Supplements to Mathematical Methods in Science and Engineering
http://www.wiley.com/WileyCDA/WileyTitle/productCd-0470041420.html

CHAPTER 16: INTEGRAL TRANSFORMS

For additional discussions and examples on Fourier series, Fourier analysis and integral transforms we recommend *Essentials of Mathematical Methods in Science and Engineering*:

http://www.wiley.com/WileyCDA/WileyTitle/productCd-0470343796.html.

I. Solutions or Hints to Selected Problems:

1. Show that the Fourier sine, $\mathcal{F}_s$, and the Fourier cosine, $\mathcal{F}_c$, transforms satisfy:

$$\mathcal{F}_c\left\{f'(t)\right\} = \omega\mathcal{F}_s\left\{f(t)\right\} - \sqrt{\frac{2}{\pi}}f(0), \tag{0.1}$$

$$\mathcal{F}_s\left\{f'(t)\right\} = -\omega\mathcal{F}_c\left\{f(t)\right\}, \tag{0.2}$$

where

$$\mathcal{F}_s\left\{f(t)\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin\omega t\ dt, \tag{0.3}$$

$$\mathcal{F}_c\left\{f(t)\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos\omega t\ dt \tag{0.4}$$

and $f(t) \to 0$ when $t \to \pm\infty$. Using these results also find

$$\mathcal{F}_c\left\{f''(t)\right\} \text{ and } \mathcal{F}_s\left\{f''(t)\right\}. \tag{0.5}$$

Solution:

Using integration by parts we obtain the first relation:

$$\mathcal{F}_c\left\{f'(t)\right\} = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{df(t)}{dt}\cos\omega t\ dt \tag{0.6}$$

$$= \sqrt{\frac{2}{\pi}}\left[f(t)\cos\omega t|_0^\infty - \int_0^\infty f(t)\frac{d\cos\omega t}{dt}dt\right] \tag{0.7}$$

$$= \sqrt{\frac{2}{\pi}}\left[-f(0) + \omega\int_0^\infty f(t)\sin\omega t\ dt\right] \tag{0.8}$$

$$= \omega\mathcal{F}_s\left\{f(t)\right\} - \sqrt{\frac{2}{\pi}}f(0). \tag{0.9}$$

For the second relation we follow similar steps. For the remaining two relations we obtain

$$\mathcal{F}_c\left\{f''(t)\right\} = -\omega^2\mathcal{F}_c\left\{f(t)\right\} - \sqrt{\frac{2}{\pi}}f'(0) \tag{0.10}$$

and

$$\mathcal{F}_s\left\{f''(t)\right\} = -\omega^2\mathcal{F}_s\left\{f(t)\right\} + \omega\sqrt{\frac{2}{\pi}}f(0). \tag{0.11}$$

2. Using the results established in the first problem, evaluate the Fourier sine transform

$$\mathcal{F}_s\left\{e^{-at}\right\}. \tag{0.12}$$

Solution:

Since

$$f(t) = e^{-at},\ f'(t) = -af(t),\ f''(t) = a^2 f(t), \tag{0.13}$$

we write

$$\mathcal{F}_s\left\{f''(t)\right\} = \mathcal{F}_s\left\{a^2 f(t)\right\} = a^2\mathcal{F}_s\left\{f(t)\right\}. \tag{0.14}$$

We now write the Fourier sine transform of the second derivative of $f(t)$ [Eq. (0.11)] as

$$\mathcal{F}_s\left\{f''(t)\right\} = -\omega^2\mathcal{F}_s\left\{f(t)\right\} + \omega\sqrt{\frac{2}{\pi}}f(0), \tag{0.15}$$

which when combined with equation (0.14) gives

$$\mathcal{F}_s\left\{e^{-at}\right\} = \sqrt{\frac{2}{\pi}}\frac{\omega}{a^2 + w^2}. \tag{0.16}$$

3. Fourier transformations in three dimensions are defined as [Eqs. (16.68) and (16.69)]

$$\Phi(\overrightarrow{k}) = \frac{1}{\sqrt[3]{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^3\overrightarrow{r}\ f(\overrightarrow{r})\ e^{i\overrightarrow{k}\cdot\overrightarrow{r}}, \tag{0.17}$$

$$f(\overrightarrow{r}) = \frac{1}{\sqrt[3]{2\pi}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} d^3\overrightarrow{k}\ \Phi(\overrightarrow{k})\ e^{-i\overrightarrow{k}\cdot\overrightarrow{r}}. \tag{0.18}$$

Write the Fourier transform of a spherically symmetric function, that is,

$$f(\overrightarrow{r}) = f(r). \tag{0.19}$$

Solution:

For spherically symmetric problems we write

$$\overrightarrow{k} \cdot \overrightarrow{r} = kr\cos\theta \tag{0.20}$$

and use the volume element:

$$d^3\overrightarrow{r} = r^2 \sin\theta\ dr d\theta d\phi, \tag{0.21}$$

to write the Fourier transform $\mathcal{F}\{f(\overrightarrow{r})\}$ as

$$\mathcal{F}\{f(\overrightarrow{r})\} = \frac{1}{(2\pi)^{3/2}} \int_0^{2\pi} d\phi \int_0^{\infty} f(r) \left[\int_0^{\pi} e^{-ikr\cos\theta} \sin\theta\ d\theta\right] r^2 dr \tag{0.22}$$

$$= \frac{1}{(2\pi)^{3/2}} \int_0^{\infty} f(r) \left[\int_0^{\pi} \frac{1}{ikr} e^{-ikr\cos\theta}\right]_0^{\pi} r^2 dr \tag{0.23}$$

$$= \sqrt{\frac{2}{\pi}} \frac{1}{k} \int_0^{\infty} f(r) r \sin kr\ dr, \tag{0.24}$$

which is now a one dimensional integral.

4. Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2} dx. \tag{0.25}$$

Solution:

We first find the Fourier transform of a square wave:

$$\Pi(t) = \begin{cases} 1, & |t| \leq 1, \\ 0, & |t| > 1. \end{cases}, \tag{0.26}$$

as

$$g(\omega) = \mathcal{F}\left\{\Pi(t)\right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Pi(t) e^{i\omega t} dt \tag{0.27}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{i\omega t} dt \tag{0.28}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2\sin\omega}{\omega}. \tag{0.29}$$

We also note that

$$\int_{-\infty}^{\infty} |\Pi(t)|^2 \, dt = 2. \tag{0.30}$$

We now write

$$\int_{-\infty}^{\infty} |g(\omega)|^2 \, d\omega = \frac{4}{2\pi} \int_{-\infty}^{\infty} \frac{\sin^2\omega}{\omega^2} d\omega \tag{0.31}$$

and use the Parceval's theorem (16.71):

$$\int_{-\infty}^{\infty} |\Pi(t)|^2 \, dt = \int_{-\infty}^{\infty} |g(\omega)|^2 \, d\omega, \tag{0.32}$$

to write

$$2 = \frac{4}{2\pi} \int_{-\infty}^{\infty} \frac{\sin^2\omega}{\omega^2} d\omega, \tag{0.33}$$

which yields the desired result as

$$\int_{-\infty}^{\infty} \frac{\sin^2\omega}{\omega^2} d\omega = \pi. \tag{0.34}$$

5. Solve the inhomogeneous Helmholtz equation,

$$y'' - k_0^2 y'(t) = f(t), \tag{0.35}$$

with the following boundary conditions:

$$y(t) \to 0, \; f(t) \to 0 \text{ as } t \to \pm\infty. \tag{0.36}$$

Solution:

We first take Fourier transform of the differential equation as

$$\mathcal{F}\left\{y'' - k_0^2 y'(t)\right\} = \mathcal{F}\left\{f(t)\right\}. \tag{0.37}$$

Since Fourier transforms are linear, we can write this as

$$\mathcal{F}\left\{y''(t)\right\} - k_0^2 \mathcal{F}\left\{y(t)\right\} = \mathcal{F}\left\{f(t)\right\}. \tag{0.38}$$

We now utilize the formula [Eq. (16.50)] which gives the transformation of a derivative, to write

$$-(\omega^2 + k_0^2)\mathcal{F}\{y(t)\} = \mathcal{F}\{f(t)\}. \tag{0.39}$$

Assuming that the Fourier transforms $\widehat{y}(\omega)$ and $\widehat{f}(\omega)$ of $y(t)$ and $f(t)$, respectively, exist, we obtain

$$\widehat{y}(\omega) = -\frac{\widehat{f}(\omega)}{(\omega^2 + k_0^2)}. \tag{0.40}$$

This is the Fourier transform of the needed solution. For $y(t)$, we need to find the inverse transform:

$$\begin{aligned} y(t) &= \mathcal{F}^{-1}\{\widehat{y}(\omega)\} && (0.41)\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \widehat{y}(\omega)e^{-i\omega t}d\omega && (0.42)\\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} -\frac{\widehat{f}(\omega)}{(\omega^2 + k_0^2)}e^{-i\omega t}d\omega. && (0.43) \end{aligned}$$

In this expression the inverse Fourier transforms of $\widehat{f}(\omega)$ and $-\frac{1}{(\omega^2+k_0^2)}$ can be written immediately as

$$\mathcal{F}^{-1}\left\{\widehat{f}(\omega)\right\} = f(t) \tag{0.44}$$

and

$$\mathcal{F}^{-1}\left\{-\frac{1}{(\omega^2 + k_0^2)}\right\} = -\frac{1}{2k_0}e^{-k_0|t|}. \tag{0.45}$$

