SUPPLEMENTS TO MATHEMATICAL METHODS IN SCIENCE AND ENGINEERING
http://www.wiley.com/WileyCDA/WileyTitle/productCd-0470041420.html

CHAPTER 3: LAGUERRE POLYNOMIALS

I. Solutions or Hints to Selected Problems:

1. **(Problem 3.2)** Derive the recursion relations

$$(n+1)\, L_{n+1}(x) = (2n+1-x)\, L_n(x) - nL_{n-1}(x), \tag{0.1}$$

$$xL_n'(x) = nL_n(x) - nL_{n-1}(x), \tag{0.2}$$

and

$$L_n'(x) = -\sum_{r=0}^{n-1} L_r(x). \tag{0.3}$$

Solution:

For the first recursion relation, differentiate the generating function [Eq. (3.25)] with respect to t and then compare equal powers of t.

For the second recursion relation, differentiate the generating function with respect to x and compare equal powers of t to obtain the relation

$$L_n'(x) - L_{n-1}'(x) = -L_{n-1}(x). \tag{0.4}$$

Now let $n \to n+1$ and then multiply the above equation with $(n+1)$ to write

$$(n+1)L_{n+1}'(x) - (n+1)L_n'(x) = -(n+1)L_n(x). \tag{0.5}$$

Finally, differentiate the first recursion relation [Eq. (0.1)] with respect to x and use the result in Equation (0.5) to obtain the desired recursion relation.

For the third recursion relation, use the fact that Laguerre polynomials form a complete set to write the expansion

$$L_n' = \sum_{k=0}^{\infty} a_k L_k, \tag{0.6}$$

and then evaluate the expansion coefficients as

$$a_k = \begin{cases} 0, & k \geq n, \\ \\ -1, & k = 0, 1, \ldots, (n-1). \end{cases} \tag{0.7}$$

Note that calculating a_k does not require long calculations and the third recursion relation [Eq. (0.3)] can also be written as

$$L'_{n+1} = -L_n - \sum_{k=0}^{n-1} L_k. \tag{0.8}$$

2. **(Problem 3.4)** Write the normalized wave function of the hydrogen atom in terms of the spherical harmonics and the associated Laguerre polynomials.

Solution:

The answer is given as

$$\Psi_{nlm}(r, \theta, \phi) = C_0 \kappa^{l+1} r^l e^{-\kappa r} L_{n-l-1}^{2l+1}(2\kappa r) Y_l^m(\theta, \phi), \tag{0.9}$$

where C_0 is the normalization constant and

$$\kappa = \sqrt{\frac{2m\,|E|}{\hbar^2}}. \tag{0.10}$$

To obtain C_0, use the normalization condition of the wave function:

$$\int_V |\Psi_{nlm}(r, \theta, \phi)|^2 \, dv = 1,$$

$$\int_0^\infty \int_0^{2\pi} \int_0^\pi |\Psi_{nlm}(r, \theta, \phi)|^2 \, r^2 \sin^2 \theta dr d\theta d\phi = 1. \tag{0.11}$$

First use the normalization condition of the spherical harmonics [Eq. (2.179)]:

$$\int_0^{2\pi} \int_0^\pi |Y_{lm}(\theta, \phi)|^2 \sin^2 \theta d\theta d\phi = 1, \tag{0.12}$$

and then show the integral

$$\int_0^\infty e^{-x} x^{k+1} \left[L_n^k(x)\right]^2 = \frac{(n+k)!(2n+k+1)}{n!}, \; n = 1, 2, \ldots \,, \tag{0.13}$$

which leads to the desired result as

$$C_0 = 2^{l+1} \sqrt{\frac{2\kappa(n-l-1)!}{2n(n+l)!}} \tag{0.14}$$

and

$$\Psi_{nlm}(r,\theta,\phi) = (2\kappa)^{3/2}\left[\frac{(n-l-1)!}{2n(n+l)!}\right]^{1/2}(2\kappa r)^l e^{-\kappa r} L_{n-l-1}^{2l+1}(2\kappa r) Y_l^m(\theta,\phi). \tag{0.15}$$

3. **(Problem 3.11)** In quantum mechanics the radial part of Schrödinger's equation for the three-dimensional harmonic oscillator is given as

$$\frac{d^2R(x)}{dx^2} + \frac{2}{x}\frac{dR(x)}{dx} + \left(\epsilon - x^2 - \frac{l(l+1)}{x^2}\right)R(x) = 0, \tag{0.16}$$

where x and ϵ are defined in terms of the radial distance r and the energy E as

$$x = \frac{r}{\sqrt{\dfrac{\hbar}{m\omega}}} \quad \text{and} \quad \epsilon = \frac{E}{\hbar\omega/2}. \tag{0.17}$$

l takes the integer values $l = 0, 1, 2...$. Show that the solutions of this equation can be expressed in terms of the associated Laguerre polynomials of argument x^2.

Solution:

First show that the above differential equation gives a three-term recursion relation. To obtain a two-term recursion relation look at the behavior of the differential equation at $\pm\infty$, which suggest the transformation

$$R(x) = e^{-x^2/2}v(x). \tag{0.18}$$

Obtain the differential equation for $v(x)$:

$$v'' + \left(\frac{2}{x} - 2x\right)v' + v\left(-3 + \epsilon - \frac{l(l+1)}{x^2}\right) = 0. \tag{0.19}$$

Associated Laguerre polynomials with the argument x^2 satisfies

$$\frac{d^2}{dx^2}L_n^k(x^2) + \left[(2k+1)\frac{1}{x} - 2x\right]\frac{d}{dx}L_n^k(x^2) + nL_n^k(x^2) = 0. \tag{0.20}$$

The last term of Equation (0.19) still does not allow us to compare the two equations, hence we analyze its behavior near the origin, which suggests the substitution

$$v(x) = x^l w(x). \tag{0.21}$$

The differential equation for $w(x)$ is obtained as

$$w'' + \left(\frac{2(l+1)}{x} - 2x \right) w' + (-3 + \epsilon - 2l)\, w = 0. \tag{0.22}$$

Comparing with Equation (0.20), we can write the solution for $w(x)$ as

$$w(x) = L^{l+1/2}_{-3+\epsilon-2l}(x^2). \tag{0.23}$$

Now the complete solution becomes

$$R(x) = x^l e^{-x^2/2} L^{l+1/2}_{-3+\epsilon-2l}(x^2). \tag{0.24}$$

II. Useful Sites

More references and other useful information about Laguerre polynomials can be found in the following sites:
`http://en.wikipedia.org/wiki/Laguerre_polynomials`,
`http://mathworld.wolfram.com/LaguerrePolynomial.html`

Selçuk Bayin (October 2008).