SUPPLEMENTS TO MATHEMATICAL METHODS IN SCIENCE AND ENGINEERING
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CHAPTER 4: HERMITE POLYNOMIALS

I. Solutions or Hints to Selected Problems:

1. **(Problem 4.3)** Quantum mechanics of the three-dimensional harmonic oscillator leads to the following differential equation for the radial part of the wave function:

$$\frac{d^2R(x)}{dx^2}+\frac{2}{x}\frac{dR(x)}{dx}+\left[\epsilon-x^2-\frac{l(l+1)}{x^2}\right]R(x)=0, \tag{0.1}$$

where x and ϵ are defined in terms of the radial distance r and the energy E as

$$x=\frac{r}{\sqrt{\frac{\hbar}{m\omega}}} \quad \text{and} \quad \epsilon=\frac{E}{\hbar\omega/2} \tag{0.2}$$

and l takes integer values $l=0,1,2...$.

i) Examine the nature of the singular point at $x=\infty$.

ii) Show that in the limit as $x\to\infty$, the solution goes as

$$R\to e^{-x^2/2}. \tag{0.3}$$

iii) Using the Frobenius method, find an infinite series solution about $x=0$ in the interval $[0,\infty]$. Check the convergence of your solution. Should your solution be finite everywhere, including the end points of your interval? why?

iv) For finite solutions everywhere in the interval $[0,\infty]$, what restrictions do you have to impose on the physical parameters of the system.

v) For $l=0,1,$ and 2 find explicitly the solutions corresponding to the three smallest values of ϵ.

Hint:

First observe that the differential equation for $R(r)$ gives a three-term recursion relation. Then use the asymptotic behavior as $x \to \pm\infty$ to make the transformation

$$R(r) = e^{-x^2/2} v(r). \tag{0.4}$$

This gives the following differential equation for $v(r)$:

$$v'' + \left(\frac{2}{x} - 2x\right) v' + v\left(-3 + \epsilon - \frac{l(l+1)}{x^2}\right) = 0. \tag{0.5}$$

Using the Frobenius method proceed with the rest of the solution.

2. **(Problem 4.7)** Show the following integrals:

$$\int_{-\infty}^{\infty} x e^{-x^2/2} H_n(x) dx = \left\{ \begin{array}{c} 0 \\ \dfrac{\sqrt{2\pi}(n+1)!}{[(n+1)/2]!} \end{array} \right\} \text{for} \left\{ \begin{array}{c} n \text{ even} \\ n \text{ odd} \end{array} \right\}, \tag{0.6}$$

$$\int_{-\infty}^{\infty} e^{-x^2/2} H_n(x) dx = \left\{ \begin{array}{c} \sqrt{2\pi} n!/(n/2)! \\ 0 \end{array} \right\} \text{for} \left\{ \begin{array}{c} n \text{ even} \\ n \text{ odd} \end{array} \right\}. \tag{0.7}$$

Solution:

We show the second one. Use the generating function definition of Hermite polynomials [Eq. (4.25)]:

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{-t^2+2xt}, \tag{0.8}$$

to write

$$\sum_{n=0}^{\infty} \left[\int_{-\infty}^{\infty} e^{-x^2/2} H_n(x) dx\right] \frac{t^n}{n!} = \int_{-\infty}^{\infty} e^{-x^2/2} e^{-t^2+2xt} dx. \tag{0.9}$$

Complete the square to evaluate the integral on the right-hand side as

$$\int_{-\infty}^{\infty} e^{-x^2/2} e^{-t^2+2xt} dx = e^{t^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-2t)^2} dx = \sqrt{2\pi} e^{t^2}. \tag{0.10}$$

We now write Equation (0.9) as

$$\sum_{n=0}^{\infty} \left[\int_{-\infty}^{\infty} e^{-x^2/2} H_n(x) dx\right] \frac{t^n}{n!} = \sqrt{2\pi} e^{t^2}. \tag{0.11}$$

Substitute the power series expansion of e^{t^2}:

$$e^{t^2} = \sum_{k=0}^{\infty} \frac{t^{2k}}{k!}, \tag{0.12}$$

to write

$$\sum_{n=0}^{\infty} \left[\int_{-\infty}^{\infty} e^{-x^2/2} H_n(x) dx \right] \frac{t^n}{n!} = \sqrt{2\pi} \sum_{k=0}^{\infty} \frac{t^{2k}}{k!}. \tag{0.13}$$

Compare equal powers of t to get the desired result.

