Supplements to Mathematical Methods in Science and Engineering
http://www.wiley.com/WileyCDA/WileyTitle/productCd-0470041420.html

CHAPTER 6: BESSEL FUNCTIONS

For additional examples, problems, and derivations of the properties of Bessel functions see Bayin (Wiley, 2008).

I. Solutions or Hints to Selected Problems:

1. **(Problem 6.2)** Write the wave equation,

$$\Box\Phi(t, r, \theta, \phi) = 0, \tag{0.1}$$

in flat spacetime using the spherical polar coordinates and find its separable solutions.

Solution:

D'Alembert (wave) operator, $\Box$, is defined as

$$\Box \equiv g_{\mu\nu}\nabla^{\mu}\nabla^{\nu},\ \ \mu, \nu = 0, 1, 2, 3, \tag{0.2}$$

where ∇_{μ} stands for the covariant derivative. The $\Box$ operator can also be written as [Eq. (10.225)]

$$\Box \equiv g^{1/2}\frac{\partial}{\partial x^{\mu}}\left[g^{1/2}g^{\mu\nu}\frac{\partial}{\partial x^{\nu}}\right], \tag{0.3}$$

where g stands for the absolute value of the determinant of the metric tensor [Eq. (10.238)]. Note that we use the Einstein summation convention, that is, the repeated indices are summed over (Chapter 10). In flat spacetime the metric tensor is defined by the line element

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^{\mu}dx^{\nu} \\ &= dt^2 - dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2, \end{aligned} \tag{0.4}$$

where

$$g = r^4\sin^2\theta. \tag{0.5}$$

Using the summation convention and the identification

$$x^0 = t,\ x^1 = r,\ x^2 = \theta,\ x^3 = \phi, \tag{0.6}$$

Equation (0.1) with Equation (0.2) can be written as

$$\left[\frac{1}{2g}\frac{\partial g}{\partial x^\mu} g^{\mu\nu}\partial_\nu + \frac{\partial g^{\mu\nu}}{\partial x^\mu}\partial_\nu + g^{\mu\nu}\partial_\mu\partial_\nu\right]\Phi = 0, \tag{0.7}$$

which eventually leads to

$$\frac{\partial^2\Phi}{\partial t^2} - \frac{\partial^2\Phi}{\partial r^2} - \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} - \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2} - \frac{2}{r}\frac{\partial\Phi}{\partial r} - \frac{\cos\theta}{r^2\sin\theta}\frac{\partial\Phi}{\partial\theta} = 0. \tag{0.8}$$

We now use the separation of variables method and substitute a solution of the form

$$\Phi(t, r, \theta, \phi) = T(t)R(r)Y(\theta, \phi), \tag{0.9}$$

to get

$$\frac{1}{T}\frac{\partial^2 T(t)}{\partial t^2} = \frac{1}{R}\left[\frac{2}{r}\frac{\partial R}{\partial r} + \frac{\partial^2 R}{\partial r^2}\right] + \frac{1}{Yr^2}\left[\frac{\partial^2 Y}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial Y}{\partial\theta}\right]. \tag{0.10}$$

Introducing two separation constants, ω^2 and $\lambda = -l(l+1)$, we obtain three ordinary differential equations:

$$\frac{1}{T}\frac{\partial^2 T(t)}{\partial t^2} = -\omega^2, \tag{0.11}$$

$$\frac{\partial^2 R}{\partial r^2} + \frac{2}{r}\frac{\partial R}{\partial r} + \left[\omega^2 - \frac{l(l+1)}{r^2}\right]R(r) = 0, \tag{0.12}$$

$$\frac{\partial^2 Y}{\partial\theta^2} + \frac{1}{\sin^2\theta}\frac{\partial^2 Y}{\partial\phi^2} + \frac{\cos\theta}{\sin\theta}\frac{\partial Y}{\partial\theta} + l(l+1)Y(\theta, \phi) = 0. \tag{0.13}$$

For the radial equation we substitute

$$R(r) = \frac{g(r)}{\sqrt{r}} \tag{0.14}$$

to get

$$r^2 g'' + rg' + [\omega^2 r^2 - (l + \frac{1}{2})^2]R(r). \tag{0.15}$$

Comparing with Equation (6.42), the general solution can be written immediately as

$$R(r) = \frac{1}{\sqrt{\omega r}}[C_0 J_{l+1/2}(\omega r) + C_1 J_{-(l+1/2)}(\omega r)]. \tag{0.16}$$

