SUPPLEMENTS TO MATHEMATICAL METHODS IN SCIENCE AND ENGINEERING
http://www.wiley.com/WileyCDA/WileyTitle/productCd-0470041420.html

CHAPTER 20: GREEN'S FUNCTIONS and PATH INTEGRALS

I. Anomalous Diffusion and Path Integrals Over Lévy Paths

Wiener's path integral approach to Brownian motion can be used to represent a wide range of stochastic processes, where the probability density, $W(x,t,x_0,t_0)$, of finding a random variable at the value x at time t is given by the Gaussian distribution [Eq. (20.8))]:

$$W(x,t,x_0,t_0) = \frac{1}{\sqrt{4\pi D(t-t_0)}} \exp\left\{-\frac{(x-x_0)^2}{4D(t-t_0)}\right\},\ t > t_0, \tag{0.1}$$

where $W(x,t,x_0,t_0)$ satisfies the diffusion equation [Eq. (20.5)]

$$\frac{\partial W}{\partial t} = D\frac{\partial^2 W}{\partial x^2} \tag{0.2}$$

with the initial condition

$$\lim_{t\to t_0} W(x,t,x_0,t_0) \to \delta(x-x_0). \tag{0.3}$$

An important feature of the Wiener process is that at all times the scaling relation

$$(x-x_0)^2 \propto (t-t_0), \tag{0.4}$$

where x_0 and t_0 are the initial values of x and t, respectively, is satisfied. To find the fractal dimension of the Brownian motion, we divide the time interval T into N slices, $T = N\Delta t$, which gives the space length of the diffusion path as

$$L = N\Delta x = \frac{T}{\Delta t}\Delta x. \tag{0.5}$$

Using the scaling property [Eq. (0.4)] we write

$$L \propto \frac{1}{\Delta x}. \tag{0.6}$$

When the spacial increment Δx goes to zero, the fractal dimension, $d_{fractal}$, is defined as (Mandelbrot)

$$L \propto (\Delta x)^{1-d_{fractal}}. \tag{0.7}$$

Comparing Equations (0.6) and (0.7) in the limit as $\Delta x \to 0$, gives the fractal dimension of the Brownian motion as

$$d_{fractal}^{Brownian} = 2. \tag{0.8}$$

In terms of the Wiener path integrals, $W(x,t,x_0,t_0)$ is expressed as [Eq. (20.24)]

$$W(x,t,x_0,t_0) = \int_{C[x_0,t_0;x,t]} d_w x(\tau), \tag{0.9}$$

where the Wiener measure, $d_w x(\tau)$, is written as

$$d_w x(\tau) = \exp\left\{-\frac{1}{4D}\int_{t_0}^{t} \dot{x}^2(\tau)d\tau\right\}\prod_{i=1}^{N}\frac{dx_i}{\sqrt{4\pi D\ d\tau}} \tag{0.10}$$

and the integral is evaluated over all continuous paths from (x_0,t_0) to (x,t) [Eqs. (20.24) and (20.25)].

In the presence of a potential, $V(x)$, the diffusion equation is written as

$$\frac{\partial W_B}{\partial t} - D\frac{\partial^2 W_B}{\partial x^2} = -V(x,t)W_B, \tag{0.11}$$

which is also called the **Bloch equation**. Using the Feynman-Kac formula [Eq. (20.38)]:

$$W_B(x,t,x_0,0) = \int_{C[x_0,0;x,t]} d_w x(\tau)\ \exp\left\{-\int_0^t d\tau\ V(x(\tau),\tau)\right\}, \tag{0.12}$$

a perturbative solution of the Bloch equation can be given as [Eq. (20.44)]

