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An Introduction to Vectors and Tensors

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Preface

These notes introduce the reader to vectors and tensors using direct notation and indicial notation. We deal almost exclusively with Cartesian tensors. The concepts introduced can be extended to general coordinate systems and spaces of higher dimension. We attempt to maintain mathematical rigor in the notes without the mathematical formalism of theorem and proof.

The notes are a revision of a set of notes by Professor Emeritus Russell Dunholter (deceased), Engineering Science and Mechanics Department, University of Cincinnati [2]. This original source furnished the framework for the current effort. The original notes were supplemented with material from lectures given by Prof. Morton Gurtin in a graduate level continuum mechanics course at Carnegie Mellon University. We have also consulted Prof. Gurtin's excellent text [3] in the preparation of these notes. Other sources have also been consulted, and several of these sources are acknowledged in the bibliography. Any of these sources may be consulted for further study.

Acknowledgments

I would like to acknowledge the work of Chris Reeder in typing the text of the original material of Professor Dunholter that was used. Ms. Margy Fotopoulos also helped with some of the initial typing and was invaluable in proofreading the completed typescript. Without their help this task would not have been completed as well or as easily as it was.

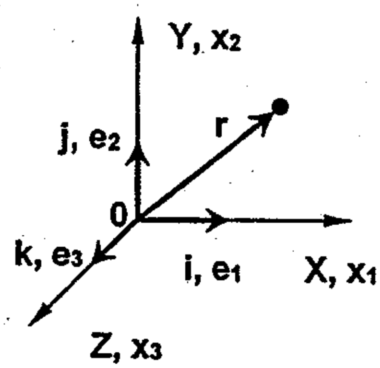

Figure 1.5.Orthonormal basis for a Euclidean vector space $\mathcal{V} = \mathbf{E}^3$.

choice of a sense is governed by the right hand sense of the definition of the vector product. Any basis consisting of three mutually perpendicular unit vectors is called an **orthonormal basis.** We choose an orthonormal basis **i**, **j**, and **k** directed along the positive X, Y, and Z axes respectively. Since X, Y, and Z is a right hand order, we have $\mathbf{i} \times \mathbf{j} = \mathbf{k}$. Then any vector **u** can be uniquely expressed as the vector sum

$$\mathbf{u} = u_x\mathbf{i}+u_y\mathbf{j}+u_z\mathbf{k} \tag{1.12}$$

The components $u_x = \mathbf{u} \cdot \mathbf{i}$, $u_y = \mathbf{u} \cdot \mathbf{j}$, and $u_z = \mathbf{u} \cdot \mathbf{k}$ can be calculated according to (1.5).

We list the nine possible scalar products of the orthonormal basis as follows

$$\begin{array}{lll} \mathbf{i}\cdot\mathbf{i}=1 & \mathbf{i}\cdot\mathbf{j}=0 & \mathbf{i}\cdot\mathbf{k}=0 \\ \mathbf{j}\cdot\mathbf{i}=0 & \mathbf{j}\cdot\mathbf{j}=1 & \mathbf{j}\cdot\mathbf{k}=0 \\ \mathbf{k}\cdot\mathbf{i}=0 & \mathbf{k}\cdot\mathbf{j}=0 & \mathbf{k}\cdot\mathbf{k}=1 \end{array} \tag{1.13}$$

These nine scalar product equations can be represented by the single index equation

$$\mathbf{e}_i \cdot \mathbf{e}_j = \delta_{ij}, \quad i, j = 1, 2, 3 \tag{1.14}$$

We identify $\mathbf{e}_1 = \mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$, and $\mathbf{e}_3 = \mathbf{k}$ and define the **Kronecker delta** symbol by

$$\delta_{ij} = \delta_{ji} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{1.15}$$