Solution for the angular part is given as the spherical harmonics [Eq. (2.182)] and the time dependence is given as

$$T(t) = C_0 e^{-iwt}, \tag{0.17}$$

thus yielding the complete solution as

$$\Phi_{\omega lm}(t, r, \theta, \phi) = \frac{1}{\sqrt{\omega r}}[C_0 J_{l+1/2}(\omega r) + C_1 J_{-(l+1/2)}(\omega r)]Y_l^m(\theta, \phi)e^{-iwt}. \tag{0.18}$$

Depending on the boundary conditions we could also use

$$\Phi_{\omega lm}(t, r, \theta, \phi) = \frac{1}{\sqrt{\omega r}}[C_0 J_{l+1/2}(\omega r) + C_1 N_{l+1/2}(\omega r)]Y_l^m(\theta, \phi)e^{-iwt}, \tag{0.19}$$

or

$$\Phi_{\omega lm}(t, r, \theta, \phi) = \frac{1}{\sqrt{\omega r}}[C_0 H^{(1)}_{l+1/2}(\omega r) + C_1 H^{(2)}_{l+1/2}(\omega r)]Y_l^m(\theta, \phi)e^{-iwt}. \tag{0.20}$$

2. Using the generating function definition [Eq. (6.47)]:

$$e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x)t^n, \tag{0.21}$$

show that

$$J_n(x) = (-1)^n J_n(-x).$$

Solution:

We write

$$\sum_{n=-\infty}^{\infty} J_n(-x)t^n = \exp\{\frac{1}{2}[-x(t - \frac{1}{t})]\} \tag{0.22}$$

$$= e^{\frac{x}{2}[-t-\frac{1}{-t}]} \tag{0.23}$$

$$= \sum_{n=-\infty}^{\infty} J_n(x)(-t)^n \tag{0.24}$$

$$= \sum_{n=-\infty}^{\infty} (-1)^n J_n(x)t^n, \tag{0.25}$$

which yields the desired result.

3. Prove that

$$J_n(x+y) = \sum_{r=-\infty}^{\infty} J_r(x) J_{n-r}(x). \tag{0.26}$$

Solution:

Use the generating function definition,

$$e^{\frac{x}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x) t^n, \tag{0.27}$$

to write

$$e^{\frac{x}{2}(t-\frac{1}{t})} e^{\frac{y}{2}(t-\frac{1}{t})} = \sum_{n=-\infty}^{\infty} J_n(x+y) t^n. \tag{0.28}$$

Rewrite the left-hand side as

$$\left(\sum_{r=-\infty}^{\infty} J_r(x) t^r\right)\left(\sum_{s=-\infty}^{\infty} J_s(y) t^s\right) = \sum_{n=-\infty}^{\infty} J_n(x+y) t^n, \tag{0.29}$$

and let

$$s = n - r \tag{0.30}$$

to write

$$\sum_{n=-\infty}^{\infty}\left[\sum_{r=-\infty}^{\infty} J_r(x) J_{n-r}(y)\right] t^n = \sum_{n=-\infty}^{\infty} J_n(x+y) t^n, \tag{0.31}$$

which yields the desired result.

4. Show the following integral:

$$\int_0^{\pi/2} J_0(x \cos t) \cos t dt = \frac{\sin x}{x}. \tag{0.32}$$

Solution:

Use the expansion

$$J_0(x \cos t) = \sum_{r=0}^{\infty} \frac{(-1)^r}{(r!)^2} \left(\frac{x \cos t}{2}\right)^{2r} \tag{0.33}$$

to obtain

$$\begin{aligned}
\int_0^{\pi/2} J_0(x\cos t)\cos t dt &= \sum_{r=0}^{\infty}\int_0^{\pi/2}\frac{(-1)^r}{(r!)^2}\left(\frac{x\cos t}{2}\right)^{2r}\cos t\ dt \\
&= \sum_{r=0}^{\infty}\frac{(-1)^r}{(r!)^2}\left(\frac{x}{2}\right)^{2r}\int_0^{\pi/2}(\cos t)^{2r+1}\ dt \\
&= \sum_{r=0}^{\infty}\frac{(-1)^r}{(r!)^2}\left(\frac{x}{2}\right)^{2r}\frac{(2r)(2r-2)\dots 4.2}{(2r+1)(2r-1)\dots 3.1} \\
&= \sum_{r=0}^{\infty}\frac{(-1)^r}{(r!)^2}\left(\frac{x}{2}\right)^{2r}\frac{2^r r! r!}{(2r+1)!} \\
&= \frac{1}{x}\sum_{r=0}^{\infty}(-1)^r\frac{x^{2r+1}}{(2r+1)!} \\
&= \frac{\sin x}{x}. \qquad (0.34)
\end{aligned}$$