$$\begin{aligned} W_B(x,t,x_0,t_0) &= W_D(x,t,x_0,t_0) \\ &- \int_{-\infty}^{\infty} dx' \int_{t_0}^{t} dt' W_D(x,t,x',t')V(x',t')W_D(x',t',x_0,t_0) \\ &+ \int_{-\infty}^{\infty} dx' \int_{t_0}^{t} dt' \int_{-\infty}^{\infty} dx'' \int_{t_0'}^{t'} dt'' W_D(x,t,x',t')V(x',t')W_D(x',t',x'',t'') \\ &\times V(x'',t'')W_D(x'',t'',x_0,t_0) + \cdots\ , \end{aligned} \tag{0.13}$$

where $W_D = W(x,t,x',t')\theta(t-t')$ [Eq. 20.35)] satisfies

$$\frac{\partial W_D}{\partial t} - D\frac{\partial^2 W_D}{\partial x^2} = \delta(x-x')\delta(t-t'), \tag{0.14}$$

and $W(x,t,x',t')$ is the solution of the homogeneous equation $\frac{\partial W_D}{\partial t} - D\frac{\partial^2 W_D}{\partial x^2} = 0$.

Even though the Wiener's mathematical theory of the Brownian motion can be used to describe a wide range of stochastic processes in nature, there also exist a lot of interesting phenomenon where the scaling law in Equation (0.4) is violated. The processes that obey the scaling rule

$$(x-x_0)^2 \propto (t-t_0)^q, \ q \neq 1, \tag{0.15}$$

are in general called **anomalous diffusion**, where the cases with $q < 1$ are classified as **subdiffusive** and the cases with $q > 1$ are called **superdiffusive**.

One of the ways to study anomalous diffusion is to use the **fractional diffusion equation**:

$$\frac{\partial W_L(x,t,x_0,t_0)}{\partial t} = D_q \nabla^q W_L(x,t,x_0,t_0), \tag{0.16}$$

where $\nabla^q \equiv \mathbf{R}_x^q$ is the Riesz fractional derivative:

$$\nabla^q = \frac{\partial^q}{\partial x^q}, \ q < 2. \tag{0.17}$$

In Equation (0.16) D_q stands for the **fractional diffusion constant**, which has the dimension $[D_q] = cm^q \sec^{-1}$. As we discussed in detail in the supplements of Chapter 14, the Riesz derivative is defined with respect to the Fourier transform as

$$\nabla^q W_L(x,t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} dk \ e^{ikx} \, |k|^q \, \overline{W}_L(k,t), \tag{0.18}$$

where $W_L(x,t)$ and its Fourier transform, $\overline{W}_L(k,t)$, are related as

$$W_L(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \ e^{ikx}\overline{W}_L(k,t), \tag{0.19}$$

$$\overline{W}_L(k,t) = \int_{-\infty}^{\infty} dx \ e^{-ikx} W_L(x,t). \tag{0.20}$$

Solution of the fractional diffusion equation [Eq. (0.16)] with the initial condition

$$\lim_{t\to t_0} W_L(x,t,x_0,t_0) = \delta(x-x_0), \tag{0.21}$$

yields the probability density

$$W_L(x,t,x_0,t_0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\ e^{ik(x-x_0)} \exp\left\{-D_q\left|k\right|^q (t-t_0)\right\}. \qquad (0.22)$$

To obtain Equation (0.22), we first take the Fourier transform of the fractional diffusion equation:

$$\frac{\partial \overline{W}_L(x,t,x_0,t_0)}{\partial t} = -D_q\left|k\right|^q \overline{W}_L(x,t,x_0,t_0), \qquad (0.23)$$

which with the initial condition

$$\overline{W}_L(k,0) = 1, \qquad (0.24)$$

can be integrated easily to yield the solution as

$$\overline{W}_L(k,t) = \exp\left(-D_q t\left|k\right|^q\right). \qquad (0.25)$$

For simplicity, we have set $x_0 = t_0 = 0$. Using Fox's H functions, we can also write $\overline{W}_L(k,t)$ as

$$\overline{W}_L(k,t) = \frac{1}{q} H_{0,1}^{1,0}\left(\left(D_q t\right)^{1/q}\left|k\right| \middle| \left(0,\frac{1}{q}\right)\right). \qquad (0.26)$$