To find the inverse Fourier transform of their product, we utilize the convolution Theorem [Eq. (16.64)]:

$$\int_{-\infty}^{\infty} a(t')b(t-t')dt' = \int_{-\infty}^{\infty} A(\omega)B(\omega)e^{-i\omega t}d\omega, \tag{0.46}$$

where $A(\omega)$ and $B(\omega)$ are the Fourier transforms of two functions, $a(t)$ and $b(t)$, respectively. In Equation [Eq. (0.46)] we take

$$\begin{aligned} B(\omega) &= -\frac{1}{(\omega^2 + k_0^2)}, && (0.47)\\ A(\omega) &= \widehat{f}(\omega), && (0.48) \end{aligned}$$

along with their inverses:

$$\begin{aligned} b(t) &= -\frac{1}{2k_0}e^{-k_0|t|}, && (0.49)\\ a(t) &= f(t), && (0.50) \end{aligned}$$

to obtain the desired solution as

$$-\frac{1}{2k_0}\int_{-\infty}^{\infty} f(t')e^{-k_0|t-t'|}dt' = \int_{-\infty}^{\infty} -\frac{\widehat{f}(\omega)}{(\omega^2+k_0^2)}e^{-i\omega t}d\omega \tag{0.51}$$

$$= y(t). \tag{0.52}$$

This gives the solution in terms of an integral which can be evaluated for a given $f(t)$.

6. **(Extra):** Show the following integral by using the Fourier transforms:

$$\int_0^{\infty} \frac{\sin^3 x}{x}dx = \frac{\pi}{4} \tag{0.53}$$

II. Conventions and Properties of Fourier Transforms

We have defined the Fourier transform as

$$g(\omega) = \mathcal{F}\{f(t)\} = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt, \tag{0.54}$$

where the inverse Fourier transform is defined as

$$f(t) = \mathcal{F}^{-1}\{g(\omega)\} = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} g(\omega)e^{-i\omega t}d\omega. \tag{0.55}$$

In some books the sign of $i\omega t$ in the exponential is reversed. In applications the final result is not affected. For the coefficients of the integrals sometimes the following asymmetric convention is adopted:

$$g(\omega) = \mathcal{F}\{f(t)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(t)e^{i\omega t}dt, \quad f(t) = \mathcal{F}^{-1}\{g(\omega)\} = \int_{-\infty}^{\infty} g(\omega)e^{-i\omega t}d\omega, \tag{0.56}$$

where the factor of $\frac{1}{2\pi}$ can also be taken to be in front of the second integral in Equation (0.56):

$$g(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{i\omega t}dt, \quad f(t) = \mathcal{F}^{-1}\{g(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)e^{-i\omega t}d\omega. \tag{0.57}$$

In spectral analysis instead of the angular frequency ω, we usually prefer to use the frequency, $\nu = \frac{\omega}{2\pi}$, to write

$$g(\nu) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{2\pi i\nu t}dt, \tag{0.58}$$

$$f(t) = \mathcal{F}^{-1}\{g(\nu)\} = \int_{-\infty}^{\infty} g(\nu)e^{-2\pi i\nu t}d\nu. \tag{0.59}$$

Note that the factors in front of the integrals have disappeared all together.

We have already mentioned that both the Fourier transform and its inverse are linear, that is,

$$\mathcal{F}\left\{c_1 f_1(t) + c_2 f_2(t)\right\} = c_1 \mathcal{F}\left\{f_1(t)\right\} + c_2 \mathcal{F}\left\{f_2(t)\right\}, \tag{0.60}$$

$$\mathcal{F}^{-1}\left\{c_1 g_1(\omega) + c_2 g_2(\omega)\right\} = c_1 \mathcal{F}^{-1}\left\{g_1(\omega)\right\} + c_2 \mathcal{F}^{-1}\left\{g_2(\omega)\right\}, \tag{0.61}$$

where c_1 and c_2 are in general complex constants. In addition to linearity, the following properties, which can be proven by direct substitution, are very useful:

Shifting:

If the time parameter t is shifted by a positive real constant, a, we get

$$\mathcal{F}\left\{f(t-a)\right\} = e^{i\omega a} g(\omega). \tag{0.62}$$

Note that shifting changes only the phase, not the magnitude of the transformation. Similarly, if the frequency is shifted by a, we obtain

$$\mathcal{F}^{-1}\left\{g(\omega-a)\right\} = e^{-iat} f(t). \tag{0.63}$$

Scaling:

If we rescale the time variable as

$$t \to at, \;\; a > 0, \tag{0.64}$$

we get

$$\mathcal{F}\left\{f(at)\right\} = \frac{1}{a} g(\frac{\omega}{a}). \tag{0.65}$$

Rescaling ω as $\omega \to a\omega$ gives

$$\mathcal{F}^{-1}\left\{g(a\omega)\right\} = \frac{1}{a} f(\frac{t}{a}). \tag{0.66}$$

Transform of an integral:

Given the integral

$$\int_{-\infty}^{t} f(t')dt', \tag{0.67}$$

we can write its Fourier transform as

$$\mathcal{F}\left\{\int_{-\infty}^{t} f(t')dt'\right\} = -\frac{1}{i\omega}\mathcal{F}\left\{f(t)\right\}. \tag{0.68}$$

Modulation:

For a given real number, ω_0, we have [Eq. (0.62)]

$$\mathcal{F}\left\{f(t)e^{-i\omega_0 t}\right\} = g(\omega - \omega_0),$$
$$\mathcal{F}\left\{f(t)e^{i\omega_0 t}\right\} = g(\omega + \omega_0), \tag{0.69}$$

which allows us to write

$$\mathcal{F}\left\{f(t)\cos(\omega_0 t)\right\} = \frac{1}{2}g(\omega - \omega_0) + \frac{1}{2}g(\omega + \omega_0), \tag{0.70}$$

$$\mathcal{F}\left\{f(t)\sin(\omega_0 t)\right\} = \frac{1}{2i}g(\omega + \omega_0) - \frac{1}{2i}g(\omega - \omega_0). \tag{0.71}$$

These are called the **modulation relations**.