5. **(Problem 6.1)** Drive the following recursion relations:

$$J_{m-1}(x) + J_{m+1}(x) = \frac{2m}{x}J_m(x) \qquad (0.35)$$

and

$$J_{m-1}(x) - J_{m+1}(x) = 2J_m'(x). \qquad (0.36)$$

Solution:

For the first one, differentiate the generating function [Eq. (6.47)] with respect to t and than equate the coefficients of the equal powers of t. Similarly, for the second recursion relation [Eq. (0.36)] differentiate with respect to x.

II. Transformations of the Bessel's Equation

Sometimes we encounter differential equations, solutions of which can be written in terms of Bessel functions. For example, consider the function

$$y(x;\alpha,\beta,\gamma) = x^{\alpha}J_n(\beta x^{\gamma}), \qquad (0.37)$$

where α,β,γ are three constant parameters. To find the differential equation that $y(x;\alpha,\beta,\gamma)$ satisfies, we substitute

$$g = \frac{y}{x^{\alpha}},\ w = \beta x^{\gamma}, \qquad (0.38)$$

to write

$$g(w) = J_n(w). \tag{0.39}$$

Hence, $g(w)$ satisfies the Bessel's equation [Eq. (6.21)]:

$$w^2 \frac{d^2 g}{dw^2} + w \frac{dg}{dw} + (w^2 - n^2) g(w) = 0, \tag{0.40}$$

which can also be written as

$$w \frac{d}{dw} \left(w \frac{dg}{dw} \right) + (w^2 - n^2) g(w) = 0. \tag{0.41}$$

We now write

$$w \frac{dg}{dw} = w \frac{dg/dx}{dw/dx} = \frac{x}{\gamma} \frac{dg}{dx}, \tag{0.42}$$

hence the first term in Equation (0.41) becomes

$$w \frac{d}{dw} \left(w \frac{dg}{dw} \right) = \frac{1}{\gamma^2} x \frac{d}{dx} \left(x \frac{dg}{dx} \right). \tag{0.43}$$

Using Equation (0.38) we can also write

$$x \frac{dg}{dx} = \frac{y'}{x^{\alpha-1}} - \frac{\alpha y}{x^{\alpha}}, \tag{0.44}$$

which leads to

$$\begin{aligned} x \frac{d}{dx} \left(x \frac{dg}{dx} \right) &= x \frac{d}{dx} \left(x \frac{d}{dx} \left[\frac{y'}{x^{\alpha-1}} - \frac{\alpha y}{x^{\alpha}} \right] \right) \\ &= \frac{y''}{x^{\alpha-2}} - \frac{(2\alpha - 1) y'}{x^{\alpha-1}} + \frac{\alpha^2 y}{x^{\alpha}}. \end{aligned} \tag{0.45}$$

Using Equations (0.43) and (0.45) in Equation (0.41), we obtain the differential equation that $y(x; \alpha, \beta, \gamma)$ satisfies as

$$\frac{d^2 y}{dx^2} - \left(\frac{2\alpha - 1}{x} \right) \frac{dy}{dx} + \left(\beta^2 \gamma^2 x^{2\gamma - 2} + \frac{\alpha^2 - n^2 \gamma^2}{x^2} \right) y(x) = 0. \tag{0.46}$$

Thus, the general solution of this equation can be written as

$$y(x) = x^{\alpha} \left[C_0 J_n(\beta x^{\gamma}) + C_1 N_n(\beta x^{\gamma}) \right]. \tag{0.47}$$

Critical Length of a Vertical Rod:

When a thin uniform vertical rod is clamped at one end, its vertical position is stable granted that its length is less than a critical length. When the rod has the critical length, the vertical position is only a neutral equilibrium position.

