(l+1)P_{l+1}^m(x) - (2l+1)xP_l^m(x) + lP_{l-1}^m(x)\\ +m(2l+1)\sqrt{1-x^2}P_{l-1}^{m-1}(x) = 0.\end{aligned} \tag{0.33}$$

Two other useful recursion relations for P_l^m can be obtained as

$$(l+1-m)P_{l+1}^m(x) - (2l+1)xP_l^m(x) + (l+m)P_{l-2}^m(x) = 0 \tag{0.34}$$

and

$$P_l^{m+2} + \frac{2(m+1)x}{\sqrt{1-x^2}}P_l^{m+1}(x) + (l-m)(l+m+1)P_l^m(x) = 0. \tag{0.35}$$

To prove the first recursion relation [Eq. (0.34)] we write

$$\frac{d}{dx}\left[P_{l+1}(x) - P_{l-1}(x)\right] = \sum_{k=0}^{l} a_k P_k(x), \tag{0.36}$$

which follows from the fact that the left-hand side is a polynomial of order l. Using the orthogonality relation of the Legendre polynomials [Eq. (2.118)], we can evaluate a_k as

$$a_k = \frac{2k+1}{2}\int_{-1}^{1} P_k(x)\frac{d}{dx}\left[P_{l+1}(x) - P_{l-1}(x)\right]dx. \tag{0.37}$$

After integration by parts and using the special values [Eq. (2.86)]:

$$P_l(1) = 1,\ \ P_l(-1) = (-1)^l, \tag{0.38}$$

we obtain

$$a_k = -\frac{2k+1}{2}\int_{-1}^{1} P_k'(x)\left[P_{l+1}(x) - P_{l-1}(x)\right]dx. \tag{0.39}$$

In this expression, $P_k'(x)$ is of order $k-1$. Since $P_{l+1}(x)$ and $P_{l-1}(x)$ are orthogonal to all polynomials of order $l-2$ or lower, $a_k = 0$ for $k = 0, 1, \ldots, (l-1)$. Hence, we obtain

$$a_l = -\frac{2l+1}{2}\int_{-1}^{1} P_l'(x)\left[P_{l+1}(x) - P_{l-1}(x)\right]dx \tag{0.40}$$

$$= \frac{2l+1}{2}\int_{-1}^{1} P_l'(x)P_{l-1}(x)dx \tag{0.41}$$

$$= \frac{2l+1}{2}\left[P_l(x)P_{l-1}(x)|_{-1}^{1} - \int_{-1}^{1} P_l(x)P_{l-1}'(x)dx\right] \tag{0.42}$$

$$= 2l+1. \tag{0.43}$$

A result, when substituted into Equation (0.36) yields

$$\frac{d}{dx}\left[P_{l+1}(x) - P_{l-1}(x)\right] = (2l+1)P_l(x). \tag{0.44}$$

Operating on this with $\dfrac{d^{m-1}}{dx^{m-1}}$ and multiplying with $(1-x^2)^{m/2}$, we finally obtain the desired result:

$$(l+1-m)P_{l+1}^m(x) - (2l+1)xP_l^m(x) + (l+m)P_{l-2}^m(x) = 0. \tag{0.45}$$

The second recursion relation [Eq. (0.35)] can be obtained by using the Legendre equation [Eq. (2.22)]:

$$(1-x^2)P_l'' - 2xP_l' + l(l+1)P_l = 0, \tag{0.46}$$

and by operating on it with

$$(1-x^2)^{m/2}\frac{d^m}{dx^m}. \tag{0.47}$$

ii) Addition Theorem for Spherical Harmonics

Spherical harmonics are defined as [Eq. (2.177)]

$$Y_l^m(\theta,\phi) = (-1)^m \sqrt{\frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!}} e^{im\phi} P_l^m(\cos\theta), \tag{0.48}$$

where the orthogonality relation is given as

$$\int_0^{2\pi}\int_0^{\pi} d\phi' d\theta' \sin\theta' Y_l^{m*}(\theta',\phi') Y_{l'}^{m'*}(\theta',\phi') = \delta_{mm'}\delta_{ll'}. \tag{0.49}$$