III. Discrete Fourier Transform

Fourier series, also called the trigonometric Fourier series, are extremely useful in analyzing a given signal, $f(x)$, in terms of sinusoidal waves. In exponential form the Fourier series are given as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi i n x/l},\ 0 < x < l, \tag{0.72}$$

where the expansion coefficients, c_n, also called the Fourier coefficients, are given as

$$c_n = \frac{1}{l}\int_0^l f(x)e^{-2\pi i n x/l}dx. \tag{0.73}$$

This series can either be used to represent a piecewise continuous function in the bounded interval $[0, l]$, or a periodic function with the period l. From the above equations everything looks straight forward. Given a signal, $f = f(x)$, we first evaluate the definite integral in Equation (0.73) to find the Fourier coefficients, c_n, which are then used to construct the Fourier series in Equation (0.72). This gives us the composition of the signal in terms of its sinusoidal components. However, in realistic situations there are many difficulties. First of all, in most cases the input signal, f, can only be given as a finite sequence numbers with N terms, $f = \{f_1, f_2, \ldots, f_N\}$, which may not always be possible to express in terms of a smooth function. Besides, even if we could find a smooth function, $f(x)$, to represent the data, the definite integral in Equation (0.73) may not be possible to evaluate analytically. In any case, to crunch out a solution we need to develop a numerical theory of Fourier analysis so that we can feed the problem into a digital computer.

We now divide the interval $[0, l]$ by introducing N evenly spaced points, x_k, as

$$x_k = \frac{kl}{N},\ 0 \le k \le N-1, \tag{0.74}$$

where each subinterval has the length

$$\Delta x_k = \frac{l}{N}\Delta k \tag{0.75}$$
$$= \frac{l}{N}. \tag{0.76}$$

We approximate the integral in Equation (0.73) by the Riemann sum:

$$\widetilde{f}_n = \frac{1}{l}\sum_{k=0}^{N-1} f(x_k)e^{-2\pi i n x_k/l}\Delta x_k \tag{0.77}$$
$$= \frac{1}{l}\sum_{k=0}^{N-1} f(\frac{kl}{N})e^{-2\pi i k n/N}\Delta x_k, \tag{0.78}$$

where we have written the left hand side as $\widetilde{f}_n$. In general, we can define the **discrete Fourier transform** of any sequence of N terms,

$$f = \{f_j\}$$
$$= \{f_0, f_1, \dots, f_{N-1}\}, \tag{0.79}$$

as the set

$$\widetilde{f} = \left\{\widetilde{f}_j\right\}$$
$$= \left\{\widetilde{f}_0, \widetilde{f}_1, \dots, \widetilde{f}_{N-1}\right\}, \tag{0.80}$$

where

$$\widetilde{f}_j = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-2\pi i k j/N}, \; j = 0, 1, \dots, N-1. \tag{0.81}$$

The **inverse discrete Fourier transform** can be written similarly as

$$f_k = \sum_{j=0}^{N-1} \widetilde{f}_j e^{2\pi i k j/N}, \; k = 0, 1, \dots, N-1. \tag{0.82}$$

To prove the inverse discrete Fourier transform we substitute $\widetilde{f}_j$ [Eq. (0.81)] into Equation (0.82):

$$f_k = \sum_{j=0}^{N-1}\left[\frac{1}{N}\sum_{l=0}^{N-1} f_l e^{-2\pi i l j/N}\right]e^{2\pi i k j/N} \tag{0.83}$$
$$= \frac{1}{N}\sum_{j=0}^{N-1}\sum_{l=0}^{N-1} f_l e^{2\pi i (k-l)j/N} \tag{0.84}$$
$$= \frac{1}{N}\sum_{l=0}^{N-1} f_l \left[\sum_{j=0}^{N-1}\left(e^{2\pi i(k-l)/N}\right)^j\right]. \tag{0.85}$$

For the inverse discrete transform to be true we need

$$f_k = \sum_{l=0}^{N-1} f_l \left[\frac{1}{N} \sum_{j=0}^{N-1} \left(e^{2\pi i(k-l)/N} \right)^j \right] \tag{0.86}$$

$$= \sum_{k=0}^{N-1} f_l \delta_{lk}, \tag{0.87}$$

where

$$\delta_{lk} = \frac{1}{N} \sum_{j=0}^{N-1} \left(e^{2\pi i(k-l)/N} \right)^j . \tag{0.88}$$