In the same spirit, we write (1.12) in the form

$$\mathbf{u} = u_1\mathbf{e}_1 + u_2\mathbf{e}_2 + u_3\mathbf{e}_3 = \sum_{i=1}^{3} u_i\mathbf{e}_i \tag{1.16}$$

or, more simply,

$$\mathbf{u} = u_i\mathbf{e}_i, \quad i = 1, 2, 3 \tag{1.17}$$

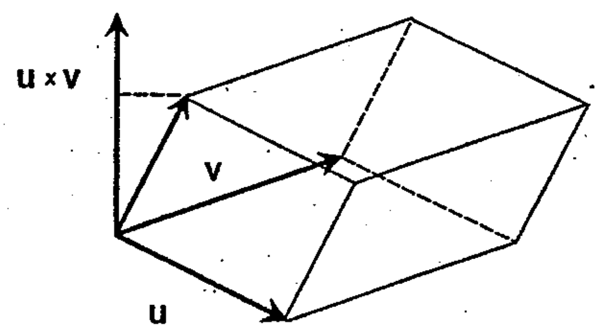

Figure 1.7.Geometrical representation of the triple scalar product $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$.

The index expression on the right of (1.22), summed on i, j, and k, represents the sum of 3^3 terms, only six of which are non-zero. In fact, (1.22) *defines* the expansion of the determinant

$$\mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{vmatrix} \tag{1.24}$$

Note that the components of the vectors u, v, and w can appear either in rows or columns. The value of the determinant, calculated by the usual rules, is the same. The scalar triple product of u, v, w is frequently denoted by $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] = \mathbf{u} \times \mathbf{v} \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{v} \times \mathbf{w}$.

The expression $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ or $\mathbf{u} \times (\mathbf{v} \times \mathbf{w})$ is called a **vector triple product**. The vector triple product is not associative. For example,

$$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

$$(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = \mathbf{0} \times \mathbf{k} = \mathbf{0}$$

We use $\mathbf{0}$ to distinguish the zero vector from the zero number 0. The vector triple product can be expressed as the difference of two vectors.

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\, \mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\, \mathbf{w}$$

The validity of this expression can be easily demonstrated using indicial notation. Let $\mathbf{u} = u_i \mathbf{e}_i$ and $\mathbf{v} \times \mathbf{w} = \varepsilon_{ijk} v_i w_j \mathbf{e}_k$. Then, changing the index letter in u to avoid ambiguity, we have

$$\begin{aligned} \mathbf{u} \times (\mathbf{v} \times \mathbf{w}) &= u_p \varepsilon_{ijk} v_i w_j \mathbf{e}_p \times \mathbf{e}_k = \varepsilon_{ijk} \varepsilon_{pkq} v_i w_j u_p \mathbf{e}_q \\ &= -\varepsilon_{ijk} \varepsilon_{pqk} v_i w_j u_p \mathbf{e}_q \end{aligned}$$

Now using $(1.21)_1$, we obtain

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\delta_{iq}\delta_{jp} - \delta_{ip}\delta_{jk})\, v_i w_j u_p \mathbf{e}_q$$

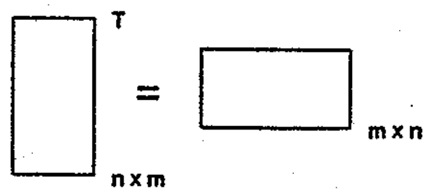

Figure 2.2.Schemetic of the transpose of an $n \times m$ matrix.

Example 1 *Let* A *be a* 2×3 *matrix*

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \end{bmatrix}$$

Then, the transpose of A, A^T *is a* 3×2 *matrix with elements*

$$A^T = \begin{bmatrix} A_{11} & A_{21} \\ A_{12} & A_{22} \\ A_{13} & A_{23} \end{bmatrix}$$

A matrix A is said to be symmetric if $A^T = A$ or $A_{ij} = A_{ji}$. A matrix A is said to be skew-symmetric if $A^T = -A$ or $A_{ij} = -A_{ji}$. Consider the case where $i = j = 1$, then $A^T_{11} = -A_{11}$ and hence $A_{11} = 0$. Similarly, $A^T_{22} = A^T_{33} = 0$. The concepts of symmetry and skew symmetry apply only to square matrices. Two matrices are equal if their elements are equal.

$$A = B \text{ implies } A_{ij} = B_{ij}$$

The zero matrix is the matrix all of whose elements are zero.