Since spherical harmonics form a complete and an orthonormal set, any sufficiently smooth function, $g(\theta,\phi)$, can be represented as the series

$$g(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} A_m^l Y_l^m(\theta,\phi), \tag{0.50}$$

where the expansion coefficients are given as

$$A_m^l = \int_0^{2\pi}\int_0^{\pi} d\phi d\theta \sin\theta g(\theta,\phi) Y_l^{m*}(\theta,\phi). \tag{0.51}$$

Substituting A_m^l back into $g(\theta,\phi)$ we write

$$g(\theta,\phi) = \int_0^{2\pi}\int_0^{\pi} d\phi' d\theta' \sin\theta' g(\theta',\phi') \sum_{l=0}^{\infty}\sum_{m=-l}^{l} Y_l^m(\theta,\phi) Y_l^{m*}(\theta',\phi'). \tag{0.52}$$

Substituting the definition of spherical harmonics, this also becomes

$$\begin{aligned} g(\theta,\phi) = {} & \int_0^{2\pi}\int_0^{\pi} d\phi' d\theta' \sin\theta' g(\theta',\phi') \\ & \times \sum_{l=0}^{\infty}\sum_{m=-l}^{l} \frac{(2l+1)}{4\pi}\frac{(l-m)!}{(l+m)!} e^{im\phi} P_l^m(\cos\theta) e^{-im\phi'} P_l^m(\cos\theta'), \end{aligned} \tag{0.53}$$

$$\begin{aligned} g(\theta,\phi) = {} & \int_0^{2\pi}\int_0^{\pi} d\phi' d\theta' \sin\theta' g(\theta',\phi') \\ & \times \sum_{l=0}^{\infty} \frac{(2l+1)}{4\pi} \sum_{m=-l}^{l} \frac{(l-m)!}{(l+m)!} e^{im(\phi-\phi')} P_l^m(\cos\theta) P_l^m(\cos\theta'). \end{aligned} \tag{0.54}$$

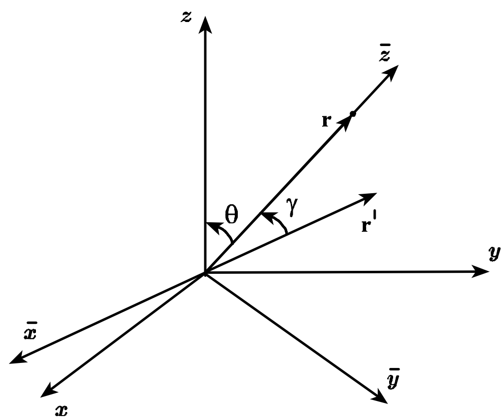

Fig. 0.1 Addition Theorem.

In this equation angular coordinates, (θ, ϕ), give the orientation of the position vector, $\overrightarrow{r} = (r, \theta, \phi)$, which is also called the field point and $\overrightarrow{r}' = (r', \theta', \phi')$ represents the source point. We now orient our axes so that the field point, $\overrightarrow{r}$, aligns with the $\overline{z}-$axis of the new coordinates. Hence, θ in the new coordinates is 0 and the angle, θ', that $\overrightarrow{r}'$ makes with the $\overline{z}-$axis is γ (Fig. 0.1). We first make a note of the following special values:

$$P_l(\cos 0) = P_l(1) = 1, \tag{0.55}$$

$$P_l^m(\cos 0) = P_l^m(1) = 0, \ m > 0. \tag{0.56}$$

From spherical trigonometry the angle, γ, between the vectors $\overrightarrow{r}$ and $\overrightarrow{r}'$, is related to θ, ϕ, θ' and ϕ' as

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi'). \tag{0.57}$$

In terms of the new orientation of our axes, we now write Equation (0.54) as

$$\begin{aligned} g(0, -) = \int_0^{2\pi}\int_0^{\pi} d\phi' d\theta' \sin\theta' g(\theta', \phi') \sum_{l=0}^{\infty} \frac{(2l+1)}{4\pi} \{P_l^0(\cos 0)P_l^0(\cos\theta') \\ + \sum_{m=1}^{l} \frac{(l-m)!}{(l+m)!} e^{-im\phi'} P_l^m(\cos 0) P_l^m(\cos\theta') \\ + \sum_{m=-l}^{-1} \frac{(l-m)!}{(l+m)!} e^{-im\phi'} P_l^m(\cos 0) P_l^m(\cos\theta')\}. \end{aligned} \tag{0.58}$$