When $l = k$, we have Equation (0.88) becomes

$$\delta_{kk} = \frac{1}{N} \sum_{j=0}^{N-1} (1)^j \tag{0.89}$$

Since the sum in the above equation is the sum of N 1's, we obtain the desired result, that is,

$$\delta_{lk} = 1, \ l = k. \tag{0.90}$$

When $k \neq l$, since k and l are integers, $k - l$ is also an integer satisfying $|k - l| < 1$, hence $\left| e^{2\pi i(k-l)/N} \right| < 1$, thus we can use the geometric sum formula:

$$\sum_{n=0}^{M} x^n = \frac{x^{M+1} - 1}{x - 1}, \ \ |x| < 1, \tag{0.91}$$

to write

$$\delta_{lk} = \frac{1}{N} \sum_{j=0}^{N-1} \frac{e^{2\pi i(k-l)N/N} - 1}{e^{2\pi i(k-l)/N} - 1}, \ l \neq k, \tag{0.92}$$

$$= \frac{1}{N} \sum_{j=0}^{N-1} \frac{e^{2\pi i(k-l)} - 1}{e^{2\pi i(k-l)/N} - 1} \tag{0.93}$$

$$= 0, \tag{0.94}$$

thus proving the inverse discrete Fourier transform formula.

In the discrete Fourier transform, the set

$$\{f_0, f_1, \dots, f_{N-1}\} \tag{0.95}$$

defines a function, $f(x)$,whose domain is the set of integers

$$\{0, 1, 2, \dots, N-1\}, \tag{0.96}$$

and the range of which is

$$\{f(0) = f_0, f(1) = f_1, \dots, f(N-1) = f_{N-1}\}. \tag{0.97}$$

In other words, in the discrete Fourier transform we either describe a function in terms of its discretization:

$$f(x) = \left\{ f(0.\frac{l}{N}), f(1.\frac{l}{N}), \dots, f((N-1).\frac{l}{N}) \right\}, \tag{0.98}$$

or deal with phenomena that can only be described by a sequence of numbers:

$$f(j) = \{f_j\} = \{f_0, f_1, \dots, f_{N-1}\}. \tag{0.99}$$

The discrete Fourier transform can also be viewed as an operation that maps the set of numbers,

$$\{f_0, f_1, \dots, f_{N-1}\}, \tag{0.100}$$

onto the set

$$\left\{ \widetilde{f}_0, \widetilde{f}_1, \dots, \widetilde{f}_{N-1} \right\}, \tag{0.101}$$

which is composed of the transformed variables.

For example, consider the set composed of two numbers $(N = 2)$:

$$f(i) = \{3, 1\}, \ i = 0, 1. \tag{0.102}$$

The discrete Fourier transform of this set can be found as

$$\widetilde{f}(0) = \frac{1}{2}\left[f(0)e^{-2\pi i 0.0/2} + f(1)e^{-2\pi i 1.0/2} \right] = 2, \tag{0.103}$$

$$\widetilde{f}(1) = \frac{1}{2}\left[f(0)e^{-2\pi i 0.1/2} + f(1)e^{-2\pi i 1.1/2} \right] = 1, \tag{0.104}$$

that is,

$$\widetilde{f}(j) = \{2, 1\}, \ j = 0, 1. \tag{0.105}$$

Using the inverse discrete Fourier transform [Eq. (0.82)] we can recover the original set as

$$f(0) = \left[\widetilde{f}(0)e^{2\pi i 0.0/2} + \widetilde{f}(1)e^{2\pi i 0.1/2} \right] = 3, \tag{0.106}$$

$$f(1) = \left[\widetilde{f}(0)e^{2\pi i 1.0/2} + \widetilde{f}(1)e^{2\pi i 1.1/2} \right] = 1. \tag{0.107}$$

This result is true in general for arbitrary N and it is usually quoted as the **reciprocity theorem**. In other words, the discrete Fourier transform possesses a unique inverse.

With the discrete Fourier transform, we now have an algorithm that can be handled by a computer. If we store the numbers $f(j), j = 0, 1, \ldots, N-1$, and $e^{-2\pi ikj/N}$, $k = 0, 1, \ldots, N-1$, into two separate registrars, R_1 and R_2, as

$$R_1 = \boxed{\; f_1 \;|\; f_2 \;|\; \cdots \;|\; f_{N-1} \;} \tag{0.108}$$

and

$$R_2 = \boxed{\; e^{-2\pi i 0 j/N} \;|\; e^{-2\pi i 1 j/N} \;|\; \cdots \;|\; e^{-2\pi i (N-1) j/N} \;}, \tag{0.109}$$

so that they can be recalled as needed, we can find how many basic operations, additions, multiplications and divisions, that a computer has to do to compute a discrete Fourier transform, that is, to completely fill a third register R_3 with the Fourier transformed values:

$$R_3 = \boxed{\; \widetilde{f}_1 \;|\; \widetilde{f}_2 \;|\; \cdots \;|\; \widetilde{f}_{N-1} \;}.$$

From Equation (0.81) it is seen that to find the jth element, $\widetilde{f}_j$, we recall the kth entry, f_k,of R_1 and then multiply it with the kth entry, $e^{-2\pi ikj/N}$, of the second registrar R_2. This establishes only one of the terms in the sum [Eq. (0.81)]. This means one multiplication for each term in the sum. Since there are N terms in the sum, the computer performs N multiplications. Then we add these N terms, which requires $N-1$ additions. Finally, we divide the result by N, that is, one more operation. All together, to evaluate the jth term, we need

$$N + (N-1) + 1 = 2N \tag{0.110}$$

basic operations. There are N such terms to be calculated, hence the computer has to perform

$$2N^2 \tag{0.111}$$

basic operations to find the discrete Fourier transform of a set with N terms. Since each basic operation takes a certain amount of time for a given computer, this is also a measure of how fast the computation will be carried out.