Finally, we find the inverse Fourier transform of $\overline{W}_L(k,t)$ to write the solution of the fractional diffusion equation [Eq. (0.16)] as (Laskin (2000a), West et. al.)

$$W_L(x,t) = \frac{\pi}{q\left|x\right|} H_{2,2}^{1,1}\left(\frac{\left|x\right|}{\left(D_q t\right)^{1/q}} \middle|_{(1,1),(1,1/2)}^{(1,1/q),(1,1/2)}\right). \qquad (0.27)$$

For large arguments, $\left|x\right|/(D_q t)^{1/q} \gg 1$, we can write the following useful series expansion (West et.al.)

$$W_L(x,t) = \sum_{l=1}^{\infty} (-1)^{l+1} \frac{\Gamma(1+lq)}{l!} \sin\left(\frac{l\pi q}{2}\right) \frac{(D_q t)^l}{\left|x\right|^{lq+1}}. \qquad (0.28)$$

For $q = 2, W_L(x,t,x_0,t_0)$ [Eq. (0.27)] reduces to the Gaussian probability distribution [Eq. (0.1)]. For $0 < q < 2$, $W_L(k,t)$ [Eq. (0.27)] is called the q-stable Lévy distribution, which possesses finite moments of order up to $m < q$, while all higher order moments diverge. Lévy processes obey the scaling rule

$$(x - x_0) \propto (t - t_0)^{1/q},\ 1 < q \leq 2, \qquad (0.29)$$

where $(x - x_0)$ is the length of the Lévy path for the time interval $(t - t_0)$. Dividing a given time interval T into N slices, $T = N\Delta t$, we write

$$L = N\Delta x = \frac{T}{\Delta t}\Delta x, \qquad (0.30)$$

where L is the length of the Lévy path and Δx is the length increment for Δt. Substituting the scaling rule [Eq. (0.29)] into the above equation gives us

$$L \propto (\Delta x)^{1-q}. \tag{0.31}$$

Considered in the limit as $\Delta x \to 0$, this yields the fractal dimension of the Lévy path as

$$d_{fractal}^{Lévy} = q, \ 1 < q \leq 2. \tag{0.32}$$

For a Lévy process obeying the fractional Bloch equation,

$$\frac{\partial W_L}{\partial t} - D_q \nabla^q W_L = -V(x,t) W_L, \tag{0.33}$$

we now write the Feynman-Kac formula as [Eq. (20.38)]

$$W_L(x,t,x_0,0) = \int_{C[x_0,0;x,t]} d_L x(\tau) \ \exp\left\{-\int_0^t d\tau \ V(x(\tau),\tau)\right\}, \tag{0.34}$$

where the Wiener measure, $d_w x(\tau)$, is replaced by the Lévy measure defined as

$$d_L x(\tau) = \lim_{N\to\infty}\left[dx_1 \cdots dx_N \ \left(\frac{1}{D_q \Delta\tau}\right)^{(N+1)/q} \prod_{i=1}^{N+1} L_q\left\{\left(\frac{1}{D_q\Delta\tau}\right)^{1/q} |x_i - x_{i-1}|\right\}\right]. \tag{0.35}$$

We have divided the interval $[t-0]$ into $N+1$ segments [Eq. (20.15)],

$$\Delta\tau = \frac{t-0}{N+1}, \tag{0.36}$$

covered in N steps and introduced the function L_q such that the Lévy distribution function, $W_L(x,t)$, is expressed in terms of the Fox's H functions as

$$\begin{aligned} W_L(x,t) &= (D_q t)^{-1/q} L_q\left\{\left(\frac{1}{D_q t}\right)^{1/q} |x|\right\} \\ &= \frac{\pi}{q\,|x|} H_{2,2}^{1,1}\left(\frac{1}{(D_q t)^{1/q}}\,|x| \middle|\begin{matrix}(1,1/q),(1,1/2) \\ \\ (1,1),(1,1/2)\end{matrix}\right). \end{aligned} \tag{0.37}$$

