SUPPLEMENTS TO MATHEMATICAL METHODS IN SCIENCE AND ENGINEERING
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CHAPTER 14: FRACTIONAL DERIVATIVES and INTEGRALS: DIFFERINTEGRALS

I. Caputo Derivative **(Problem 14.7)**

Laplace transform of a differintegral is given as [Eq. (14.222)]

$$\begin{aligned} \pounds\left\{\frac{d^q f(t)}{dt^q}\right\} &= s^q\ \widetilde{f}(s) - \sum_{k=0}^{n-1} s^k \frac{d^{q-1-k} f}{dx^{q-1-k}}(0) \\ &= s^q\ \widetilde{f}(s) - f^{(q-1)}(0) - sf^{(q-2)}(0) - \cdots - s^{n-1} f^{(q-n)}(0), \end{aligned} \tag{0.1}$$

where n is an integer satisfying $n-1 < q \leq n$. For $0 < q < 1$, we take $n = 1$, thus obtaining

$$\pounds\left\{\frac{d^q f(t)}{dt^q}\right\} = s^q\ \widetilde{f}(s) - f^{(q-1)}(0) - sf^{(q-2)}(0) - \cdots - f^{(q-1)}(0). \tag{0.2}$$

Due to the difficulty in imposing boundary conditions with fractional derivatives, Caputo defined the Laplace transform for

$$0 < q < 1 \tag{0.3}$$

as

$$\begin{aligned} \pounds\left\{\frac{d^q f(t)}{dt^q}\right\} &= s^q\ \widetilde{f}(s) - s^{q-1} f(0), \\ &= s^{q-1}\ \left(s\widetilde{f}(s) - f(0)\right), \end{aligned} \tag{0.4}$$

the inverse of which gives

$$\frac{d^q f(t)}{dt^q} = \pounds^{-1}\left\{s^{q-1}\ \left(s\widetilde{f}(s) - f(0)\right)\right\}. \tag{0.5}$$

Using the convolution theorem [Eq. (16.156)]:

$$\int_0^t f(u)g(t-u)du = \pounds^{-1}\left\{F(s)G(s)\right\}, \tag{0.6}$$

with the definitions

$$F(s) = s\widetilde{f}(s) - f(0), \tag{0.7}$$
$$G(s) = s^{q-1}, \tag{0.8}$$

yields the fractional derivative known as the **Caputo derivative**:

$$\left[\frac{d^q f(t)}{dt^q}\right]_C = \frac{1}{\Gamma(1-q)}\int_0^t \left(\frac{df(\tau)}{d\tau}\right)\frac{d\tau}{(t-\tau)^q},\ 0<q<1, \tag{0.9}$$

which was used by him to model dissipation effects in linear viscosity.

II. Caputo Derivative and the Riemann-Liouville Derivative

We now write the Riemann-Liouville derivative [Eq. (14.71)] for $0 < q < 1$ as

$$\left[\frac{d^{q+1} f(t)}{dt^{q+1}}\right]_{R-L} = \frac{d^n}{dt^n}\left[\frac{1}{\Gamma(n-q-1)}\int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{q+1-n+1}}\right], \tag{0.10}$$

where n is a positive integer satisfying $n-q-1>0$. Choosing $n=2$ yields

$$\left[\frac{d^{q+1} f(t)}{dt^{q+1}}\right]_{R-L} = \frac{d^2}{dt^2}\left[\frac{1}{\Gamma(1-q)}\int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{q}}\right]. \tag{0.11}$$

Similarly, we write

$$\left[\frac{d^{1+q} f(t)}{dt^{1+q}}\right]_{R-L} = \frac{d^n}{dt^n}\left[\frac{1}{\Gamma(n-1-q)}\int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{1+q-n+1}}\right] \tag{0.12}$$

and choose $n=2$:

$$\left[\frac{d^{1+q} f(t)}{dt^{1+q}}\right]_{R-L} = \frac{d^2}{dt^2}\left[\frac{1}{\Gamma(1-q)}\int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{q}}\right], \tag{0.13}$$

thus verifying the relation

$$\left[\frac{d^{q+1} f(t)}{dt^{q+1}}\right]_{R-L} = \left[\frac{d^{1+q} f(t)}{dt^{1+q}}\right]_{R-L}. \tag{0.14}$$

Returning to the Caputo derivative [Eq. (0.9)], we write

$$\frac{d}{dt}\left[\frac{d^q f(t)}{dt^q}\right]_C = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t \frac{df(\tau)}{d\tau}\frac{d\tau}{(t-\tau)^q}. \tag{0.15}$$

As in the Riemann-Liouville and Grünwald definitions, we impose the condition [Eqs. (14.40) and (14.67)]

$$\frac{d}{dt}\left[\frac{d^q f(t)}{dt^q}\right] = \frac{d^{1+q} f(t)}{dt^{1+q}}, \tag{0.16}$$

to get

$$\left[\frac{d^{1+q} f(t)}{dt^{1+q}}\right]_C = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^t \frac{df(\tau)}{d\tau}\frac{d\tau}{(t-\tau)^q}. \tag{0.17}$$

Definition of the Riemann-Liouville derivative and Equation (0.14) allows us to write

$$\left[\frac{d^{1+q} f(t)}{dt^{1+q}}\right]_C = \left[\frac{d^q}{dt^q}\left(\frac{df(t)}{dt}\right)\right]_{R-L}. \tag{0.18}$$

Using the composition rule [Eq. (14.158)] of differintegrals:

$$d^q d^Q f = d^{q+Q} f - d^{q+Q}[f - d^{-Q} d^Q f], \tag{0.19}$$

we write the right-hand side of Equation (0.18) as

$$\left[\frac{d^q}{dt^q}\left(\frac{df(t)}{dt}\right)\right]_{R-L} = \left[\frac{d^{q+1} f(t)}{dt^{q+1}}\right]_{R-L} - \left[\frac{d^{q+1}}{dt^{q+1}}\right]_{R-L}\left[f(t) - \frac{d^{-1}}{dt^{-1}}\frac{d}{dt}f(t)\right]. \tag{0.20}$$

Also using Equation (14.152):

$$\frac{d^{-n} f^{(N)}(t)}{[d(t-a)]^{-n}} = f^{(N-n)}(t) - \sum_{k=0}^{n-1}\frac{[t-a]^k}{k!} f^{(N+k-n)}(a), \tag{0.21}$$

with

$$n = 1, \tag{0.22}$$
$$N = 1, \tag{0.23}$$
$$a = 0, \tag{0.24}$$

we have

$$d^{-1} d^1 f(t) = f(t) - f^{(1+0-1)}(0) \tag{0.25}$$
$$= f(t) - f(0). \tag{0.26}$$

Thus,

$$\left[\frac{d^q}{dt^q}\left(\frac{df(t)}{dt}\right)\right]_{R-L} = \left[\frac{d^{q+1}f(t)}{dt^{q+1}}\right]_{R-L} - \left[\frac{d^{q+1}}{dt^{q+1}}\right]_{R-L}[f(t)-f(t)+f(0)] \tag{0.27}$$

$$= \left[\frac{d^{q+1}f(t)}{dt^{q+1}}\right]_{R-L} - \left[\frac{d^{q+1}}{dt^{q+1}}\right]_{R-L} f(0). \tag{0.28}$$

Using this in Equation (0.18) we write

$$\left[\frac{d^{1+q}f(t)}{dt^{1+q}}\right]_{C} = \left[\frac{d^{q+1}f(t)}{dt^{q+1}}\right]_{R-L} - \left[\frac{d^{q+1}}{dt^{q+1}}\right]_{R-L} f(0). \tag{0.29}$$

Also using Equation (0.14):

$$\left[\frac{d^{q+1}f(t)}{dt^{q+1}}\right]_{R-L} = \left[\frac{d^{1+q}f(t)}{dt^{1+q}}\right]_{R-L}, \tag{0.30}$$

and the Riemann-Liouville derivative of a constant [Eq. (14.190)]:

$$\frac{d^{q+1}}{dt^{q+1}}f(0) = \frac{t^{-q-1}f(0)}{\Gamma(-q)}, \tag{0.31}$$

we finally obtain the relation between the Riemann-Liouville derivative and the Caputo derivative as

$$\left[\frac{d^q f(t)}{dt^q}\right]_{C} = \left[\frac{d^q f(t)}{dt^q}\right]_{R-L} - \frac{t^{-q}f(0)}{\Gamma(1-q)},\ 0 < q < 1. \tag{0.32}$$