IV. Fast Fourier Transform

We start with a sequence of N terms, $\{f(j)\}$, with the discrete Fourier transform, $\{\widetilde{f}(j)\}$, where $j = 0, 1, \ldots, N-1$. Let us assume that N is even so that we can write $\frac{N}{2} = M$, where M is an integer. We now split $\{f(j)\}$ into two new sequences

$$\{f_1(j)\} = \{f(2j)\} \tag{0.112}$$

and

$$\{f_2(j)\} = \{f(2j+1)\}, \tag{0.113}$$

where $j = 0, 1, \ldots, M-1$. Note that both $\{f_1(j)\}$ and $\{f_2(j)\}$ are periodic sequences with the period M. We can now use Equation (0.81) to write their discrete Fourier transforms as

$$\{\widetilde{f}_1(j)\} = \frac{1}{M}\sum_{k=0}^{M-1} f_1(k)e^{-2\pi ikj/M},\ j = 0, 1, \ldots, M-1, \tag{0.114}$$

$$\{\widetilde{f}_2(j)\} = \frac{1}{M}\sum_{k=0}^{M-1} f_2(k)e^{-2\pi ikj/M},\ j = 0, 1, \ldots, M-1. \tag{0.115}$$

We now return to the discrete Fourier transform of the full set $\{f(j)\}$ and write

$$\{\widetilde{f}(j)\} = \frac{1}{N}\sum_{k=0}^{N-1} f(k)e^{-2\pi ikj/N}, \tag{0.116}$$

which can be rearranged as

$$\{\widetilde{f}(j)\} = \frac{1}{N}\sum_{k=0}^{M-1} f(2k)e^{-2\pi i(2k)j/N} + \frac{1}{N}\sum_{k=0}^{M-1} f(2k+1)e^{-2\pi i(2k+1)j/N}. \tag{0.117}$$

Using the relations

$$e^{-2\pi i(2k)j/N} = e^{-2\pi ikj/M}, \tag{0.118}$$

$$e^{-2\pi i(2k)j/N} = e^{-2\pi ikj/M}e^{-2\pi ij/N}, \tag{0.119}$$

which the reader should show, we write Equation (0.117) as

$$\{\widetilde{f}(j)\} = \frac{1}{N}\sum_{k=0}^{M-1} f_1(k)e^{-2\pi ikj/M} + \frac{e^{-2\pi ij/N}}{N}\sum_{k=0}^{M-1} f_2(k)e^{-2\pi ikj/M}, \tag{0.120}$$

where $j = 0, 1, \ldots, N-1$. This is nothing but

$$\{\widetilde{f}(j)\} = \frac{\{\widetilde{f}_1(j)\}}{2} + \frac{e^{-2\pi ij/N}\{\widetilde{f}_2(j)\}}{2},\ j = 0, 1, \ldots, N-1. \tag{0.121}$$

Since both $\{\widetilde{f}_1(j)\}$ and $\{\widetilde{f}_2(j)\}$ are periodic with the period M, that is,

$$\{\widetilde{f}_{1\mathrm{or}2}(j+M)\} = \{\widetilde{f}_{1\mathrm{or}2}(j)\}, \tag{0.122}$$

we have extended the range of the index j to $N-1$.

We have seen that the sum in Equation (0.81) requires $2N^2$ basic operations to yield the discrete Fourier transform $\{\widetilde{f}(j)\}$. All we have done in Equation (0.121) is to split the original sum into two parts. Let us see what advantage comes out of this. In order to compute the discrete Fourier transform, $\{\widetilde{f}(j)\}$, via the rearranged expression [Eq. (0.121)], we first have to construct the transforms $\{\widetilde{f}_1(j)\}$ and $\{\widetilde{f}_2(j)\}$, each of which requires $2M^2$ basic operations. Next, we need M multiplications to establish the product of the elements of $\{\widetilde{f}_2(j)\}$ with $e^{-2\pi ij/N}$, which will be followed by the M additions of the elements of the sets $\{\widetilde{f}_1(j)\}$ and $e^{-2\pi ij/N}\{\widetilde{f}_2(j)\}$, each of which has M elements. Finally, each element of the sum, $\{\widetilde{f}_1(j)\}+e^{-2\pi ij/N}\{\widetilde{f}_2(j)\}$ has to be divided by 2, that is, N divisions to yield the final result:

$$\{\widetilde{f}(j)\} = \frac{1}{2}\left[\{\widetilde{f}_1(j)\} + e^{-2\pi ij/N}\{\widetilde{f}_2(j)\}\right],\ j=0,1,\ldots,N-1. \tag{0.123}$$

All together, this means

$$2M^2 + 2M^2 + M + M + N = N^2 + 2N \tag{0.124}$$

operations, where we have substituted $N = M/2$ in the last step.