Note that $i = 1$ marks the initial point while $t = N+1$ is the end point of the path. Since the particle is certain to be somewhere in the internal $x \in [-\infty,\infty]$, we have

$$\int_{-\infty}^{\infty} dx \int_{[x_0,t_0;x,t]} d_L x(\tau) = 1. \tag{0.38}$$

In this regard, the dimension of $d_L x(\tau)$ and the propagator

$$W_L(x,t) = \int_{[x_0,t_0;x,t]} d_L x(\tau) \tag{0.39}$$

is $1/cm$, a point that will be needed shortly.

II. Fox's H-Functions

In 1961 Fox introduced the H-functions, which are special functions of very general nature. They allow treatment of several phenomenon, among which is anomalous diffusion, in elegant and efficient formalism. Fox's H functions are generalizations of the Meijer's G-functions and they are defined with respect to a Mellin-Barnes type integral (Fox, Srivastava et.al., West et.al.):

$$H^{m,n}_{p,q}(z) = H^{m,n}_{p,q}\left(z\Big|^{(a_p,A_p)}_{(b_q,B_q)}\right) = H^{m,n}_{p,q}\left(z\Big|^{(a_1,A_1),\dots,(a_p,A_p)}_{(b_1,B_1),\dots,(b_q,B_q)}\right) \tag{0.40}$$

$$= \frac{1}{2\pi i}\int_C h(s) z^s ds, \tag{0.41}$$

where

$$h(s) = \frac{\prod_{j=1}^m \Gamma(b_j - B_j s)\prod_{j=1}^n \Gamma(1 - a_j + A_j s)}{\prod_{j=n+1}^p \Gamma(a_j - A_j s)\prod_{j=m+1}^q \Gamma(1 - b_j + B_j s)}, \tag{0.42}$$

m, n, p, q are positive integers satisfying $0 \le n \le p$, $1 \le m \le q$, and empty products are taken as unity. Also, A_j, $j = 1, \dots, p$, and B_j, $j = 1, \dots, q$, are positive numbers, and a_j, $j = 1, \dots, p$, and b_j, $j = 1, \dots, q$, are complex numbers satisfying

$$A_j(b_h + \nu) \neq B_h(a_j - \lambda - 1),\ \nu, \lambda = 0, 1, \dots\ ;\ h = 1, \dots, m,\ j = 1, \dots, n. \tag{0.43}$$

The contour C is such that the poles of $\Gamma(b_j - B_j s)$, $j = 1, \dots, m$, are separated from the poles of $\Gamma(1 - a_j + A_j s)$, $j = 1, \dots, n$. The poles of the integrand are assumed to be simple. The H-function is an analytic function of z, if either

$$\mu = \sum_{j=1}^q B_j - \sum_{j=1}^p A_j > 0 \text{ and } 0 < |z| < \infty \tag{0.44}$$

or

$$\mu = 0 \text{ and } 0 < |z| < \prod_{j=1}^p A_j^{-A_j} \prod_{j=1}^q B_j^{B_j}. \tag{0.45}$$

Fox's H-functions are very useful in solving fractional diffusion equation (Glöckle and Nonnenmacher, West et. al.). Remembering the definitions for the Riemann-Liouville fractional integral:

$$ {}_0\mathbf{D}_t^{-q}[f(t)] = {}_0\mathbf{I}_t^q[f(t)] = \frac{1}{\Gamma(q)} \int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{1-q}}, \; q > 0, \tag{0.46} $$

and the Riemann-Liouville fractional derivative as

$$ {}_0\mathbf{D}_t^q[f(t)] = \frac{d^n}{dt^n}\left({}_0\mathbf{I}_t^{n-q}[f(t)]\right), \; n > q, \tag{0.47} $$

we can write the following useful fractional derivative of the H-function for arbitrary q:

$$ {}_0\mathbf{D}_z^q\left[z^a H_{p,q}^{m,n}\left((cz)^b\Big|_{(b_j,B_j)}^{(a_j,A_j)}\right)\right] = z^{a-q} H_{p+1,q+1}^{m,n+1}\left((cz)^b\Big|_{(b_j,B_j),(q-a,b)}^{(-a,b),(a_j,A_j)}\right), \tag{0.48} $$

where $a, b > 0$ and $a + b\min(b_j/B_j) > -1$, $1 \leq j \leq m$. Solutions of the fractional diffusion equation can be obtained by formally manipulating the parameters in the above formula.