Example 2 *A symmetric matrix* A *has components*

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{12} & A_{22} & A_{23} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

A skew symmetric matrix A *has components*

$$A = \begin{bmatrix} 0 & A_{12} & A_{13} \\ -A_{12} & 0 & A_{23} \\ -A_{13} & -A_{23} & 0 \end{bmatrix}$$

The addition and multiplication of matrices are defined as follows. Addition of matrices is only valid for matrices with the same number of rows and columns.

we see that det A is the completely contracted product of the components of five tensors. This is a scalar, a tensor of order zero. This implies that det A is a number that is the same for all representations of A.

We can also consider this in another way. We have

$$A^*_{ij} = Q_{ri}Q_{sj}A_{rs} = Q_{ri}A_{rs}Q_{sj}$$

In matrix notation, this has the form

$$\mathsf{A}^* = \mathsf{Q}^\mathsf{T}\mathsf{A}\mathsf{Q}$$

and

$$\det \mathsf{A}^* = \det\left(\mathsf{Q}^\mathsf{T}\mathsf{A}\mathsf{Q}\right) = \det\left(\mathsf{Q}^\mathsf{T}\right)\ \det(\mathsf{A})\ \det(\mathsf{Q}) = \det \mathsf{A}$$

since $\det(\mathsf{Q}) = \det\left(\mathsf{Q}^\mathsf{T}\right) = 1$.

We considered the scalar invariants of the tensor **A** and adopted the notation

$$\begin{aligned} \mathrm{I}_\mathbf{A} &= \mathrm{tr}\,\mathbf{A} = A_{11} + A_{22} + A_{33} \\ \mathrm{II}_\mathbf{A} &= \tfrac{1}{2}\left[(\mathrm{tr}\,\mathbf{A})^2 - \mathrm{tr}\left(\mathbf{A}^2\right)\right] \\ \mathrm{III}_\mathbf{A} &= \det\mathbf{A} \end{aligned} \tag{3.67}$$

3.5 The Algebraic Eigenvalue Problem

Let **A** be a second order tensor. We consider solutions of the equation

$$\mathbf{Ar} = \lambda\mathbf{r} \tag{3.68}$$

where λ is a scalar. If we write $\mathbf{r} = x_i\mathbf{e}_i$, then the index form of (3.68) is

$$(A_{ij} - \lambda\delta_{ij})\,x_j = 0 \tag{3.69}$$

There exist non-zero solutions of (3.69) $\{x_1, x_2, x_3\}$ if and only if

$$\det(A_{ij} - \lambda\delta_{ij}) = 0$$

Expansion of this equation leads to the characteristic equation for the tensor **A**.

$$\lambda^3 - \mathrm{I}_\mathbf{A}\lambda^2 + \mathrm{II}_\mathbf{A}\lambda - \mathrm{III}_\mathbf{A} = 0 \tag{3.70}$$

Since the coefficients of (3.70) are scalar invariants of **A**, equation (3.70) holds for all reference frames. Solutions of (3.68) will exist only for the three roots λ_1, λ_2, λ_3 of (3.70). Since (3.70) is a cubic, there is at least one real root. The other two roots may or may not be complex. The roots are called eigenvalues (proper values, characteristic values, or principal values) and the corresponding vectors **r** are called eigenvectors.

