Supplements to Mathematical Methods in Science and Engineering
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CHAPTER 7: HYPERGEOMETRIC FUNCTIONS

I. Solutions or Hints to Selected Problems:

1. **(Problems 7.1, 7.2, 7.4–7.7)** Using the Pochhammer symbol:

$$(\alpha)_r = \alpha(\alpha+1)\cdots(\alpha+r-1) \tag{0.1}$$

$$= \frac{\Gamma(\alpha+r)}{\Gamma(\alpha)}, \tag{0.2}$$

where r is a positive integer and $(\alpha)_0 = 1$, we can write the hypergeometric function [Eq. (7.12)] as

$$F(a,b,c;x) = \sum_{r=0}^{\infty} \frac{(a)_r(b)_r}{(c)_r}\frac{x^r}{r!}. \tag{0.3}$$

Hypergeometric functions are also written as ${}_2F_1(a,b,c;x)$, which follows from the general definition

$$\begin{aligned} &{}_mF_n(a_1,a_2,\ldots,a_m,b_1,b_2,\ldots,b_n;x) \\ &= \sum_{r=0}^{\infty} \frac{(a_1)_r\cdots(a_m)_r}{(b_1)_r\cdots(b_n)_r}\frac{x^r}{r!}. \end{aligned} \tag{0.4}$$

Hypergeometric function ${}_2F_1(a,b,c;x)$ satisfies the hypergeometric equation [Eq. (7.1)]:

$$x(1-x)y'' + [c-(a+b+1)x]y' - aby(x) = 0. \tag{0.5}$$

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Many of the special functions of physics and engineering can be written in terms of the hypergeometric function:

$$P_l(x) = {}_2F_1(-l, l+1, 1; \frac{1-x}{2}), \tag{0.6}$$

$$P_l^m(x) = \frac{(l+m)!}{(l-m)!} \frac{(1-x^2)^{m/2}}{2^m m!} {}_2F_1(m-l, m+l+1, m+1; \frac{1-x}{2}), \tag{0.7}$$

$$C_n^\lambda(x) = \frac{\Gamma(n+2\lambda)}{n!\Gamma(2\lambda)} {}_2F_1(-n, n+2\lambda, \lambda+\frac{1}{2}; \frac{1-x}{2}), \tag{0.8}$$

$$U_n(x) = n\sqrt{1-x^2}\, {}_2F_1(-n+1, n+1, \frac{3}{2}; \frac{1-x}{2}), \tag{0.9}$$

$$T_n(x) = {}_2F_1(-n, n, \frac{1}{2}; \frac{1-x}{2}). \tag{0.10}$$

Similarly, using the confluent hypergeometric function, $M(a, c; x)$, which is also written as

$${}_1F_1(a, c; x) = \sum_{r=0}^{\infty} \frac{(a)_r}{(c)_r} \frac{x^r}{r!}, \tag{0.11}$$

we can write

$$J_n(x) = \frac{e^{-ix}}{n!} (\frac{x}{2})^n {}_1F_1(n+\frac{1}{2}, 2n+1; 2ix), \tag{0.12}$$

$$H_{2n}(x) = (-1)^n \frac{(2n)!}{n!} {}_1F_1(-n, \frac{1}{2}; x^2), \tag{0.13}$$

$$H_{2n+1}(x) = (-1)^n \frac{2(2n+1)!x}{n!} {}_1F_1(-n, \frac{3}{2}; x^2), \tag{0.14}$$

$$L_n(x) = {}_1F_1(-n, 1; x), \tag{0.15}$$

$$L_n^k(x) = \frac{\Gamma(n+k+1)}{n!\Gamma(k+1)} {}_1F_1(-n, k+1; x). \tag{0.16}$$

Show these relations.

Solution:

To prove these relations we can write the series expressions for the hypergeometric functions and then compare with the series representations of the corresponding function. For example, consider

$$P_l(x) = {}_2F_1(-l, l+1, 1; \frac{1-x}{2}). \tag{0.17}$$

We write the hypergeometric function as

$${}_2F_1(-l, l+1, 1; \frac{1-x}{2}) = \sum_{r=0}^{\infty} \frac{(-n)_r (n+1)_r}{(1)_r} \frac{[(1-x)/2]^r}{r!}. \tag{0.18}$$

Using

$$(-n)_r = \begin{cases} (-1)^r \dfrac{n!}{(n-r)!}, & r \leq n, \\ 0, & r \geq (n+1), \end{cases} \tag{0.19}$$

and

$$(n+1)_r = \frac{(n+r)!}{n!},\ (1)_r = r!, \tag{0.20}$$

we obtain

$${}_2F_1(-l, l+1, 1; \frac{1-x}{2}) = \sum_{r=0}^{\infty} \frac{(n+r)!}{2^r (n-r)!(r!)^2} (x-1)^r. \tag{0.21}$$

To obtain the desired result, we need the Taylor series expansion of $P_l(x)$ around $x = 1$ as

$$P_l(x) = \sum_{r=0}^{\infty} P_l^{(r)}(1) \frac{(x-1)^r}{r!}, \tag{0.22}$$

where $P_l^{(r)}(1)$ stands for the r^{th} derivative of $P_l(x)$ evaluated at $x = 1$. Using the generating function [Eq. (2.65)] we can evaluate these derivatives as

$$P_l^{(r)}(1) = \begin{cases} \dfrac{1}{2^r r!} \dfrac{(r+l)!}{(l-r)!}, & l \geq r \\ 0, & l < r \end{cases}, \tag{0.23}$$

which when substituted into Equation (0.22) and compared with Equation (0.21) yields the desired result. Of course, we can also use the method in Section (7.2). That is, by making an appropriate transformation of the independent variable in the hypergeometric equation and then by comparing the result with the equation at hand to find the parameters.