Note that in the new orientation of our axes we are still using primes to denote the coordinates of the source point $\overrightarrow{r}'$. That is, the angular variables, θ' and

ϕ', in Equation (0.58) are now measured in terms of the new orientation of our axes. Naturally, rotation does not affect the magnitudes of $\overrightarrow{r}$ and $\overrightarrow{r}'$. Since $g(\theta, \phi)$ is a scalar function on the surface of a sphere, it's numerical value at a given point on the sphere is independent of the orientation of our axes. Hence, in the new orientation of our axes, the numerical value of g, that is $g(0, -)$, is still equal to $g(\theta, \phi)$, where in $g(\theta, \phi)$ the angles are measured in terns of the original orientation of our axes. Hence we can write

$$g(\theta, \phi) = g(0, -) = \int_0^{2\pi} \int_0^{\pi} d\phi' d\theta' \sin\theta' g(\theta', \phi') \sum_{l=0}^{\infty} \frac{(2l+1)}{4\pi} \{P_l(1) P_l(\cos\gamma)$$
$$+ \sum_{m=1}^{l} \frac{(l-m)!}{(l+m)!} e^{-im\phi'} P_l^m(1) P_l^m(\cos\gamma) + \sum_{m=-l}^{1} \frac{(l-m)!}{(l+m)!} e^{-im\phi'} P_l^m(1) P_l^m(\cos\gamma)\}. \tag{0.59}$$

Substituting the special values in Equations (0.55) and (0.56), this becomes

$$g(\theta, \phi) = \int_0^{2\pi} \int_0^{\pi} d\phi' d\theta' \sin\theta' g(\theta', \phi') \sum_{l=0}^{\infty} \frac{(2l+1)}{4\pi} P_l(\cos\gamma), \tag{0.60}$$

Comparison of Equations (60) and (52) gives us the **addition theorem** of spherical harmonics:

$$\frac{(2l+1)}{4\pi} P_l(\cos\gamma) = \sum_{m=-l}^{l} Y_l^m(\theta, \phi) Y_l^{m*}(\theta', \phi'). \tag{0.61}$$

Sometimes we need the addition theorem written in terms of $P_l^m(\cos\theta)$ as

$$P_l(\cos\gamma) = P_l(\cos\theta) P_l(\cos\theta') + 2 \sum_{m=-l}^{m=l} \frac{(l-m)!}{(l+m)!} P_l^m(\cos\theta) P_l^m(\cos\theta') \cos m(\phi - \phi'). \tag{0.62}$$

If we set $\gamma = 0$, the result is the *sum rule*

$$\frac{(2l+1)}{4\pi} = \sum_{m=-l}^{l} |Y_l^m(\theta, \phi)|^2 . \tag{0.63}$$

Another derivation of the addition theorem using the rotation matrices is given in Section (11.11.12).

Note: In spherical coordinates a general solution of Laplace equation, $\overrightarrow{\nabla}^2 \Phi(r, \theta, \phi) = 0$, can be written as

$$\Phi(r, \theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} [A_{lm} r^l + B_{lm} r^{-(l+1)}] Y_{lm}(\theta, \phi), \tag{0.64}$$

where A_{lm} and B_{lm} are to be evaluated using the appropriate boundary conditions and the orthogonality condition of the spherical harmonics. The fact that under rotations $\Phi(r,\theta,\phi)$ remains to be solution of the Laplace operator follows from the fact that the Laplace operator, $\overrightarrow{\nabla}^2 = \overrightarrow{\nabla} \cdot \overrightarrow{\nabla}$, is invariant under rotations. That is, $\overrightarrow{\nabla}^2 = \overrightarrow{\nabla}'^2$. On the surface of a sphere, $r = R$, the angular part of the Laplace equation reduces to

$$\left[\frac{1}{\sin\theta}\frac{\partial}{\partial}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right] Y_{lm}(\theta,\phi) + l(l+1)Y_{lm}(\theta,\phi) = 0, \quad (0.65)$$

which is the differential equation that the spherical harmonics satisfy.