From the above equation, Caputo and the Riemann-Liouville derivatives agree when $f(0) = 0$. Furthermore, the Caputo derivative of a constant, C_0, is zero:

$$\left[\frac{d^q C_0}{dt^q}\right]_{C} = \left[\frac{d^q C_0}{dt^q}\right]_{R-L} - \frac{t^{-q}C_0}{\Gamma(1-q)}$$

$$= \frac{t^{-q}C_0}{\Gamma(1-q)} - \frac{t^{-q}C_0}{\Gamma(1-q)} = 0. \tag{0.33}$$

To display the clear distinction between the two definitions of fractional derivatives, we use the Riemann-Liouville definition of fractional integrals [Eq. (14.70)] to introduce the fractional integral operator ${}_0\mathbf{I}_t^q$:

$${}_0\mathbf{I}_t^q[f(t)] = \frac{1}{\Gamma(q)}\int_0^t \frac{f(\tau)d\tau}{(t-\tau)^{1-q}},\ q > 0, \tag{0.34}$$

which allows us to define the **Riemann-Liouville** and the **Caputo derivatives** of arbitrary order, $q > 0$, respectively, as

$$\left[\frac{d^q f(t)}{dt^q}\right]_{R-L} = \frac{d^n}{dt^n}\left({}_0\mathbf{I}_t^{n-q}[f(t)]\right),\ n > q, \tag{0.35}$$

$$\left[\frac{d^q f(t)}{dt^q}\right]_{C} = {}_0\mathbf{I}_t^{n-q}\left[\frac{d^n}{dt^n}f(t)\right],\ n > q, \tag{0.36}$$

where n is the smallest integer greater than q, that is, $n-1<q<n$. Notice how the order of the $\frac{d^n}{dt^n}$ and the ${}_0\mathbf{I}_t^{n-q}$ operators reverses. We can also write the above equations as

$$ {}_0^{R-L}\mathbf{D}_t^q f(t) = \frac{d^n}{dt^n}\left({}_0\mathbf{I}_t^{n-q}[f(t)]\right), \tag{0.37}$$

$$ {}_0^{C}\mathbf{D}_t^q f(t) = {}_0\mathbf{I}_t^{n-q}\left[\frac{d^n}{dt^n}f(t)\right]. \tag{0.38}$$

Taking the Laplace transform of these derivatives yields, respectively,

$$ \pounds\left\{{}_0^{R-L}\mathbf{D}_t^q f(t)\right\} = s^q\widetilde{f}(s) - \sum_{k=0}^{n-1} s^k \left({}_0^{R-L}\mathbf{D}_t^{q-k-1} f(t)\right)\Big|_{t=0}, \ n-1<q\le n, \tag{0.39}$$

$$ \pounds\left\{{}_0^{C}\mathbf{D}_t^q f(t)\right\} = s^q\widetilde{f}(s) - \sum_{k=0}^{n-1} s^{q-k-1} \left.\frac{d^k f(t)}{dt^k}\right|_{t=0}, \ n-1<q\le n. \tag{0.40}$$

Since the Laplace transform of the Caputo derivative requires only the values of the function and its ordinary derivatives at $t=0$, it has a clear advantage over the Riemann-Liouville derivative, when it comes to imposing the initial conditions. Equation (0.32) can be generalized for all $q>0$ as

$$ {}_0^{C}\mathbf{D}_t^q f(t) = {}_0^{R-L}\mathbf{D}_t^q f(t) - \sum_{k=0}^{n-1} \frac{t^{k-q}}{\Gamma(k-q+1)} f^{(k)}(0^+), \ n-1<q<n. \tag{0.41}$$

In other words, the two derivatives are not equal unless $f(t)$ and its first $n-1$ derivatives vanish at $t=0$ (Gorenflo and Mainardi).

III. Differintegral of 1/x

In applications we frequently need the differintegral of $1/x$. Let us start with the differintegral of $\ln x$ [Eq. (14.216) and see Example (0.2) of this supplement.]:

$$ \frac{d^q(\ln x)}{dx^q} = \frac{x^{-q}}{\Gamma(1-q)}[\ln x - \gamma - \Psi(1-q)], \tag{0.42}$$

which is valid for all q (Fig. 0.1). The gamma and the $\Psi(x)$ functions are related as

$$ \Psi(x) = \frac{1}{\Gamma(x)}\frac{d\Gamma(x)}{dx}, \tag{0.43}$$

$$ \gamma = -\Psi(1) = 0.5772157, \ \Psi(n+1) = \Psi(1) + \sum_{j=1}^{n}\frac{1}{j}. \tag{0.44}$$

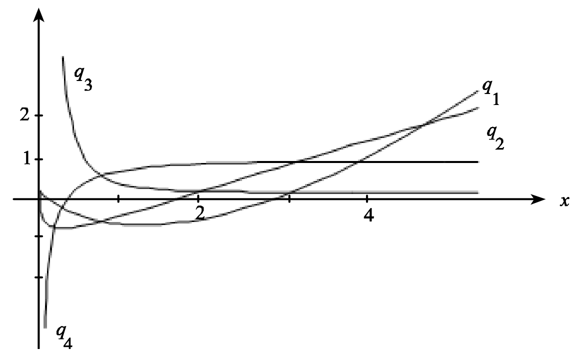

Fig. 0.1 Graph of the differintegral $\frac{d^q \ln x}{dx^q} = \frac{x^{-q}}{\Gamma(1-q)}[\ln x - \gamma - \Psi(1-q)]$ for the $q_1 = -1.1$, $q_2 = -0.5$, $q_3 = 1.5$, $q_4 = 2.4$ values.

Using the identity

$$\frac{\Psi(1-n)}{\Gamma(1-n)} = (-1)^n \Gamma(n), \tag{0.45}$$

where n is an integer, Equation (0.42) reproduces the usual expressions:

$$\frac{d^{-|n|}(\ln x)}{dx^{-|n|}} = \frac{x^{|n|}}{|n|!}\left[\ln x - \sum_{j=1}^{|n|}\frac{1}{j}\right] \tag{0.46}$$

and

$$\frac{d^{|n|}(\ln x)}{dx^{|n|}} = -\Gamma(|n|)[-x]^{-|n|}. \tag{0.47}$$

Using the integral representation of $\ln x$:

$$\ln x = \int_1^x \frac{dx}{x},\ \ x \in [1, \infty], \tag{0.48}$$

we can write

$$\frac{d^q(\ln x)}{[d(x-1)]^q} = \frac{d^q}{[d(x-1)]^q}\left[\int_1^x \frac{dx}{x}\right],\ \ x \in [1, \infty]. \tag{0.49}$$

Since

$$\frac{1}{x} - \frac{d}{[d(x-1)]}\frac{d^{-1}}{[d(x-1)]^{-1}}\left(\frac{1}{x}\right) = 0, \tag{0.50}$$

Equation (14.158) allows us to write this as

$$\frac{d^q(\ln x)}{[d(x-1)]^q} = \frac{d^{q-1}}{[d(x-1)]^{q-1}}\left[\frac{1}{x}\right]. \tag{0.51}$$

Equation (14.185) gives the dependence of a differintegral on the lower limit as

$$\delta(a,b;f(x)) = \frac{d^q f(x)}{[d(x-a)]^q} - \frac{d^q f(x)}{[d(x-b)]^q} \tag{0.52}$$

$$= \sum_{l=1}^{\infty} \frac{d^{q+l}(1)}{[d(x-b)]^{q+l}} \frac{d^{-l} f(b)}{[d(b-a)]^{-l}}. \tag{0.53}$$

Using this with $a = 0$, $b = 1$ and $f(x) = \ln x$, we write Equation (0.51) as

$$\frac{d^{q-1}}{[d(x-1)]^{q-1}} \left[\frac{1}{x}\right] = \frac{d^q(\ln x)}{dx^q} - \delta(0,1;\ln x). \tag{0.54}$$