Introducing the notation

$$ H(t) = H_{p,q}^{m,n}\left(t\Big|_{(b_j,B_j)}^{(a_j,A_j)}\right), \tag{0.49} $$

we can express the Laplace transform of $H(t)$ in terms of another H-function as (West et.al.)

$$ \widetilde{H}(s) = \pounds\{H(t)\} = \begin{cases} \frac{1}{s} H_{q,p+1}^{n+1,m}\left(s\Big|_{(1,1),(1-a_j,A_j)}^{(1-b_j,B_j)}\right) & , \quad 0 \leq \mu \leq 1, \\ \frac{1}{s} H_{p+1,q}^{m,n+1}\left(\frac{1}{s}\Big|_{(b_j,B_j)}^{(0,1),(a_j,A_j)}\right) & , \quad \mu \geq 1, \end{cases} \tag{0.50} $$

where μ is defined in Equation (0.44). On the other hand, given the Laplace transform

$$ \widetilde{H}(s) = H_{p,q}^{m,n}\left(s\Big|_{(b_j,B_j)}^{(a_j,A_j)}\right), \tag{0.51} $$

we can write the inverse transform as

$$ H(s) = \pounds^{-1}\{\widetilde{H}(s)\} = \begin{cases} \frac{1}{t} H_{q,p+1}^{n,m}\left(t\Big|_{(1-a_j,A_j),(1,1)}^{(1-b_j,B_j)}\right) & , \quad 0 \leq \mu \leq 1, \\ \frac{1}{t} H_{p+1,q}^{m,n}\left(\frac{1}{t}\Big|_{(b_j,B_j)}^{(a_j,A_j),(0,1)}\right) & , \quad \mu \geq 1. \end{cases} \tag{0.52} $$

These relations [Eqs. (0.50) and (0.52)] hold for $\lambda > 0$ and for

$$\max_{1\leq j\leq n} \mathrm{Re}\left(\frac{a_j - 1}{A_j}\right) < \min_{1\leq j\leq m} \mathrm{Re}\left(\frac{b_j}{B_j}\right). \quad (0.53)$$

III. Fractional Quantum Mechanics

It is well known that fractional diffusion equation is a convenient approach to anomalous diffusion, which involves global interactions and memory effects. In this regard, it is important that we also develop the basic equations of fractional quantum mechanics. We first remember that under Wick's rotation, $t \to -it$, the one dimensional Schrödinger equation for a free particle:

$$\frac{\partial \Psi}{\partial t} = \frac{i}{2m}\frac{\partial^2 \Psi}{\partial x^2}, \ \hbar = 1, \quad (0.54)$$

transforms into the diffusion equation:

$$\frac{\partial \Psi}{\partial t} = \widetilde{D}\frac{\partial^2 \Psi}{\partial x^2}, \quad (0.55)$$

where $\widetilde{D}$ is equal to $1/2m$. Just like we have written the anomalous diffusion equation [Eq. (0.16)], we generalize the above equation so that it is now written in terms of the Riesz fractional space derivative as

$$\frac{\partial \Psi}{\partial t} = \widetilde{D}_q \nabla^q \Psi, \ 0 < q < 2, \quad (0.56)$$

where $\widetilde{D}_q$ is the *generalized fractional quantum diffusion constant* and ∇^q is the Riesz derivative. The inverse Wick rotation, $t \to it$, gives the fractional version of the Schrödinger equation as

$$\frac{\partial \Psi}{\partial t} = i\widetilde{D}_q \nabla^q \Psi. \quad (0.57)$$