iii) Asymptotic Forms

In many applications and in establishing the convergence properties of Legendre series, we need to know the asymptotic forms of the Legendre polynomials for large l. For this, we first write the Legendre equation [Eq. (2.22)] as

$$P_l''(\cos\theta) + \cot\theta P_l'(\cos\theta) + l(l+1)P_l(\cos\theta) = 0 \quad (0.66)$$

and then substitute

$$P_l(\cos\theta) = \frac{u(\theta)}{\sqrt{\sin\theta}}, \quad (0.67)$$

to obtain

$$u''(\theta) + \left[\left(l + \frac{1}{2}\right)^2 + \frac{1}{4\sin^2\theta}\right] u(\theta) = 0. \quad (0.68)$$

For sufficiently large values of l, we can neglect the term $1/4\sin^2\theta$ and write the above equation as

$$u''(\theta) + \left(l + \frac{1}{2}\right)^2 u(\theta) \approx 0, \quad (0.69)$$

solution of which can be written immediately as

$$P_l(\cos\theta) \approx \frac{A_l \cos\left[\left(l + \frac{1}{2}\right)\theta + \delta_l\right]}{\sqrt{\sin\theta}}. \quad (0.70)$$

In this asymptotic solution, the amplitude, A_l, and the phase, δ_l, may depend on l. To determine A_l, we use the asymptotic solution in the normalization condition [Eq. (2.105)]:

$$\int_0^\pi \sin\theta[P_l(\cos\theta]^2 d\theta = \frac{2}{2l+1}, \quad (0.71)$$

to find

$$A_l \approx \sqrt{\frac{2}{\pi l}}. \tag{0.72}$$

To determine the phase, we make use of the generating function definition [Eq. (2.65)] for $\theta = \pi/2$:

$$\frac{1}{\sqrt{1+t^2}} = \sum_{l=0}^{\infty} P_l(0) t^n. \tag{0.73}$$

If we use the binomial expansion for the left-hand side, for the odd values of l we find $P_l(0) = 0$ and for the even values of l the sign of $P_l(0)$ alternates. This allows us to deduce the value of δ_l as $-\pi/4$, thus allowing us to write the asymptotic solution as

$$P_l(\cos\theta) \approx \sqrt{\frac{2}{l\pi \sin\theta}} \cos\left[\left(l + \frac{1}{2}\right)\theta - \frac{\pi}{4}\right] \tag{0.74}$$

for the sufficiently large values of l for a given θ.

iv) Real Spherical Harmonics

As in "spherical harmonic lighting", in some applications we require only the real valued spherical harmonics:

$$y_l^m = \begin{cases} \sqrt{2}\,\mathrm{Re}(Y_l^m) = \sqrt{2} N_l^m \cos(m\phi) P_l^m(\cos\theta), & m > 0, \\ Y_l^0 = N_l^0 P_l^0(\cos\theta), & m = 0, \\ \sqrt{2}\,\mathrm{Im}(Y_l^m) = \sqrt{2} N_l^{|m|} \sin(|m|\,\phi) P_l^{|m|}(\cos\theta), & |m| < 0, \end{cases} \tag{0.75}$$

where

$$N_l^m = \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}. \tag{0.76}$$

As can be investigated in the following site:
`http://www.quantum-physics.polytechnique.fr/en/pages/p0500.html`, the spherical harmonics with $m = 0$ define zones parallel to the equator on the unit sphere. Hence, they are called **zonal harmonics**. Spherical harmonics of the form $Y_{|m|}^m$ are called **sectoral harmonics**, while all the other spherical harmonics are called **tesseral harmonics**, which usually divide the unit sphere into several blocks in latitude and longitude.

III. Applications to Computer Graphics and Useful Sites

Aside from applications to classical physics and quantum mechanics, spherical harmonics have found interesting applications in computer graphics and cinematography in terms of a technique called the "spherical harmonic lighting". For the details we refer the reader to Robin Green's article: *Spherical Harmonic Lighting: The Gritty Details*, SCEA Research and Development, 2003. This interesting article can be obtained from the site

`http://www.research.scea.com/gdc2003/spherical-harmonic-lighting.pdf`.

More references and other useful information about spherical harmonics and Legendre polynomials can be found in the following sites:

`http://en.wikipedia.org/wiki/Spherical_harmonics`,

`http://mathworld.wolfram.com/SphericalHarmonic.html`.

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