Substituting

$$\delta(0,1;\ln x) = \sum_{l=1}^{\infty} \frac{d^{q+l}(1)}{[d(x-1)]^{q+l}} \left(\frac{d^{-l}(\ln x)}{[d(x-0)]^{-l}} \right)_{x=1} \tag{0.55}$$

$$= \sum_{l=1}^{\infty} \frac{d^{q+l}(1)}{[d(x-1)]^{q+l}} \left(\frac{d^{-l}(\ln x)}{dx^{-l}} \right)_{x=1}, \tag{0.56}$$

and using Equation (0.42), we write

$$\frac{d^{q-1}}{[d(x-1)]^{q-1}} \left[\frac{1}{x}\right] = \frac{x^{-q}}{\Gamma(1-q)}[\ln x - \gamma - \Psi(1-q)]$$

$$- \sum_{l=1}^{\infty} \frac{d^{q+l}(1)}{[d(x-1)]^{q+l}} \left(\frac{d^{-l}(\ln x)}{dx^{-l}} \right)_{x=1}. \tag{0.57}$$

Since this is valid for all q, we let $q-1 \to q$, thus obtaining the desired formula:

$$\frac{d^q(1/x)}{[d(x-1)]^q} = \frac{x^{-(q+1)}}{\Gamma(-q)}[\ln x - \gamma - \Psi(-q)] - \sum_{l=1}^{\infty} \frac{d^{q+1+l}(1)}{[d(x-1)]^{q+1+l}} \left(\frac{d^{-l}(\ln x)}{dx^{-l}} \right)_{x=1} \tag{0.58}$$

$$= \frac{x^{-(q+1)}}{\Gamma(-q)}[\ln x - \gamma - \Psi(-q)] - \sum_{l=1}^{\infty} \frac{(x-1)^{-(q+1+l)}}{\Gamma(-q-l)} \left(\frac{d^{-l}(\ln x)}{dx^{-l}} \right)_{x=1}. \tag{0.59}$$

To simplify, we use Equation (0.42) to write

$$\frac{d^{-l}(\ln x)}{dx^{-l}} = \frac{x^l}{\Gamma(1+l)}[\ln x - \gamma - \Psi(1+l)], \tag{0.60}$$

which gives

$$\left.\frac{d^{-l}(\ln x)}{dx^{-l}}\right|_{x=1} = \frac{-1}{\Gamma(1+l)}[\gamma + \Psi(1+l)]. \tag{0.61}$$

Equation (0.59) can now written as

$$\frac{d^q(1/x)}{[d(x-1)]^q} = \frac{x^{-(q+1)}}{\Gamma(-q)}[\ln x - \gamma - \Psi(-q)] + \sum_{l=1}^{\infty}(x-1)^{-(q+1+l)}\ \frac{[\gamma + \Psi(1+l)].}{\Gamma(-q-l)l!}. \tag{0.62}$$

One can easily check that for the integer values of q, this expression reduces to the usual results [Eqs. (0.46) and (0.47)]. The power series on the right-hand side converges uniformly and absolutely for $|x-1| > 1$.

To obtain the differintegral $\frac{d^q(1/x)}{dx^q}$, we apply Equation (0.52) one more time with the substitutions $a = 0$, $b = 1$, $f(x) = 1/x$ and write

$$\frac{d^q(1/x)}{dx^q} = \frac{d^q(1/x)}{[d(x-1)]^q} + \delta(0,1;1/x), \tag{0.63}$$

where

$$\delta(0,1;1/x) = \sum_{l=1}^{\infty} \frac{d^{q+l}(1)}{[d(x-1)]^{q+l}} \left(\frac{d^{-l}(1/x)}{dx^{-l}}\right)_{x=1}. \tag{0.64}$$

Along with Equation (0.59) these yield

$$\frac{d^q(1/x)}{dx^q} = \frac{x^{-(q+1)}}{\Gamma(-q)}[\ln x - \gamma - \Psi(-q)] - \sum_{l=1}^{\infty}\frac{(x-1)^{-(q+1+l)}}{\Gamma(-q-l)}\left(\frac{d^{-l}(\ln x)}{dx^{-l}}\right)_{x=1}$$
$$+\sum_{l=1}^{\infty}\frac{(x-1)^{-(q+l)}}{\Gamma(1-q-l)}\left(\frac{d^{-l}(1/x)}{dx^{-l}}\right)_{x=1}. \tag{0.65}$$

To simplify the first sum, we use Equation (0.61) for $\left(\frac{d^{-l}(\ln x)}{dx^{-l}}\right)_{x=1}$. For $\left(\frac{d^{-l}(1/x)}{dx^{-l}}\right)_{x=1}$, we use the composition rule in Equation (14.135) to write

$$\frac{d^{-l}(1/x)}{dx^{-l}} = \frac{d^{-l+1}}{dx^{-l+1}}\left(\frac{d^{-1}(1/x)}{dx^{-1}}\right), \tag{0.66}$$

where

$$\frac{d^{-1}(1/x)}{dx^{-1}} = \int_0^x \frac{1}{x'}dx' \tag{0.67}$$
$$= \ln x - \ln 0, \tag{0.68}$$

which diverges logarithmically. We use the limit

$$\frac{d^{-1}(1/x)}{dx^{-1}} = \ln x - \lim_{\varepsilon\to 0}(\ln\varepsilon), \tag{0.69}$$

to write Equation (0.66) as

$$\frac{d^{-l}(1/x)}{dx^{-l}} = \frac{d^{-l+1}(\ln x)}{dx^{-l+1}} - \theta_\varepsilon \frac{d^{-l+1}(1)}{dx^{-l+1}}, \tag{0.70}$$

where

$$\theta_\varepsilon = \lim_{\varepsilon \to 0} (\ln \varepsilon). \tag{0.71}$$

Using Equations (0.42) and (14.190) we can now write

$$\frac{d^{-l}(1/x)}{dx^{-l}} = \frac{x^{l-1}}{\Gamma(l)} [\ln x - \gamma - \Psi(l)] - \theta_\varepsilon \frac{x^{l-1}}{\Gamma(l)}, \tag{0.72}$$

thus obtaining

$$\left(\frac{d^{-l}(1/x)}{dx^{-l}} \right)_{x=1} = \frac{-1}{\Gamma(l)} [\gamma + \Psi(l)] - \theta_\varepsilon \frac{1}{\Gamma(l)}. \tag{0.73}$$

Substituting Equations (0.61) and (0.73) into Equation (0.65), we finally obtain

$$\begin{aligned} \frac{d^q(1/x)}{dx^q} &= \frac{x^{-(q+1)}}{\Gamma(-q)} [\ln x - \gamma - \Psi(-q)] + \sum_{l=1}^{\infty} (x-1)^{-(q+1+l)} \left[\frac{\Psi(l+1) + \gamma}{\Gamma(-q-l) l!} \right] \\ &\quad - \sum_{l=1}^{\infty} (x-1)^{-(q+l)} \left[\frac{\Psi(l) + \gamma}{\Gamma(1-q-l)\Gamma(l)} \right] - \theta_\varepsilon \sum_{l=1}^{\infty} (x-1)^{-(q+l)} \left[\frac{1}{\Gamma(1-q-l)\Gamma(l)} \right]. \end{aligned} \tag{0.74}$$

All of the three series in this expression converge absolutely and uniformly for $|x-1| > 1$. However, the entire expression diverges logarithmically as

$$\theta_\varepsilon = \lim_{\varepsilon \to 0} (\ln \varepsilon) : \frac{d^q(1/x)}{dx^q} \sim \theta_\varepsilon \left(\sum_{l=1}^{\infty} (x-1)^{-(q+l)} \left[\frac{1}{\Gamma(1-q-l)\Gamma(l)} \right] \right). \tag{0.75}$$

Example 0.1: Prove the relation

$$\frac{\Psi(1-n)}{\Gamma(1-n)} = (-1)^n \Gamma(n), \tag{0.76}$$

where n is an integer.