In the presence of interactions, the Schrödinger equation [Eq. (20.14)]:

$$\frac{\partial \Psi}{\partial t} = \frac{i}{2m}\frac{\partial^2 \Psi}{\partial x^2} - iV(x)\Psi(x,t), \quad (0.58)$$

is now generalized as

$$\frac{\partial \Psi}{\partial t} = i\widetilde{D}_2 \nabla^q \Psi - iV(x)\Psi(x,t). \quad (0.59)$$

After a Wick rotation, this becomes the Bloch equation:

$$\frac{\partial \Psi}{\partial t} = \widetilde{D}_q \nabla^q \Psi - V(x)\Psi(x,t). \quad (0.60)$$

When $q=2$, the generalized fractional quantum diffusion constant becomes $\widetilde{D}_2 = 1/2m$. We now follow the steps described in Section 20.7 that lead to the Feynman path integral formulation of quantum mechanics and replace D_q with $i\widetilde{D}_q$ in $d_L x(\tau)$ [Eq. (0.35)] to write

$$d_L^{Feynman} x(\tau) = \lim_{N\to\infty} \left[dx_1 \cdots dx_N \quad \left(\frac{1}{i\widetilde{D}_q \Delta\tau} \right)^{-N+1)/q} \times \prod_{i=1}^{N+1} L_q \left\{ \left(\frac{1}{i\widetilde{D}_q \Delta\tau} \right)^{1/q} |x_i - x_{i-1}| \right\} \right]. \tag{0.61}$$

Instead of the Feynman measure $d_F x(\tau)$ [Eq. (20.143)], we now use $d_L^{Feynman} x(\tau)$, since the path integrals are to be evaluated over the Lévy paths, and $\widetilde{D}_q$ is the generalized fractional diffusion constant of fractional quantum mechanics. To convert these equations into physical dimensions, we have to introduce the proper powers of $\hbar$ into $d_L^{Feynman} x(\tau)$. We first note that the physical unit of $\left(\frac{1}{i\widetilde{D}_q \Delta\tau}\right)^{1/q}$ is $1/cm$, hence the unit of $(i\widetilde{D}_q \Delta\tau)^{-(N+1)/q}$ must be cm^{N-1}. We now write

$$\hbar^a \left(\frac{\hbar^b}{i\widetilde{D}_q \Delta\tau} \right)^{1/q}, \tag{0.62}$$

where a and b are to be determined. Using

$$[\hbar] = ergs\ cm, \tag{0.63}$$
$$[\Delta\tau] = \sec, \tag{0.64}$$

and

$$\left[\hbar^a \left(\frac{\hbar^b}{i\widetilde{D}_q \Delta\tau} \right)^{1/q} \right] = \frac{1}{cm} \tag{0.65}$$

we get

$$[\widetilde{D}_q]^{1/q} = erg^{aq+1} cm^q \sec^{aq+b-1}, \tag{0.66}$$
$$= gm^{aq+b} cm^{2aq+q+2b} \sec^{-aq-b-1}. \tag{0.67}$$

Since when $q=2$, Equation (0.56) gives the dimension of $\widetilde{D}_2$ as

$$[\widetilde{D}_2] = \left[\frac{1}{2m} \right] = gm^{-1}, \tag{0.68}$$

we require the following set of equations:

$$\left.\begin{array}{r}aq+b=-1,\\2aq+q+2b=0,\\-aq-b-1=0,\end{array}\right|_{q=2}, \tag{0.69}$$

to be true at $q=2$, which yields a and b as

$$a=-1, \tag{0.70}$$

$$b=1. \tag{0.71}$$

Thus the physical dimension of $\widetilde{D}_q$ is now obtained as $ergs^{1-q}cm^q \sec^{-q}$, and the Feynman measure over the Lévy paths with the physical dimensions becomes