In summary, calculating $\{\widetilde{f}(j)\}$, $j = 0,1,\ldots,N-1$, directly requires $2N^2$ basic operations, while the new approach, granted that N is even, requires N^2+2N operations. The fractional reduction in the number of operations is

$$\frac{N^2+2N}{2N^2} = \frac{1}{2} + \frac{1}{N}, \tag{0.125}$$

which approaches to $\frac{1}{2}$ as N gets very large. Since each operation takes a certain time in a computer, a reduction in the number of operations by half implies a significant reduction in the operation time of the computer.

Wait! we can do even better with this *divide and conquer* strategy. If N is divisible by 4, we can further subdivide the sequences $\{f_1(j)\}$ and $\{f_2(j)\}$ into 4 new sequences with $M/2$ terms each. This will reduce the number of operations further. In fact, it can be shown that when N is divisible by 2^p, $p>0$ integer, we can reduce the number of elementary operations to

$$(4p-1)N = 4N\log_2 N - N. \tag{0.126}$$

For large N, compared to $2N^2$, this is remarkably small, hence will result in significant reduction of the running time of our computer. In cases where our sequence do not have the desired number of terms, we can always add sufficient number of zeros to match the required number.

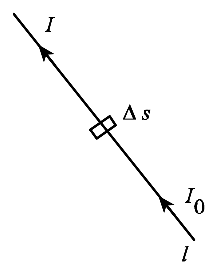

Fig. 0.1 A narrow beam going through a homogeneous material of thickness Δs.

This procedure which tremendously shortens the number of operations needed to compute a discrete Fourier transform was first introduced by Tukey and Cooley in 1965. It is now called the **fast Fourier transform** and it is considered to be one of the most significant contributions to the field of numerical analysis.

V. Radon Transforms

Radon transforms were introduced by an Austrian mathematician, Johann Radon, in 1917. They are extremely useful in medical technology and establish the mathematical foundations of computational axial tomography, that is, CAT scanning. Radon transforms are also very useful in electron microscopy and reflection seismology.

To introduce the basic properties of the two-dimensional Radon transforms, consider a narrow beam of X-ray travelling along a straight line (Fig. 0.1). As the beam passes through a homogeneous material of length Δs, the initial intensity, I_0, will decrease exponentially according to the formula

$$I = I_0 e^{-\alpha\rho\Delta s}, \tag{0.127}$$

where, ρ is the linear density along the direction of propagation and α is a positive constant depending on other physical parameters of the medium. If the beam is going through a series of parallel layers described by α_i, ρ_i, and Δs_i, where the index, $i = 1, 2, \ldots, n$, denotes the ith layer, we can write the final intensity as

$$I = I_0 e^{-[\alpha_1\rho_1\Delta s_1 + \alpha_2\rho_2\Delta s_2 + \cdots + \alpha_n\rho_n\Delta s_n]}. \tag{0.128}$$

In the continuum limit we can write this as

$$I = I_0 e^{-\int_l \alpha(\overrightarrow{x})\rho(\overrightarrow{x})ds_{\overrightarrow{x}}}, \tag{0.129}$$

where $\overrightarrow{x} = (x, y)$ is a point on the ray l, and

$$\int_l \alpha(\overrightarrow{x})\rho(\overrightarrow{x})ds_{\overrightarrow{x}} \tag{0.130}$$

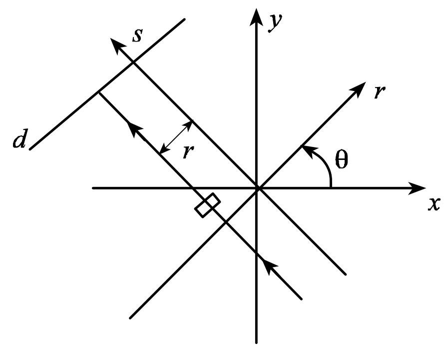

Fig. 0.2 Reference axes s and r, projection angle θ and the detector plane d.

is a line integral taken over a straight line representing the path of the X-ray. We usually write

$$f(x, y) = \alpha(\overrightarrow{x})\rho(\overrightarrow{x}), \tag{0.131}$$

where $f(x, y)$ represents the attenuation coefficient of the object, hence Equation (0.129) becomes

$$-\ln\left(\frac{I}{I_0}\right) = \int_l f(x, y) ds_l. \tag{0.132}$$

The line integral on the right-hand side is called the **Radon transform** of $f(x, y)$. Along the path of the X-ray, which is a straight line with the equation $y_l = y_l(x)$, $f(x, y_l(x))$ represents the attenuation coefficient along the path of the X-ray.