Proof: We start with the identity

$$\Gamma(-x)\Gamma(x+1) = -\pi \csc(\pi x) \tag{0.77}$$

or

$$\Gamma(x)\Gamma(1-x) = \pi \csc(\pi x) \tag{0.78}$$

and then differentiate:

$$\frac{d\Gamma(x)}{dx}\Gamma(1-x) + \Gamma(x)\frac{d\Gamma(1-x)}{dx} = -\pi^2 \frac{\cos(\pi x)}{\sin^2(\pi x)}, \tag{0.79}$$

$$\frac{d\Gamma(x)}{dx}\Gamma(1-x) + \Gamma(x)\frac{d\Gamma(1-x)}{d(1-x)}\frac{d(1-x)}{dx} = -\pi^2 \cos(\pi x)\csc^2(\pi x), \tag{0.80}$$

$$\frac{d\Gamma(x)}{dx}\Gamma(1-x) - \Gamma(x)\frac{d\Gamma(1-x)}{d(1-x)} = -\pi^2 \cos(\pi x)\csc^2(\pi x), \tag{0.81}$$

$$\frac{1}{\Gamma(x)}\frac{d\Gamma(x)}{dx} - \frac{1}{\Gamma(1-x)}\frac{d\Gamma(1-x)}{d(1-x)} = -\frac{\pi^2 \cos(\pi x)}{\Gamma(1-x)\Gamma(x)}\csc^2(\pi x), \tag{0.82}$$

$$\frac{1}{\Gamma(x)}\frac{d\Gamma(x)}{dx} - \frac{1}{\Gamma(1-x)}\frac{d\Gamma(1-x)}{d(1-x)} = -\frac{\pi^2 \cos(\pi x)}{\Gamma(1-x)\Gamma(x)}\csc^2(\pi x), \tag{0.83}$$

$$\frac{1}{\Gamma(x)}\frac{d\Gamma(x)}{dx} - \frac{1}{\Gamma(1-x)}\frac{d\Gamma(1-x)}{d(1-x)} = -\frac{\pi^2 \cos(\pi x)}{\Gamma(1-x)\Gamma(x)}\frac{\Gamma^2(1-x)\Gamma^2(x)}{\pi^2}, \tag{0.84}$$

$$\frac{1}{\Gamma(x)}\frac{d\Gamma(x)}{dx} - \frac{1}{\Gamma(1-x)}\frac{d\Gamma(1-x)}{d(1-x)} = -\cos(\pi x)\Gamma(1-x)\Gamma(x). \tag{0.85}$$

This is nothing but

$$\Psi(x) - \Psi(1-x) = -\cos(\pi x)\Gamma(1-x)\Gamma(x). \tag{0.86}$$

For $x = n$, where $n = 1, 2, 3, \ldots$, this becomes

$$\frac{\Psi(n)}{\Gamma(1-n)} - \frac{\Psi(1-n)}{\Gamma(1-n)} = -(-1)^n\Gamma(n),\ n = 1, 2, 3, \ldots\ . \tag{0.87}$$

Since $\Psi(n)$ is finite for $n = 1, 2, 3, \ldots$, due to the fact that the gamma function with a negative integer argument is infinity, the first term vanishes, thus proving the desired identity:

$$\frac{\Psi(1-n)}{\Gamma(1-n)} = (-1)^n\Gamma(n),\ n = 1, 2, 3, \ldots\ . \tag{0.88}$$

Problem 0.1: Using Equation (0.76) show that Equation (0.62) reduces to the usual results for the integer values of q . For the negative integer values of q show only for the first three values: $q = -1, -2, -3$

Example 0.2: Differintegral of $\ln x$: To find the differintegral of $\ln x$, we first write the Riemann-Liouville derivative:

$$\frac{d^q \ln x}{dx^q} = \frac{1}{\Gamma(-q)}\int_0^x \frac{\ln x' dx'}{[x-x']^{q+1}},\ q < 0, \tag{0.89}$$

and then make the substitution

$$y = \frac{x - x'}{x}, \tag{0.90}$$

to write

$$\frac{d^q \ln x}{dx^q} = \frac{x^{-q} \ln x}{\Gamma(-q)} \int_0^1 \frac{dy}{y^{q+1}} + \frac{x^{-q}}{\Gamma(-q)} \int_0^1 \frac{\ln(1-y)dy}{y^{q+1}}. \tag{0.91}$$

The first integral is easily evaluated as $1/(-q)$. Using integration by parts, the second integral can be written as

$$\int_0^1 \frac{\ln(1-y)dy}{y^{q+1}} = \frac{1}{q} \int_0^1 \ln(1-y)d(1-y^{-q}) \tag{0.92}$$

$$= \frac{(1-y^{-q})\ln(1-y)}{q}\bigg|_0^1 + \frac{1}{q} \int_0^1 \frac{(1-y^{-q})dy}{1-y} \tag{0.93}$$

$$= \frac{1}{q} \int_0^1 \frac{1-y^{-q}}{1-y} dy. \tag{0.94}$$

Using the integral definition of $\Psi(x)$, also called the **digamma function**:

$$\Psi(x+1) = -\gamma + \int_0^1 \frac{1-t^x}{1-t} dt, \tag{0.95}$$

we find

$$\int_0^1 \frac{\ln(1-y)dy}{y^{q+1}} = \frac{1}{q}[\gamma + \Psi(1-q)]. \tag{0.96}$$

Using these in Equation (0.91) we obtain

$$\frac{d^q \ln x}{dx^q} = \frac{x^{-q} \ln x}{(-q)\Gamma(-q)} + \frac{x^{-q}}{(-q)\Gamma(-q)}[-\gamma - \Psi(1-q)] \tag{0.97}$$

$$= \frac{x^{-q}}{\Gamma(1-q)}[\ln x - \gamma - \Psi(1-q)]. \tag{0.98}$$

Even though this result is obtained for $q < 0$, Using analyticity we can use it for all q (Oldham and Spanier).

IV. Mittag-Leffler Function and the Caputo Derivative

The Mittag-Leffler function, $E_q(x)$, is the generalization of the exponential function

$$e^x = \sum_{n=0}^{\infty} \frac{t^n}{n!} \tag{0.99}$$

$$= \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n+1)} \tag{0.100}$$

as

$$E_q(x) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(qn+1)}, \ q > 0. \tag{0.101}$$

We now consider the following fractional differential equation:

$${}_0^C\mathbf{D}_x^q y(x) = \omega y(x), \ y(0) = y_0, \ 0 < q < 1, \tag{0.102}$$

where ${}_0^C\mathbf{D}_x^q$ stands for the Caputo derivative, and write its Laplace transform:

$$s^q \widetilde{y} - s^{q-1} y_0 = \omega \widetilde{y}, \tag{0.103}$$

as

$$\widetilde{y}(s) = \frac{s^{q-1} y_0}{s^q - \omega}. \tag{0.104}$$

Using the geometric series, $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, we can write $\widetilde{y}$ as

$$\widetilde{y}(s) = y_0 \sum_{n=0}^{\infty} \frac{\omega^n}{s^{1+qn}}, \tag{0.105}$$

which can be inverted easily to yield the solution as

$$y(x) = y_0 \sum_{n=0}^{\infty} \frac{\omega^n x^{qn}}{\Gamma(qn+1)} \tag{0.106}$$

$$= y_0 E_q(\omega x^q). \tag{0.107}$$

We also use the notation

$$y(x) = y_0 E_q(w; x), \tag{0.108}$$

where $E_q(w; x)$ satisfies

$${}_0^C\mathbf{D}_x^q E_q(w; x) = \omega E_q(w; x), \ E_q(w; 0) = 1. \tag{0.109}$$

V. Euler Equation For the Mittag-Leffler Function

Euler equation for the exponential function is given as

$$y(t) = e^{i\omega t} = \cos \omega t + i \sin \omega t, \tag{0.110}$$

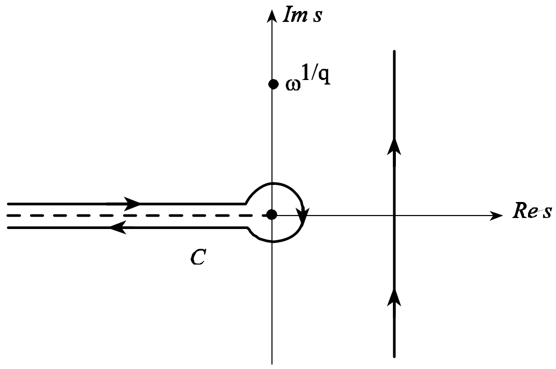

Fig. 0.2 Modified Bromwich contour.