If **A** is symmetric, then the characteristic equation will always have three real roots λ, and there will exist three mutually perpendicular eigenvectors **r**. This is demonstrated in the following way. Let $\lambda = \alpha + i\beta$ and $\mathbf{r} = \mathbf{u} + i\mathbf{v}$. Here $i = \sqrt{-1}$ is the complex constant.

order tensor $\mathbf{A}$. The tensor $\mathbf{A}$ is

$$\mathbf{A} = \mathbf{B} \otimes \mathbf{u} = (B_{ij}\mathbf{e}_i \otimes \mathbf{e}_j) \otimes u_k \mathbf{e}_k = B_{ij}u_k (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k)$$

The components of $\mathbf{A}$ are

$$A_{ijk} = B_{ij}u_k$$

Similarly, if $\mathbf{A}$ and $\mathbf{B}$ are second order tensors, the outer product is a fourth order tensor $\mathbf{E}$.

$$\mathbf{E} = \mathbf{A} \otimes \mathbf{B} = (A_{ij}\mathbf{e}_i \otimes \mathbf{e}_j) \otimes (B_{kl}\mathbf{e}_k \otimes \mathbf{e}_k) = A_{ij}B_{kl} (\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_k)$$

This gives the component form as

$$E_{ijkl} = A_{ij}B_{kl}$$

We note that a fourth order tensor is a linear transformation that maps second order tensors into second order tensors. Notationally, we write this as

$$\mathbf{T} = \mathsf{C}[\mathbf{E}]$$

Here $\mathbf{T}$ and $\mathbf{E}$ are second order tensors and C is a fourth order tensor. To see how these mappings operate, we define the following tensor product operator. Let $\mathbf{a}$, $\mathbf{b}$, $\mathbf{s}$, $\mathbf{t}$, $\mathbf{u}$, $\mathbf{v}$ be vectors. Then $[\mathbf{s} \otimes \mathbf{t} \otimes \mathbf{u} \otimes \mathbf{v}]$ forms a fourth order tensor and $[\mathbf{a} \otimes \mathbf{b}]$ is a second order tensor. The operation of the fourth order tensor on the second order tensor is defined by

$$[\mathbf{s} \otimes \mathbf{t} \otimes \mathbf{u} \otimes \mathbf{v}][\mathbf{a} \otimes \mathbf{b}] = (\mathbf{u} \cdot \mathbf{a})(\mathbf{v} \cdot \mathbf{b})[\mathbf{s} \otimes \mathbf{t}]$$

This has particular significance when the vectors are all base orthonormal base vectors. Then we have

$$\begin{aligned}[\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l][\mathbf{e}_m \otimes \mathbf{e}_p] &= (\mathbf{e}_k \cdot \mathbf{e}_m)(\mathbf{e}_l \cdot \mathbf{e}_p)[\mathbf{e}_i \otimes \mathbf{e}_j] \\ &= \delta_{km}\delta_{lp}[\mathbf{e}_i \otimes \mathbf{e}_j]\end{aligned}$$

The operation of the fourth order tensor C on $\mathbf{E}$ can be written as

$$\begin{aligned}T_{ij}(\mathbf{e}_i \otimes \mathbf{e}_j) &= \mathbf{T} = \mathsf{C}[\mathbf{E}] = C_{ijkl}[\mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k \otimes \mathbf{e}_l] E_{mp}[\mathbf{e}_m \otimes \mathbf{e}_p] \\ &= C_{ijkl}E_{mp}\delta_{km}\delta_{lp}[\mathbf{e}_i \otimes \mathbf{e}_j] = C_{ijkl}E_{kl}[\mathbf{e}_i \otimes \mathbf{e}_j]\end{aligned}$$

And the components form of the operation is

$$T_{ij} = C_{ijkl}E_{kl}$$

These types of mappings are important in the theories of behavior of fluids and elastic solids.