2. Using the fact that confluent hypergeometric series is convergent for all x, which can be verified by standard methods, find the solutions of

$$x^2 y'' + \{-\frac{x^2}{4} + kx + \frac{1}{4} - m^2\} y(x) = 0 \tag{0.24}$$

for the interval $x \in [0, \infty]$.

Solution:

First obtain the transformation,

$$y(x) = x^{(\frac{1}{2} - m)} e^{-x/2} w(x), \tag{0.25}$$

that reduces the above differential equation into a differential equation with a two-term recursion relation and then find the solution for $w(x)$ in terms of the hypergeometric functions.

3. Show that the transformation

$$t \to 1 - t \tag{0.26}$$

transforms the basic integral representation of the hypergeometric function [Eq. (7.27)]:

$${}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}dt}{(1-tx)^a}, \quad \mathrm{real}\, c > \mathrm{real}\, b > 0, \tag{0.27}$$

into an integral of the same form and then prove that

$${}_2F_1(a,b,c;x) = (1-x)^{-a}\ {}_2F_1(a,c-b,c;\frac{x}{x-1}) \tag{0.28}$$

Solution:

Substituting $t \to 1-t$ in the integral definition [Eq. (0.27)]:

$${}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{(1-t)^{b-1}t^{c-b-1}dt}{(1-(1-t)x)^a}, \tag{0.29}$$

and rearranging terms as

$$\begin{aligned}&{}_2F_1(a,b,c;x)\\ &= \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-t)^{b-1}t^{c-b-1}(1-x)^{-a}\left(1-\frac{xt}{x-1}\right)^{-a} dt\end{aligned} \tag{0.30}$$

yields the desired result.

4. Drive the following relation.

$${}_2F_1(a,b,c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\ . \tag{0.31}$$

Solution:

Use the integral definition [Eq. (0.27)] and the definition of the beta function and the properties of the gamma function given in Problem 7.10.

5. **(Problem 7.10)** Prove the integral representations

$${}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-tx)^a}dt, \quad \mathrm{real}\, c > \mathrm{real}\, b > 0. \tag{0.32}$$

and

$$ {}_1F_1(a,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{xt} t^{a-1}(1-t)^{c-a-1} dt, \ \ \text{real}\, c > \text{real}\, a > 0. \tag{0.33} $$

Solution:

We show the first one. Start with the basic series definition of the hypergeometric equation and write

$$ {}_2F_1(a,b,c;x) = \sum_{r=0}^{\infty} \frac{(a)_r (b)_r}{(c)_r} \frac{x^r}{r!} \tag{0.34} $$

$$ = \sum_{r=0}^{\infty} \frac{\Gamma(a+r)\Gamma(b+r)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c+r)} \frac{x^r}{r!}. \tag{0.35} $$

Using the relation between the beta and the gamma functions,

$$ B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \tag{0.36} $$

we write this as

$$ {}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-b)} \sum_{r=0}^{\infty} \Gamma(a+r) \left[\frac{\Gamma(c-b)\Gamma(b+r)}{\Gamma(c+r)} \right] \frac{x^r}{r!} $$

$$ = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-b)} \sum_{r=0}^{\infty} \Gamma(a+r) B(b+r, c-b) \frac{x^r}{r!}. \tag{0.37} $$

We now use the integral definition of the beta function:

$$ B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt, \ \ p > 0, \ \ q > 0, \tag{0.38} $$

to write

$$ {}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(c-b)} \sum_{r=0}^{\infty} \Gamma(a+r) \int_0^1 t^{b+r-1}(1-t)^{c-b-1} dt \frac{x^r}{r!}. \tag{0.39} $$

Finally, rearrange and use the binomial expansion

$$ (1-xt)^{-a} = \sum_{r=0}^{\infty} \frac{\Gamma(a+r)}{\Gamma(a)} \frac{(xt)^r}{r!} \tag{0.40} $$

to get the desired result.

6. **(Problem 7.3)** Derive the Kummer formula

$$M(a, c; x) = e^x M(c - a, c; -x). \tag{0.41}$$

Solution:

We write the integral definition:

$${}_1F_1(a, c; x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 e^{xt} t^{a-1} (1-t)^{c-a-1} dt, \quad \mathrm{real}\, c > \mathrm{real}\, a > 0, \tag{0.42}$$

and make the transformation

$$t \to 1 - t. \tag{0.43}$$

Comparing the result with the integral definition above gives the Kummer formula.

Remember that

$$M(a, c; x) \equiv {}_1F_1(a, c; x). \tag{0.44}$$

II. Useful Sites

More references and other useful information about the hypergeometric functions can be found in the following sites:

`http://mathworld.wolfram.com/HypergeometricFunction.html`,
`http://en.wikipedia.org/wiki/Hypergeometric_series`,

For applications with Mathematica® one can use the site
`http://functions.wolfram.com/HypergeometricFunctions/`.

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