where $y(t)$ satisfies the differential equation

$$\frac{dy}{dt} = i\omega y(t),\ y(0) = 1. \tag{0.111}$$

We now consider the following fractional differential equation for the Caputo derivative:

$${}_0^C\mathbf{D}_t^q y(t) = \omega i^q y(t),\ y(0) = y_0,\ 0 < q < 1, \tag{0.112}$$

the solution of which is given as

$$y(t) = y_0 E_q(\omega i^q; t). \tag{0.113}$$

Laplace transform of the differential equation gives the transform of the solution:

$$\widetilde{y}(s) = \frac{s^{q-1} y_0}{s^q - \omega i^q}, \tag{0.114}$$

which when inverted yields the solution, $y(t)$, as

$$y(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \left[\frac{e^{st} s^{q-1} y_0}{s^q - \omega i^q} \right] ds. \tag{0.115}$$

Since the integrand has a branch point at $s = 0$, the Bromwich contour has to be modified as in Figure 0.2. We have located the branch cut along the negative real axis and the contour around the branch cut is called the Hankel contour. There are two contributions to this integral, one of which is due to the pole at

$$s = \omega^{1/q} i, \tag{0.116}$$

and the other one comes from the straight line segments of the contour above and below the branch cut. We can now write

$$y(t) = [\text{residue at } s = \omega^{1/q} i\ ] + \frac{y_0}{2\pi i} \int_{\text{Hankel}} \left[\frac{e^{st} s^{q-1}}{s^q - \omega i^q} \right] ds. \tag{0.117}$$

The residue is evaluated easily as

$$\text{residue} = \lim_{s \to s_0} \frac{(s - s_0) y_0 e^{st} s^{q-1}}{s^q - \omega i^q},\ s_0 = \omega^{1/q} i, \tag{0.118}$$

$$= \frac{e^{i\omega^{1/q} t}}{q}\ . \tag{0.119}$$

The remaining integral over the Hankel contour can be written as

$$-\frac{y_0 \omega i^q}{\pi} \int_0^\infty \frac{(\sin q\pi) e^{-xt} x^{q-1} dx}{x^{2q} - 2\omega i^q (\cos q\pi) x^q + (\omega i^q)^2}. \tag{0.120}$$

We can now put together the final solution as

$$y(t) = y_0 \left[\frac{e^{i\omega^{1/q} t}}{q} - \frac{\omega i^q (\sin q\pi)}{\pi} \int_0^\infty \frac{e^{-xt} x^{q-1} dx}{x^{2q} - 2\omega i^q (\cos q\pi) x^q + (\omega i^q)^2} \right]. \tag{0.121}$$

The first term on the right-hand side is oscillatory, while the second term contains an exponentially decaying term. As $q \to 1$, the above expression reduces to the Euler equation. Defining the function

$$F_q(\sigma; t) = \frac{\sigma(\sin q\pi)}{\pi} \int_0^\infty \frac{e^{-xt} x^{q-1} dx}{x^{2q} - 2\sigma(\cos q\pi) x^q + \sigma^2}, \tag{0.122}$$

which is monotonically decreasing, we can write the solution of the differential equation:

$${}_0^C\mathbf{D}_t^q y(t) = \omega i^q y(t),\ y(0) = y_0,\ 0 < q < 1, \tag{0.123}$$

as

$$y(t) = y_0 \left[\frac{1}{q} e^{i\omega^{1/q} t} - F_q(\omega i^q; t) \right], \tag{0.124}$$

where ${}_0^C\mathbf{D}_t^q$ stands for the Caputo derivative. Following Naber, we can now write the analog of the Euler equation in fractional calculus as

$$E_q(\omega i^q; t) = \frac{1}{q} e^{i\omega^{1/q} t} - F_q(\omega i^q; t),\ 0 < q < 1, \tag{0.125}$$

which satisfies the differential equation

$${}_0^C\mathbf{D}_t^q E_q(\omega i^q; t) = \omega i^q E_q(\omega i^q; t),\ 0 < q < 1. \tag{0.126}$$

For $q = 1$, $E_q(\omega i^q; t)$ reduces to the Euler equation:

$$E_1(i\omega; t) = e^{i\omega t}. \tag{0.127}$$

Equation (0.115) also allows us to write the integral representation of the Mittag-Leffler equation as

$$E_q(\omega i^q; t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \left[\frac{e^{st} s^{q-1}}{s^q - \omega i^q} \right] ds, \tag{0.128}$$

which with the substitution

$$x = \omega i^q,\ t = 1, \tag{0.129}$$

is also written as

$$E_q(x) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \left[\frac{e^{s} s^{q-1}}{s^q - x} \right] ds,\ q > 0, \tag{0.130}$$

In applications we frequently need the asymptotic forms:

$$E_q(x) \sim \frac{1}{q} e^{x^{1/q}} - \sum_{k=1}^{\infty} \frac{x^{-k}}{\Gamma(1 - qk)},\ \ |x| \to \infty,\ 0 < q < 2, \tag{0.131}$$

$$E_q(x) \sim - \sum_{k=1}^{\infty} \frac{x^{-k}}{\Gamma(1 - qk)},\ \ |x| \to \infty,\ q < 0, \tag{0.132}$$

$$E_q(x) \sim \frac{1}{q} \sum_{m} e^{\left(x^{1/2} e^{2\pi i m/q}\right)} - \sum_{k=1}^{\infty} \frac{x^{-k}}{\Gamma(1 - qk)},\ \ |x| \to \infty,\ q \geq 2, \tag{0.133}$$

where m takes all the integer value such that $-q\pi/2 < 2\pi m < q\pi/2$ for $x > 0$. For analytic continuation of these expressions, [Eqs. (0.130)-(0.133)], we refer the reader to Mainardi and Gorenflo.

VI. Right- and Left-Handed Operators

We can generalize the fractional Riemann-Liouville integral,

$${}_0\mathbf{I}_t^q[f(t)] = \frac{1}{\Gamma(q)} \int_0^t \frac{f(\tau) d\tau}{(t - \tau)^{1-q}},\ q > 0, \tag{0.134}$$

to define the **right-** and the **left- handed Riemann-Liouville integrals**, respectively, as [Eqs. (14.178) and (14.180)]

$${}_{a^+}\mathbf{I}_t^q[f(t)] = \frac{1}{\Gamma(q)} \int_a^t (t - \tau)^{q-1} f(\tau) d\tau, \tag{0.135}$$

$${}_{b^-}\mathbf{I}_t^q[f(t)] = \frac{1}{\Gamma(q)} \int_t^b (\tau - t)^{q-1} f(\tau) d\tau, \tag{0.136}$$

where $a < t < b$ and $q > 0$. In applications we frequently encounter cases with $a = -\infty$ or $b = \infty$. Fractional integrals with either the lower or the upper limit is taken as infinity are also called the **Weyl fractional integral**. Some authors may reverse the definitions of the right- and the left- handed derivatives. Sometimes ${}_{a^+}\mathbf{I}_t^q$ and ${}_{b^-}\mathbf{I}_t^q$ are also called **progressive** and **regressive**, respectively.