3. **(Problem 4.10)** Prove that

$$\int_{-\infty}^{\infty} e^{-a^2x^2} H_{2n}(x) dx = \frac{(2n)!}{n!} \frac{\sqrt{\pi}}{a} \left[\frac{1-a^2}{a^2} \right]^n, \tag{0.14}$$

where

$$\operatorname{Re} a^2 > 0 \text{ and } n = 0, 1, 2, \ldots . \tag{0.15}$$

Solution:

Use the generating function definition to write

$$\sum_{n=0}^{\infty} \left[\int_{-\infty}^{\infty} e^{-a^2x^2} H_n(x) dx \right] \frac{t^n}{n!} = \int_{-\infty}^{\infty} e^{-a^2x^2 - t^2 + 2xt} dx. \tag{0.16}$$

Since

$$\int_{-\infty}^{\infty} e^{-a^2x^2 - t^2 + 2xt} dx = \frac{\sqrt{\pi}}{a} \exp \left[\frac{1-a^2}{a^2} t^2 \right], \tag{0.17}$$

write the right-hand side of Equation (0.16) in terms of powers of t as

$$\sum_{n=0}^{\infty} \left[\int_{-\infty}^{\infty} e^{-a^2x^2} H_n(x) dx \right] \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{\sqrt{\pi}}{a} \left(\frac{1-a^2}{a^2} \right)^2 \frac{t^{2n}}{n!}. \tag{0.18}$$

Let $n \to 2n$ on the right-hand side and then compare equal powers of t to get the desired result.

4. **(Problem 4.15)** Derive the following recursion relations:

$$H_{n+1}(x) = 2xH_n(x) - 2nH_{n-1}(x) \tag{0.19}$$

and

$$H_n'(x) = 2nH_{n-1}(x). \tag{0.20}$$

Solution:

For the first one differentiate the generating function with respect to t and then equate powers of t. Similarly, for the second one differentiate the generating function with respect to x.

5. Evaluate the integral

$$I = \int_{-\infty}^{\infty} x e^{-x^2} H_n(x) H_m(x) dx. \tag{0.21}$$

Solution:

Use the recursion relation [Eq. (0.19)]

$$H_n\left(x\right) = \frac{1}{x}\left[\frac{1}{2}H_{n+1}\left(x\right) + nH_{n-1}\left(x\right)\right], \tag{0.22}$$

to write

$$\begin{aligned} I &= \int_{-\infty}^{\infty} x e^{-x^2} \frac{1}{x}\left[\frac{1}{2}H_{n+1}\left(x\right) + nH_{n-1}\left(x\right)\right] H_m(x) dx \\ &= \int_{-\infty}^{\infty} e^{-x^2} \left[\frac{1}{2}H_{n+1}H_m + nH_{n-1}H_m\right] dx. \end{aligned} \tag{0.23}$$

Finally, use the orthogonality relation [Eq. (4.46)]:

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x) dx = \sqrt{\pi} 2^n n! \delta_{nm}, \tag{0.24}$$

to obtain

$$\int_{-\infty}^{\infty} x e^{-x^2} H_n(x) H_m(x) dx = \sqrt{\pi} 2^{n-1} n! \left[2(n+1)\delta_{m,n+1} + \delta_{m,n-1}\right]. \tag{0.25}$$

6. Verify the relation

$$\frac{d^m}{dx^m} H_n(x) = \frac{2^n n!}{(n-m)!} H_{n-m}(x), \;\; m < n. \tag{0.26}$$

Solution:

Using the generating function:

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{-t^2+2xt}, \tag{0.27}$$

we write

$$\sum_{n=0}^{\infty}\left[\frac{d^m}{dx^m}H_n(x)\right]\frac{t^n}{n!} = \frac{d^m}{dx^m}\left[e^{-t^2+2xt}\right]$$
$$= (2t)^m e^{-t^2+2xt}$$
$$= 2^m t^m \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}$$
$$= 2^m \sum_{n=0}^{\infty} H_n(x)\frac{t^{n+m}}{n!}. \tag{0.28}$$

Let $n+m=k$ to get

$$\sum_{n=0}^{\infty}\left[\frac{d^m}{dx^m}H_n(x)\right]\frac{t^n}{n!} = 2^m \sum_{k=m}^{\infty} H_{k-m}(x)\frac{t^k}{(k-m)!}, \tag{0.29}$$

which implies

$$\frac{d^m}{dx^m}H_n(x) = \frac{2^n n!}{(n-m)!}H_{n-m}(x),\ m<n. \tag{0.30}$$

II. Useful Sites

More references and other useful information about Hermite polynomials can be found in the following sites:
`http://en.wikipedia.org/wiki/Hermite_polynomials`,
`http://scienceworld.wolfram.com/biography/Hermite.html`.

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