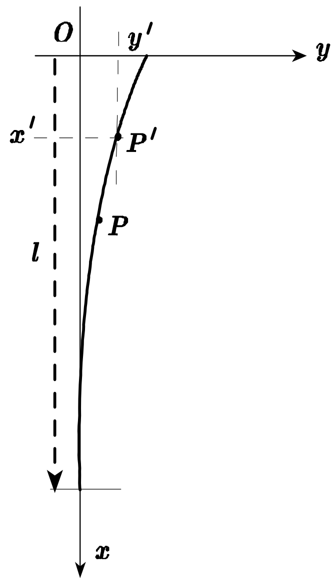

Fig. 0.1 Bending of a rod.

That is, the rod stays in the displaced position after it has been displaced slightly (Greenhill, Proc. Camb. Phil. Soc., IV, 1881. Also see, F. Bowman, Introduction to Bessel Functions, Dover, 1958).

Let the rod be in equilibrium when deviating slightly from the vertical position (Fig. 0.1). Let l be the length of the rod, a be the radius of its cross section and ρ_0 be the uniform density. Let P be an arbitrary point on the rod and P' be a point above it (Fig. 0.1). We now consider the part of the rod in equilibrium above the point P. If we take a mass element, $\rho_0 dx'$, at P', the torque acting on it due to the weight of the upper part of the rod will be will be the integral

$$\int_0^x \rho_0 g(y' - y)dx, \tag{0.48}$$

where g is the acceleration of gravity. This will be balanced by the torque from the elastic forces acting on the rod. From the theory of elasticity, this torque is equal to

$$EI\frac{d^2y}{dx^2}, \tag{0.49}$$

where E is the Young's modulus and we take $I = \frac{1}{4}\pi a^2$. Equating the two torques we get

$$EI\frac{d^2y}{dx^2} = \int_0^x \rho_0 g(y'(x') - y(x))dx', \tag{0.50}$$

Differentiating this with respect to x gives (use Eq. (18.5))

$$EI\frac{d^3y}{dx^3} = -\rho_0 g x\frac{dy}{dx}. \tag{0.51}$$

We rewrite this as

$$\frac{d^3y}{dx^3} + k^2\frac{dy}{dx} = 0, \tag{0.52}$$

where

$$k^2 = \frac{\rho_0 g}{EI}. \tag{0.53}$$

Comparing with Equation (0.46) we see that the solution for $\frac{dy}{dx}$ can be written in terms of Bessel functions as

$$\frac{dy}{dx} = \sqrt{x}\left(C_0 J_{-1/3}\left(\frac{2k}{3}x^{2/3}\right) + C_1 J_{1/3}\left(\frac{2k}{3}x^{2/3}\right)\right). \tag{0.54}$$

For the desired solution we have to satisfy the following boundary conditions:

i) Since there is no torque at the top, where $x = 0$, we need to have

$$\left(\frac{d^2y}{dx^2}\right)_{x=0} = 0. \tag{0.55}$$

ii) At the bottom, where the rod is fixed and vertical, we need to satisfy

$$\left(\frac{dy}{dx}\right)_{x=l} = 0. \tag{0.56}$$

To satisfy the first boundary condition we set $C_1 = 0$, thus obtaining

$$\frac{dy}{dx} = C_0\sqrt{x}J_{-1/3}\left(\frac{2k}{3}x^{2/3}\right). \tag{0.57}$$

The second condition can be satisfied with $C_0 = 0$, which is the trivial solution. For a nontrivial solution, $C_0 \neq 0$, we set

$$J_{-1/3}\left(\frac{2k}{3}l^{2/3}\right) = 0 \tag{0.58}$$

and take the smallest root as the physical solution:

$$\frac{2k}{3}l^{3/2} = 1.8663. \tag{0.59}$$

For a steel rod of radius 0.15 cm, $E = 84,000$ $tons/cm^2$ and density 7.9 g/cm^3, we find $l \cong 1.15m$.

Question: Can you use this as an example for spontaneous symmetry breaking and how? Below are some useful sites on symmetry breaking:

`http://superstringtheory.com/experm/exper3a.html`,
`http://en.wikipedia.org/wiki/Spontaneous_symmetry_breaking`,
`http://cosmicvariance.com/2005/10/24/hidden-symmetries/`.

III. Useful Sites

More references and other useful information about Bessel functions can be found in the following sites:

`http://en.wikipedia.org/wiki/Bessel_function`,
`http://mathworld.wolfram.com/BesselFunctionoftheFirstKind.html`.

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