The **right-** and the **left- handed Riemann-Liouville derivatives** of order $q > 0$ are defined as [Eq. (14.71)]

$$ {}_{a^+}\mathbf{D}_t^q f(t) = \frac{d^n}{dt^n} \left({}_{a^+}\mathbf{I}_t^{n-q}[f(t)]\right), \tag{0.137} $$

$$ {}_{b^-}\mathbf{D}_t^q f(t) = (-1)^n \frac{d^n}{dt^n} \left({}_{b^-}\mathbf{I}_t^{n-q}[f(t)]\right), \tag{0.138} $$

where $a < t < b$ and $n > q$. The following composition rules hold for the d^n/dt^n and the ${}_{a^+}\mathbf{I}_t^n f(t)$ operators [Eqs. (14.150) and (14.151)]:

$$ \frac{d^n}{dt^n}\left[{}_{a^+}\mathbf{I}_t^n f(t)\right] = f(t), \tag{0.139} $$

$$ \left[{}_{a^+}\mathbf{I}_t^n\right] \frac{d^n f(t)}{dt^n} = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a^+)}{k!}(t-a)^k. \tag{0.140} $$

The corresponding equations for the left-handed integrals are given as

$$ \frac{d^n}{dt^n}\left[{}_{b^-}\mathbf{I}_t^n f(t)\right] = (-1)^n f(t) \tag{0.141} $$

and

$$ \left[{}_{b^-}\mathbf{I}_t^n\right] \frac{d^n f(t)}{dt^n} = (-1)^n \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(b^-)}{k!}(b-t)^k\right]. \tag{0.142} $$

The **right-handed Caputo derivative for** $q > 0$ is defined as

$$ \begin{aligned} {}_{a^+}^{C}\mathbf{D}_r^q f(r) &= {}_{a^+}\mathbf{I}_r^{n-q} f^{(n)}(x) \\ &= \frac{1}{\Gamma(n-q)} \int_a^r \frac{f^{(n)}(\tau)d\tau}{(r-\tau)^{1-n+q}}, \end{aligned} \tag{0.143} $$

where n is the next integer higher than q. Note that for $0 < q < 1$, $a = 0$ and $n = 1$, we obtain Equation (0.9).

The **left-handed Caputo derivative for** $q > 0$ is defined as

$$ \begin{aligned} {}_{b^-}^{C}\mathbf{D}_r^q f(r) &= (-1)^n {}_{b^-}\mathbf{I}_r^{n-q} f^{(n)}(x) \\ &= \frac{(-1)^n}{\Gamma(n-q)} \int_r^b \frac{f^{(n)}(\tau)d\tau}{(\tau-r)^{1-n+q}}, \end{aligned} \tag{0.144} $$

where n is again the next integer higher than q (El-Sayed and Gaber). We reserve the letter a for the lower limit of the integral operators and the letter b for the upper limit, hence we will ignore the superscripts in a^+ and b^-.

Example (0.3) Left-handed Caputo derivative of $1/r$ **:**

We calculate the left-handed Caputo derivative of $1/r$ for $0 < q < 1$, $k = 1$ and $b = \infty$:

$$\begin{aligned} {}^{C}_{\infty}\mathbf{D}^{q}_{r}\left(\frac{1}{r}\right) &= \frac{-1}{\Gamma(1-q)}\int_{r}^{\infty}\frac{(-1/\tau^2)d\tau}{(\tau-r)^q} \\ &= \frac{-1}{\Gamma(1-q)}\int_{r}^{\infty}(-1)\tau^{-(q+2)}\left(1-\frac{r}{\tau}\right)^{-q}d\tau. \end{aligned} \quad (0.145)$$

Defining

$$t = \frac{r}{\tau}, \quad (0.146)$$

we write

$${}^{C}_{\infty}\mathbf{D}^{q}_{r}\left(\frac{1}{r}\right) = \frac{1}{\Gamma(1-q)}\int_{1}^{0}\left(\frac{r}{t}\right)^{-(q+2)}(1-t)^{-q}\left(-\frac{rdt}{t^2}\right) \quad (0.147)$$

$$= \frac{-r^{-(q+1)}}{\Gamma(1-q)}\int_{1}^{0}\frac{(1-t)^{-q}}{t^{-q}}dt \quad (0.148)$$

$$= \frac{r^{-(q+1)}}{\Gamma(1-q)}\int_{0}^{1}t^{q}\,(1-t)^{-q}\,dt. \quad (0.149)$$

Using the definition of the beta function [Eq. (13.151) and Eq. (13.146)] this gives

$${}^{C}_{\infty}\mathbf{D}^{q}_{r}\left(\frac{1}{r}\right) = r^{-(q+1)}\Gamma(1+q),\ 0 < q < 1. \quad (0.150)$$

Note that as $q \to 1$ one does not get the expected result, that is, $D^1_r(1/r) = -1/r^2$. Actually, in general one has $\lim_{q\to n}\left[{}^{C}_{b}\mathbf{D}^{q}_{r}f(r)\right] = (-1)^n f^{(n)}(r)$. For this and other properties of the Caputo derivatives see El-Sayed and Gaber.

VII. Riesz Fractional Integral and Derivative

Riesz fractional integral:

Another fractional derivative commonly encountered in applications is the Riesz derivative. It is useful in generalizing the standard diffusion equation by replacing the second-order space derivative. Since the Riesz derivative is defined through the Fourier transform of a function, we start with a review of the basic properties of the Fourier transforms. Fourier transform of an absolutely integrable function, $f(t)$, in the interval $(-\infty, \infty)$ is defined as [Bayin (2006), Bayin (2008)]

$$F(\omega) = \mathcal{F}\{f(t)\} = \int_{-\infty}^{\infty}e^{-i\omega t}f(t)dt,\ \omega \text{ is real}, \quad (0.151)$$

where the inverse transform is given as

$$f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t}F(\omega)d\omega. \tag{0.152}$$

If $F(\omega)$ and $G(\omega)$ are the Fourier transforms of $f(t)$ and $g(t)$, respectively, the convolution of f with g, $f * g$, is defined as

$$f * g = \int_{-\infty}^{\infty} f(t-\tau)g(\tau)d\tau = \int_{-\infty}^{\infty} f(\tau)g(t-\tau)d\tau. \tag{0.153}$$

The Fourier transform of a convolution, $\mathcal{F}\{f * g\}$, is equal to the product of the Fourier transforms $F(\omega)$ and $G(\omega)$:

$$\mathcal{F}\{f * g\} = F(\omega)\cdot G(\omega). \tag{0.154}$$

Granted that all the required derivatives, $f(f), f'(t), \ldots, f^{(n-1)}(t)$, vanish as $t \to \pm\infty$, the Fourier transform of a derivative, $\mathcal{F}\{f^{(n)}(t)\}$, is given as

$$\mathcal{F}\{f^{(n)}(t)\} = (i\omega)^n F(\omega). \tag{0.155}$$

To find the Fourier transform of the fractional Riemann-Liouville integral [Eq. (14.70) with $a = -\infty$.]

$$_{-\infty}\mathbf{I}_t^q g(t) = _{-\infty}\mathbf{D}_t^{-q} g(t) = \frac{1}{\Gamma(q)}\int_{-\infty}^{t} (t-\tau)^{q-1} g(\tau)d\tau,\ q > 0, \tag{0.156}$$

we first write the Laplace transform of the function [Eq. (16.113)]

$$h(t) = \frac{t^{q-1}}{\Gamma(q)},\ q > 0: \tag{0.157}$$

$$\pounds\{h(t)\} = \frac{1}{\Gamma(q)}\int_{0}^{\infty} t^{q-1}e^{-st}dt = s^{-q}, \tag{0.158}$$

and then substitute $s = i\omega$ to obtain the Fourier transform of

$$h_+(t) = \begin{cases} \dfrac{t^{q-1}}{\Gamma(q)}, & t > 0, \\ 0 & t \le 0 \end{cases} \tag{0.159}$$

as

$$\mathcal{F}\{h_+(t)\} = (i\omega)^{-q}. \tag{0.160}$$

In this case the convergence of the integral in Equation (0.158) restricts q to $0 < q < 1$. We now write the convolution of $g(t)$ with $h_+(t)$, which is nothing but ${}_{-\infty}\mathbf{D}_t^{-q} g(t)$, as

$$h_+(t) * g(t) = \int_{-\infty}^{\infty} h_+(t-\tau) g(\tau) d\tau \tag{0.161}$$

$$= \int_{-\infty}^{\infty} (t-\tau)^{q-1} g(\tau) d\tau \tag{0.162}$$

$$=_{-\infty} \mathbf{D}_t^{-q} g(t). \tag{0.163}$$

We finally use Equation (0.160) to obtain

$$\mathcal{F}\{{}_{-\infty}\mathbf{D}_t^{-q} g(t)\} = (i\omega)^{-q} G(\omega), \tag{0.164}$$

where $G(\omega)$ is the Fourier transform of $g(t)$.