The method used in the first scanners was to use a system of parallel lines that represents the X rays that scan a certain slice of a three dimensional object, where $f(x, y)$ represents the attenuation coefficient of the slice. For a mathematical description of the problem, we parametrize the parallel rays in terms of their perpendicular distances to a reference line, s, and the projection angle θ (Fig. 0.2). Now the scanning data consists of a series of Radon transforms of the attenuation coefficient, $f(x, y)$, projected onto the plane of the detector (Fig. 0.3). The **projection-slice theorem** says that given an infinite number of one-dimensional projections of an object taken from infinitely many directions, one could perfectly reconstruct the original object, that is, $f(x, y)$.

The Radon transform for a family of parallel lines, l, is shown as

$$R_2[f](l) = \int_l f(x, y_l) ds_l, \tag{0.133}$$

where the subscript 2 indicates that this is a two-dimensional Radon transform and $R_2[f]$ is a function of lines. In general, $f(x, y)$ is a continuous function

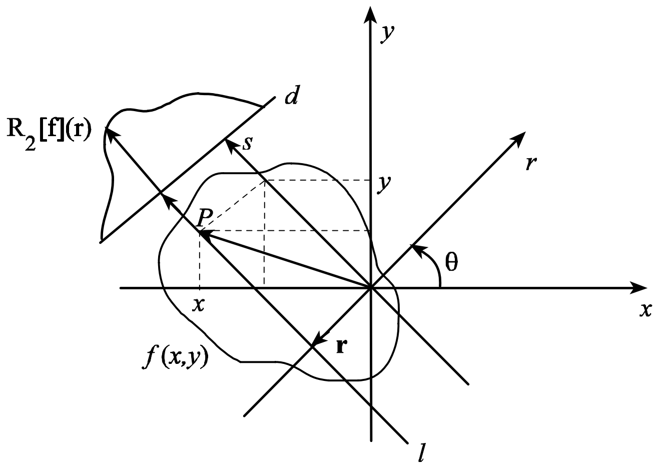

Fig. 0.3 Projection of $f(x, y)$ onto the detector surface d.

on the plane that vanishes outside a finite region. For a given ray, l, we parametrize a point on the ray as

$$x = s\cos(\pi/2 + \theta) + r\cos(\pi + \theta), \tag{0.134}$$

$$y = s\sin(\pi/2 + \theta) + r\sin(\pi + \theta), \tag{0.135}$$

or as

$$x = -s\sin\theta - r\cos\theta, \tag{0.136}$$

$$y = s\cos\theta - r\sin\theta. \tag{0.137}$$

Hence, we can write Equation (0.133) as

$$R_2[f](\theta, r) = \int_{-\infty}^{\infty} f(-s\sin\theta - r\cos\theta, s\cos\theta - r\sin\theta)ds. \tag{0.138}$$

Note that on a given ray, that is, a straight line in the family of parallel lines, r is fixed and s is the variable.

To find the desired quantity that represents the physical characteristics of the object, $f(x, y)$, we need to find the inverse Radon transform. This corresponds to integrating the Radon transform at (x, y) for all angles:

$$f(x, y) = \int_{0}^{2\pi} R_2[f](\theta, -x\cos\theta + y\sin\theta)d\theta, \tag{0.139}$$

where using Figure (0.3) we have substituted

$$r = -x\cos\theta + y\sin\theta, \tag{0.140}$$

$$|\overrightarrow{x}| = x, \ \ |\overrightarrow{y}| = x. \tag{0.141}$$

This method of inversion is proven to be rather noisy and unstable with respect to noisy data, hence in applications an efficient algorithm in terms of its dicretized version, called the filtered back-projection, is preferred. A lot of research has been done in improving the performance of CAT scanners and improving the practical means of inverting Radon transforms. Radon transforms can also be defined in dimensions higher than two (Walker).

V. Additional References and Useful Links

For an interesting application of the fast Fourier transforms to financial mathematics, we recommend the paper entitled *Option Pricing and Fast Fourier Transform* by Peter Carr and Dilip Madan (Journal of Computational Finance, Summer 1999, 2, no 4, pp. 61-73), where more references to papers on applications of Fourier transforms to determine option prices can be found. A pdf file of this article can be found in Prof. Carr's website:

`http://www.math.nyu.edu/research/carrp/research.html`.

Other useful information and links to relevant sites on integral transforms can be found in the following sites:

`http://www.intmath.com/Laplace-transformation/Intro.php`,
`http://en.wikipedia.org/wiki/Integral_transform`,
`http://mathworld.wolfram.com/IntegralTransform.html`,
`http://eqworld.ipmnet.ru/en/auxiliary/aux-inttrans.htm`,
`http://en.wikipedia.org/wiki/Filtered_back_projection#Filtered_back-projection`,
`http://en.wikipedia.org/wiki/Tomographic_reconstruction`,
`http://mathworld.wolfram.com/RadonTransform.html`.

We also recommend the following books:

Stade, E., *Fourier Analysis*, Wiley, Hoboken, NJ, 2005.

Tang, K.T., *Mathematical Methods for Engineers and Scientists 3*, Springer, Berlin, 2007.

Walker, J.S., *Fourier Analysis*, Oxford University Press, New York, 1988.

Weaver, H.J., *Applications of Discrete and Continuous Fourier Analysis*, Wiley, New York, 1983.

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