3.2.2 Contraction

Contraction is the operation of equating any two indices in the components of a tensor of *and*

$$\det \mathbf{Q} = 1$$

This demonstrates that Q *is proper orthogonal.*

A change of basis is equivalent to a change of reference frame and induces a change in the components of a position vector or, what is the same thing, a change in the coordinates of a point. The transformation formula is obtained as follows. We write $\mathbf{r} = x_j^* \mathbf{e}_j^* = x_i \mathbf{e}_i$. Using (2.62) in this expression, we find

$$x_j^* \mathbf{e}_j^* = x_i \mathbf{Q}^T \mathbf{e}_i^* = x_i Q_{ij} \mathbf{e}_j^* \quad \text{or} \quad x_j^* = Q_{ij} x_i \tag{2.63}$$

Similarly, we find that the components u_i of a vector or T_{ij} of a second order tensor transform according to the rules

$$\begin{aligned} u_i^* &= Q_{ji} u_j \\ T_{ij}^* &= Q_{ri} Q_{sj} T_{rs} \end{aligned} \tag{2.64}$$

In general, if $A_{i_1 i_2 \cdots i_r}$ are the components of a tensor of order r, then the identity

$$\mathbf{A} = A^*_{i_1 i_2 \cdots i_r} \mathbf{e}^*_{i_1} \otimes \mathbf{e}^*_{i_1} \otimes \cdots \otimes \mathbf{e}^*_{i_r} = A_{j_1 j_2 \cdots j_r} \mathbf{e}_{j_1} \otimes \mathbf{e}_{j_1} \otimes \cdots \otimes \mathbf{e}_{j_r} \tag{2.65}$$

leads to the transformation formula

$$A^*_{i_1 i_2 \cdots i_r} = Q_{j_1 i_1} Q_{j_2 i_2} \cdots Q_{j_r i_r} A_{j_1 j_2 \cdots j_r} \tag{2.66}$$

Conversely, if the ordered set of numbers transforms according to the pattern (2.66) following a change of basis (2.60), then the identity (2.65) holds, and we can say that $A_{j_1 j_2 \cdots j_r}$ represents a tensor of order r. Thus (2.66) is not only a formula for transforming tensor components, it is also a test of whether the set $A_{j_1 j_2 \cdots j_r}$ is the representation of a tensor. It is customary to speak of the set of components u_i as a vector, or the components T_{ij} as a tensor of order two, etc. This means that these sets of components represent the vector or tensor relative to a reference frame with a given orthonormal set of base vectors.

We write the identity

$$\mathbf{A} = \frac{1}{2}\left(\mathbf{A}+\mathbf{A}^T\right)+\frac{1}{2}\left(\mathbf{A}-\mathbf{A}^T\right) \tag{2.57}$$

The first tensor on the right is symmetric and the second is skew-symmetric.

2.4.5 Tensor Product Properties

The tensor product has the following properties.

TP1. $(\mathbf{a}\otimes\mathbf{b})^T = \mathbf{b}\otimes\mathbf{a}$

TP2. $(\mathbf{a}\otimes\mathbf{b})(\mathbf{c}\otimes\mathbf{d}) = (\mathbf{b}\cdot\mathbf{c})\,\mathbf{a}\otimes\mathbf{d}$

TP3. $(\mathbf{e}_i\otimes\mathbf{e}_i)(\mathbf{e}_j\otimes\mathbf{e}_j) = \begin{cases} 0 & i\neq j \\ \mathbf{e}_i\otimes\mathbf{e}_i & i=j \end{cases}$

TP4. $\sum_i \mathbf{e}_i\otimes\mathbf{e}_i = \mathbf{I}$

2.4.6 Matrix of a Tensor

The matrix of a tensor is

$$[\mathbf{S}] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix}$$

The rules of matrix operations apply, and we may write

$$\begin{aligned} [\mathbf{S}^T] &= [\mathbf{S}]^T \\ [\mathbf{ST}] &= [\mathbf{S}][\mathbf{T}] \end{aligned}$$

The matrix of the identity tensor is

$$[\mathbf{I}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The operations of 3×3 matrices and tensors are equivalent. Vectors can be represented by column matrices. We record some equivalent operations for matrices, tensors and tensor components.