To find the Fourier transform of

$$_{\infty}\mathbf{D}_t^{-q} g(t) = \frac{1}{\Gamma(q)} \int_t^{\infty} (\tau - t)^{q-1} g(\tau) d\tau,\ 0 < q < 1, \tag{0.165}$$

we again make use of the Laplace transform

$$\pounds\{h(t)\} = \frac{1}{\Gamma(q)} \int_0^{\infty} t^{q-1} e^{-st} dt = s^{-q},\ q > 0, \tag{0.166}$$

and substitute $s = -i\omega$ to get

$$\pounds\{h(t)\} = \frac{1}{\Gamma(q)} \int_0^{\infty} t^{q-1} e^{i\omega t} dt$$

$$= (-i\omega)^{-q},\ 0 < q < 1. \tag{0.167}$$

We then let $t \to -t$ in the above integral to write

$$\pounds\{h(t)\} = \frac{1}{\Gamma(q)} \int_{-\infty}^{0} (-t)^{q-1} e^{-i\omega t} dt \tag{0.168}$$

$$= (-i\omega)^{-q},\ 0 < q < 1. \tag{0.169}$$

This is nothing but the Fourier transform of the function

$$h_-(t) = \begin{cases} 0, & t \geq 0, \\ \dfrac{(-t)^{q-1}}{\Gamma(q)}, & t < 0 \end{cases} \tag{0.170}$$

as

$$\mathcal{F}\{h_-(t)\} = \frac{1}{\Gamma(q)} \int_{-\infty}^{\infty} h_-(t) e^{-i\omega t} dt \tag{0.171}$$

$$= \frac{1}{\Gamma(q)} \left[\int_{-\infty}^{0} h_-(t) e^{-i\omega t} dt + \int_0^{\infty} h_-(t) e^{-i\omega t} dt \right] \tag{0.172}$$

$$= \frac{1}{\Gamma(q)} \int_{-\infty}^{0} (-t)^{q-1} e^{-i\omega t} dt \tag{0.173}$$

$$= (-i\omega)^{-q}. \tag{0.174}$$

We now employ the convolution theorem to write

$$h_-(t) * g(t) = \int_{-\infty}^{\infty} h_-(t-\tau)g(\tau)d\tau \tag{0.175}$$
$$= \int_{-\infty}^{0} h_-(t-\tau)g(\tau)d\tau + \int_{0}^{\infty} h_-(t-\tau)g(\tau)d\tau \tag{0.176}$$
$$= \frac{1}{\Gamma(q)} \int_{0}^{\infty} (\tau - t)^{\chi-1} g(\tau)d\tau \tag{0.177}$$
$$= \mathcal{F}\{{}_{\infty}\mathbf{D}_t^{-q} g(t)\} \tag{0.178}$$
$$= H_-(\omega)G(\omega) \tag{0.179}$$
$$= (-i\omega)^{-q} G(\omega). \tag{0.180}$$

Summary:
We have obtained the following Fourier transforms of fractional integrals:

$$\mathcal{F}\{{}_{-\infty}\mathbf{D}_t^{-q} g(t)\} = (i\omega)^{-q} G(\omega), \tag{0.181}$$
$$\mathcal{F}\{{}_{\infty}\mathbf{D}_t^{-q} g(t)\} = (-i\omega)^{-q} G(\omega). \tag{0.182}$$

We can combine these equations to write

$$\mathcal{F}\{\left[{}_{-\infty}\mathbf{D}_t^{-q} +_{\infty} \mathbf{D}_t^{-q}\right] g(t)\} = \left[(i\omega)^{-q} + (-i\omega)^{-q}\right] G(\omega) \tag{0.183}$$
$$= |\omega|^{-q} \left[i^{-q} + (-i)^{-q}\right] G(\omega) \tag{0.184}$$
$$= \left(2\cos\frac{q\pi}{2}\right) |\omega|^{-q} G(\omega) \tag{0.185}$$

and

$$\mathcal{F}\left\{ \frac{\left[{}_{-\infty}\mathbf{D}_t^{-q} +_{\infty} \mathbf{D}_t^{-q}\right]}{2\cos\left(\frac{q\pi}{2}\right)} g(t) \right\} = |\omega|^{-q} G(\omega). \tag{0.186}$$

The combined expression (El-Sayed and Gaber):

$$\mathbf{R}_t^{-q} g(t) = \frac{\left[{}_{-\infty}\mathbf{D}_t^{-q} +_{\infty} \mathbf{D}_t^{-q}\right] g(t)}{2\cos\left(\frac{q\pi}{2}\right)} \tag{0.187}$$
$$= \frac{1}{2\Gamma(q)\cos\left(\frac{q\pi}{2}\right)} \int_{-\infty}^{\infty} (t-\tau)^{q-1} g(\tau)d\tau,\ q > 0,\ q \neq 1, 3, 5, \dots, \tag{0.188}$$

is called the **Riesz fractional integral** or the **Riesz potential**. The Riesz fractional integral for $0 < q < 1$ is evaluated through its Fourier transform as

$$\mathcal{F}\{\mathbf{R}_t^{-q} g(t)\} = |\omega|^{-q} G(\omega), 0 < q < 1. \tag{0.189}$$

Problem 0.2: Verify Equation (0.185).

Riesz fractional derivative:

To find the Fourier transform of fractional derivatives, we write the Riemann-Liouville definition [Eq. (14.17)] with $a = -\infty$ as

$$
\begin{aligned}
{}_{-\infty}\mathbf{D}_t^q g(t) &= \frac{1}{\Gamma(n-q)} \int_{-\infty}^{t} (t-\tau)^{-q-1+n} g^{(n)}(\tau) d\tau, \ q > 0, && (0.190) \\
&= {}_{-\infty} \mathbf{D}_t^{q-n} g^{(n)}(t), \ n-1 < q < n. && (0.191)
\end{aligned}
$$

We have assumed reasonable behavior of $g(t)$ and its derivatives, and used $-\infty$ for the lower limit in Section II of this supplement. In other words, due to the boundary conditions used, Riemann-Liouville and the Caputo definitions of the fractional derivatives agree (El-Sayed and Gaber, also see Podlubny). Since $q - n < 0$, we can write the Fourier transform of ${}_{-\infty}\mathbf{D}_t^q g(t)$ as

$$
\begin{aligned}
\mathcal{F}\{{}_{-\infty}\mathbf{D}_t^q g(t)\} &= (i\omega)^{q-n} \mathcal{F}\{g^{(n)}(t)\}, \ q > 0, && (0.192) \\
&= (i\omega)^{q-n} (i\omega)^n G(\omega) && (0.193) \\
&= (i\omega)^q G(\omega), && (0.194)
\end{aligned}
$$

where we have used the result in Equation (0.155). Similarly, we can show that

$$
\mathcal{F}\{{}_{\infty}\mathbf{D}_t^q g(t)\} = (-i\omega)^q G(\omega). \tag{0.195}
$$

We now combine the results in Equations (0.194) and (0.195) to write

$$
\begin{aligned}
\mathcal{F}\{[{}_{-\infty}\mathbf{D}_t^q +_{\infty} \mathbf{D}_t^q]\, g(t)\} &= [(i\omega)^q + (-i\omega)^q]\, G(\omega) && (0.196) \\
&= \left(2 \cos \frac{q\pi}{2}\right) |\omega|^q \, G(\omega). && (0.197)
\end{aligned}
$$

For $0 < q \leq 2$, $q \neq 1$, the **Riesz fractional derivative** is defined as (Herrmann)

$$
\mathbf{R}_t^q g(t) = -\frac{[{}_{-\infty}\mathbf{D}_t^q +_{\infty} \mathbf{D}_t^q]\, g(t)}{2 \cos\left(\frac{q\pi}{2}\right)}, \tag{0.198}
$$

which can be found through its Fourier transform:

$$
\mathcal{F}\{\mathbf{R}_t^q g(t)\} = |\omega|^q \, G(\omega), \ 0 < q \leq 2, \tag{0.199}
$$

as

$$
\mathbf{R}_t^q g(t) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} |\omega|^q \, G(\omega) e^{i\omega t} d\omega. \tag{0.200}
$$