$$\begin{aligned}d_L^{Feynman}x(\tau) = \lim_{N\to\infty}&\left[dx_1\cdots dx_N\ \frac{1}{\hbar^{N+1}}\left(\frac{\hbar}{i\widetilde{D}_q\Delta\tau}\right)^{(N+1)/q}\right.\\&\left.\times\prod_{i=1}^{N+1}L_q\left\{\frac{1}{\hbar}\left(\frac{\hbar}{i\widetilde{D}_q\Delta\tau}\right)^{1/q}|x_i-x_{i-1}|\right\}\right].\end{aligned} \tag{0.72}$$

Note that the physical dimension of $d_L^{Feynman}x(\tau)$ is $1/cm$.

We now modify the Feynman-Kac formula to be evaluated over the Lévy paths to write the propagator:

$$K(x,t,x_0,t_0)=\int_{[x_0,t_0,x,t]}d_L^{Feynman}x(\tau)\exp\left\{-\frac{i}{\hbar}\int_{t_0}^{t}d\tau\ V[x(\tau)]\right\}. \tag{0.73}$$

Using the propagator $K(x,t,x_0,t_0)$, we can write the solution of the fractional Schrödinger equation [Eq. (0.59)] as

$$\Psi(x,t)=\int K(x,t,x',t')\Psi(x',t')dx'. \tag{0.74}$$

With the proper factors of $\hbar$ introduced, the fractional Schrödinger equation in physical dimensions becomes

$$i\hbar\frac{\partial\Psi}{\partial t}=-\widetilde{D}_q\left(\hbar\nabla\right)^q\Psi+V(x)\Psi(x,t). \tag{0.75}$$

Laskin (2000a) has shown that indeed $\Psi(x,t)$ [Eq. (0.74)] satisfies the fractional Schrödinger equation [Eq. (0.75)]. Based on the Lévy paths and the generalization of the Riesz fractional derivative to three dimensions, Laskin (2002) also gave the three dimensional version of the fractional Schrödinger equation.

IV. References and Useful Sites

References:

Fox, C., *The G and H-Functions as Symmetric Kernels*, Tran.Am. Math. Soc., **98**, 395, (1961).

Glöckle, W.G. and T.F. Nonnenmacher, J. Stat. Phys. **71**, 741, (1993).

Guo, X. and M. Xu, *Some Applications of Fractional Schrödinger Equation*, J. Math. Phys. **47**, 082104, (2006).

Laskin, N., *Fractional Quantum Mechanics*, Phys. Rev. **E62**, 3135, (2000a), (Also available online: http://arxiv.org/abs/0811.1769.)

Laskin, N., *Fractional Quantum Mechanics and Lěvy Path Integrals*, Phys. Lett. **A268**, 298, (2000b).

Laskin, N., *Fractional Schrödinger equation*, Physical Review, **E66**, 056108, (2002). (Also available online: http://arxiv.org/abs/quant-ph/0206098.)

Mandelbrot, B.B., *The Fractal Geometry of Nature*, Freeman, New York, (1982).

Mathai, A. M., R.K. Saxena, *The H-Function With Applications in Statistics and Other Disciplines*, Wiley Eastern, New Delphi, (1978).

Naber, M., *Time Fractional Schrödinger Equation*, J. Math. Physics, **45**, 3339, (2004).

Srivastava, H.M., R.K. Saxena, *Operators of Fractional Integration and Their Applications*, Appl. Math. and Comp., **118**,1, (2001).

West, B.J., P. Grigolini, R. Metzler and F. Nonnenmacher, *Fractional Diffusion and Lévy Stable Processes*, Phys. Rev. **E55**, 99, (1977).

Useful links:

`http://mathworld.wolfram.com/FoxH-Function.html`,
`http://en.wikipedia.org/wiki/Levy_distribution`,
`http://en.wikipedia.org/wiki/Wiener_process`,
`http://en.wikipedia.org/wiki/Levy_process`.

Selçuk Bayin (August, 2009)