$\mathsf{Au} = \mathsf{v}$	$\mathbf{Au} = \mathbf{v}$	$A_{ij}u_j = v_i$
$\mathsf{AB} = \mathsf{C}$	$\mathbf{AB} = \mathbf{C}$	$A_{ij}B_{jk} = C_{ik}$
$\mathsf{A}^\mathsf{T}\mathsf{B} = \mathsf{C}$	$\mathbf{A}^T\mathbf{B} = \mathbf{C}$	$A_{ji}B_{jk} = C_{ik}$
$\mathsf{A} = \mathsf{uv}^\mathsf{T}$	$\mathbf{A} = \mathbf{u}\otimes\mathbf{v}$	$A_{ij} = u_iv_j$

A tensor $\mathbf{T}$ has an inverse denoted by $\mathbf{T}^{-1}$ such that

$$\mathbf{TT}^{-1} = \mathbf{T}^{-1}\mathbf{T} = \mathbf{I}$$

The inverse is unique and exists if $\det \mathbf{T} \neq 0$. Compare this with the inverse of a matrix. The

2.3 The Tensor Product

The tensor product[2] of two vectors $\mathbf{u}$, $\mathbf{v}$ in $\mathcal{V}$ is denoted by $\mathbf{u} \otimes \mathbf{v}$. The tensor product defines a linear transformation by the requirement that

$$(\mathbf{u} \otimes \mathbf{v})\,\mathbf{w} = (\mathbf{v} \cdot \mathbf{w})\,\mathbf{u} \tag{2.49}$$

for any vector $\mathbf{w}$ in $\mathcal{V}$. The tensor product is also called a tensor of order two. Equation (2.49) defines a linear transformation because

$$\begin{aligned}(\mathbf{u} \otimes \mathbf{v})(\mathbf{w}+\mathbf{x}) &= [\mathbf{v}\cdot(\mathbf{w}+\mathbf{x})]\,\mathbf{u} = (\mathbf{v}\cdot\mathbf{w}+\mathbf{v}\cdot\mathbf{x})\,\mathbf{u} \\ &= (\mathbf{v}\cdot\mathbf{w})\,\mathbf{u} + (\mathbf{v}\cdot\mathbf{x})\,\mathbf{u} \\ &= (\mathbf{u}\otimes\mathbf{v})\,\mathbf{w} + (\mathbf{u}\otimes\mathbf{v})\,\mathbf{x}\end{aligned}$$

and

$$\begin{aligned}(\mathbf{u} \otimes \mathbf{v})(\alpha\mathbf{w}) &= (\mathbf{v}\cdot\alpha\mathbf{w})\,\mathbf{u} = \alpha(\mathbf{v}\cdot\mathbf{w})\,\mathbf{u} \\ &= \alpha(\mathbf{u}\otimes\mathbf{v})\,\mathbf{w}\end{aligned}$$

The tensor product of two vectors $\mathbf{u} \otimes \mathbf{v}$ has the following properties:

$$\begin{aligned}&\text{(i)} && \mathbf{u}\otimes(\mathbf{v}+\mathbf{w}) = \mathbf{u}\otimes\mathbf{v}+\mathbf{u}\otimes\mathbf{w} \\ &\text{(ii)} && (\mathbf{u}+\mathbf{v})\otimes\mathbf{w} = \mathbf{u}\otimes\mathbf{w}+\mathbf{v}\otimes\mathbf{w} \\ &\text{(iii)} && \mathbf{u}\otimes(\alpha\mathbf{v}) = (\alpha\mathbf{u})\otimes\mathbf{v} = \alpha(\mathbf{u}\otimes\mathbf{v})\end{aligned} \tag{2.50}$$

These properties follow from the definition of the tensor product (2.49). To prove (i), for example, consider

$$\begin{aligned}[\mathbf{u}\otimes(\mathbf{v}+\mathbf{w})]\,\mathbf{x} &= [(\mathbf{v}+\mathbf{w})\cdot\mathbf{x}]\,\mathbf{u} = (\mathbf{v}\cdot\mathbf{x})\,\mathbf{u}+(\mathbf{w}\cdot\mathbf{x})\,\mathbf{u} \\ &= (\mathbf{u}\otimes\mathbf{v})\,\mathbf{x}+(\mathbf{u}\otimes\mathbf{w})\,\mathbf{x}\end{aligned}$$

for any $\mathbf{x}$ in $\mathcal{V}$. It follows that (i) is true.