The minus sign in the definition of the Riesz derivative [Eq. (0.198)] is introduced to recover the $q = 2$ case as

$$\mathbf{R}_t^2 g(t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} |\omega|^2 G(\omega) e^{i\omega t} d\omega \tag{0.201}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega)\left[\frac{d^2}{dt^2} e^{i\omega t}\right] d\omega \tag{0.202}$$

$$= \frac{d^2}{dt^2}\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} G(\omega) e^{i\omega t} d\omega\right] \tag{0.203}$$

$$= \frac{d^2}{dt^2} g(t). \tag{0.204}$$

As in the fractional integral case, because of the boundary conditions that $g(t)$ satisfies, the Riemann-Liouville, Grünwald, Caputo and Riesz derivatives all agree. In general, the Riesz derivative is related to the $q/2$ power of the positive definite operator

$$-\mathbf{D}_t^2 g(t) = -\frac{d^2}{dt^2} g(t) \tag{0.205}$$

as

$$-\mathbf{R}_t^q g(t) = \left(-\frac{d^2}{dt^2}\right)^{q/2} g(t). \tag{0.206}$$

Problem 0.3: Justify Equation (0.195) and show that the Riemann-Liouville, Grünwald, Caputo and the Riesz definitions of the fractional derivative agree. Note the following important relation between the left-handed Riemann-Liouville and the Caputo derivatives (El-Sayed and Gaber):

$${}_b^{R-L}\mathbf{D}_t^q g(t) = {}_b^{C}\,\mathbf{D}_t^q g(t) + \sum_{k=0}^{n-1} \frac{(-1)^{n-k}(b-t)^{k-q}}{\Gamma(k-q+1)} \left[\mathbf{D}_t^k g(t)\right]_{t=b}, \tag{0.207}$$

where $q \in (n-1, n]$. When $g(t)$ satisfies the boundary conditions

$$\mathbf{D}_t^k g(b) = 0,\ \ k = 0, 1, \ldots, n-1, \tag{0.208}$$

Equation (0.207) implies

$${}_b^{R-L}\mathbf{D}_t^q g(t) \equiv {}_b^{C}\,\mathbf{D}_t^q g(t). \tag{0.209}$$

Another representation of the Riesz derivative:

To find another useful representation of the Riesz derivative, we use Equations (0.137) and (0.138) to write

$${}_{-\infty}\mathbf{D}_t^q f(t) = \frac{d}{dt}\left[{}_{-\infty}\mathbf{I}_t^{1-q} f(t)\right],\ \ 0 < q < 1, \tag{0.210}$$

and

$$ {}_{\infty}\mathbf{D}_t^q f(t) = (-1)\frac{d}{dt}\left[{}_{\infty}\mathbf{I}_t^{1-q} f(t)\right],\ 0 < q < 1. \tag{0.211} $$

The first equation [Eq. (0.210)]:

$$ {}_{-\infty}\mathbf{D}_t^q f(t) = {}_{-\infty}\mathbf{I}_t^{-q} f(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_{-\infty}^{t} (t-\tau)^{-q} f(\tau) d\tau, \tag{0.212} $$

which after a variable change can be written as

$$ {}_{-\infty}\mathbf{D}_t^q f(t) = \frac{1}{\Gamma(1-q)}\frac{d}{dt}\int_0^{\infty} \xi^{-q} f(t-\xi) d\xi. \tag{0.213} $$

Furthermore, using the integral

$$ \xi^{-q} = q\int_{\xi}^{\infty} \frac{d\eta}{\eta^{1+q}}, \tag{0.214} $$

we can also write

$$ {}_{-\infty}\mathbf{D}_t^q f(t) = \frac{q}{\Gamma(1-q)}\frac{d}{dt}\int_0^{\infty} \frac{\partial f(t-\xi)}{\partial t}\int_{\xi}^{\infty} \frac{1}{\eta^{1+q}} d\eta\ d\xi. \tag{0.215} $$

Since

$$ \frac{\partial f(t-\xi)}{\partial t} = -\frac{\partial f(t-\xi)}{\partial \xi}, \tag{0.216} $$

we write

$$ {}_{-\infty}\mathbf{D}_t^q f(t) = \frac{q}{\Gamma(1-q)}\left[-\int_0^{\infty} \frac{\partial f(t-\xi)}{\partial \xi}\left(\int_{\xi}^{\infty} \frac{d\eta}{\eta^{1+q}}\right) d\xi\right]. \tag{0.217} $$

Integration by parts yields

$$ \begin{aligned} {}_{-\infty}\mathbf{D}_t^q f(t) &= \frac{q}{\Gamma(1-q)}\left\{\left[-\left(f(t-\xi)-f(t)\right)\left(\int_{\xi}^{\infty} \frac{d\eta}{\eta^{1+q}}\right)\right]_{\xi=0}^{\infty}\right. \qquad (0.218) \\ &\quad \left. + \int_0^{\infty} \left(f(t-\xi)-f(t)\right)\left(\frac{d}{d\xi}\int_{\xi}^{\infty} \frac{d\eta}{\eta^{1+q}}\right) d\xi\right\} \\ &= \frac{q}{\Gamma(1-q)}\int_0^{\infty} \frac{f(t)-f(t-\xi)}{\xi^{1+q}} d\xi,\ 0 < q < 1. \qquad (0.219) \end{aligned} $$

Following similar steps and using Equation (0.211), we obtain

$$ \begin{aligned} {}_{\infty}\mathbf{D}_t^q f(t) &= {}_{\infty}\mathbf{I}_t^{-q} f(t) = -\frac{d}{dt}\left[{}_{\infty}\mathbf{I}_t^{1-q} f(t)\right],\ 0 < q < 1, \qquad (0.220) \\ &= \frac{q}{\Gamma(1-q)}\int_0^{\infty} \frac{f(t+\xi)-f(t)}{\xi^{1+q}} d\xi. \qquad (0.221) \end{aligned} $$

Similar results can be obtained for the interval $1 < q < 2$. Recalling the definition of the Riesz derivative [Eq. (0.198)]:

$$\mathbf{R}_t^q g(t) = -\frac{\left[{}_{-\infty}\mathbf{D}_t^q + {}_{\infty}\mathbf{D}_t^q\right] f(t)}{2\cos\left(\frac{q\pi}{2}\right)} \tag{0.222}$$

and using the identity

$$\frac{q}{\Gamma(1-q)} = -\frac{1}{\Gamma(-q)} = \Gamma(1+q)\frac{\sin(q\pi/2)}{\pi}, \tag{0.223}$$

we get the following regularized representation of the Riesz derivative, which is also valid for $q = 1$ (Herrmann, Scalas et.al.):

$$\mathbf{R}_t^q f(t) = \frac{\Gamma(1+q)\sin(q\pi/2)}{\pi}\int_0^{\infty}\frac{f(t+\xi)-2f(t)+f(t-\xi)}{\xi^{1+q}}d\xi,\ 0 < q < 2. \tag{0.224}$$

Fractional Laplacian:
Using the following definitions of the three-dimensional Fourier transforms:

$$\Phi(\overrightarrow{k},t) = \int_{-\infty}^{\infty} d^3\overrightarrow{r}\ \Psi(\overrightarrow{r},t)e^{-i\overrightarrow{k}\cdot\overrightarrow{r}}, \tag{0.225}$$

$$\Psi(\overrightarrow{r},t) = \frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3\overrightarrow{k}\ \Phi(\overrightarrow{k},t)e^{i\overrightarrow{k}\cdot\overrightarrow{r}}, \tag{0.226}$$

we can introduce the fractional Laplacian as (Laskin (2002))

$$\Delta^{q/2}\Psi(\overrightarrow{r},t) = -\frac{1}{(2\pi)^3}\int_{-\infty}^{\infty} d^3\overrightarrow{k}\ \Phi(\overrightarrow{k},t)\,|k|^q\,e^{i\overrightarrow{k}\cdot\overrightarrow{r}}. \tag{0.227}$$

VIII. Useful Sites and References

Useful Sites on Fractional Calculus:

`http://mathworld.wolfram.com/FractionalCalculus.html`
`http://en.wikipedia.org/wiki/Fractional_calculus`.

Sites On Mittag-Leffler Functions:

`http://mathworld.wolfram.com/Mittag-LefflerFunction.html`
`http://en.wikipedia.org/wiki/Mittag-Leffler_funcion`.

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Revised version: R_1

Revisions made in R_1:

♦ Section on the right- and the left- handed Caputo derivatives is revised.
♦ Section on the Riesz fractional integrals and derivatives is added.