Every tensor product $\mathbf{u} \otimes \mathbf{v}$ is by definition a linear transformation or tensor of order two, but not every linear transformation or tensor of order two can be expressed as the tensor product of two vectors. If $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ is a basis for $\mathcal{V}$, it is a non-coplanar set with $[\mathbf{u}\ \mathbf{v}\ \mathbf{w}] \neq 0$. Then, a linear transformation $\mathbf{T} : \mathcal{V} \to \mathcal{V}$ is uniquely defined by the assignments $\mathbf{T}(\mathbf{u}) = \mathbf{a}$, $\mathbf{T}(\mathbf{v}) = \mathbf{b}$, $\mathbf{T}(\mathbf{w}) = \mathbf{c}$ where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are any three vectors in $\mathcal{V}$. If we define the sum and scalar multiplication of linear transformations by

$$\begin{aligned}[\mathbf{T}+\mathbf{S}](\mathbf{u}) &= \mathbf{T}(\mathbf{u})+\mathbf{S}(\mathbf{u}) \\ [\alpha\mathbf{T}](\mathbf{u}) &= \alpha\mathbf{T}(\mathbf{u})\end{aligned}$$

then the general tensor product form of a linear transformation is given by

$$\mathbf{T} = \mathbf{a}\otimes\mathbf{u}^{*} + \mathbf{b}\otimes\mathbf{v}^{*} + \mathbf{c}\otimes\mathbf{w}^{*} \tag{2.51}$$

[2] The tensor product of two vectors is sometimes called the dyadic product. Some authors use the notation $\mathbf{uv}$ for the tensor product.

Since $\mathbf{Ar} = \lambda\mathbf{r}$, we have $\mathbf{A}(\mathbf{u} + i\mathbf{v}) = (\alpha + i\beta)(\mathbf{u} + i\mathbf{v})$. We separate this equation into real and imaginary parts.

$$\begin{aligned}\mathbf{Au} &= \alpha\mathbf{u} - \beta\mathbf{v} \\ \mathbf{Av} &= \beta\mathbf{u} - \alpha\mathbf{v}\end{aligned}$$

Since $\mathbf{A}$ is symmetric, $\mathbf{v}\cdot(\mathbf{Au}) = \mathbf{u}\cdot(\mathbf{Av})$. This gives $\beta\left(\mathbf{u}^2 + \mathbf{v}^2\right) = 0$. Since $\mathbf{u}^2 + \mathbf{v}^2 \neq 0$, β must be zero. Thus the roots λ are real and hence so are the corresponding vectors $\mathbf{r}$.

Let λ_1, λ_2 be two distinct roots with eigenvectors $\mathbf{r}_1$ and $\mathbf{r}_2$. Then

$$\begin{aligned}\mathbf{Ar}_1 &= \lambda_1\mathbf{r}_1 \\ \mathbf{Ar}_2 &= \lambda_2\mathbf{r}_2\end{aligned}$$

From this pair we obtain

$$\begin{aligned}\lambda_1\mathbf{r}_1\cdot\mathbf{r}_2 - \lambda_2\mathbf{r}_2\cdot\mathbf{r}_1 &= 0 \\ \mathbf{r}_1\cdot\mathbf{r}_2(\lambda_1 - \lambda_2) &= 0\end{aligned}$$

Since λ_1 and λ_2 are distinct, it follows that $\mathbf{r}_1$ is perpendicular to $\mathbf{r}_2$. Thus, three distinct roots $\lambda_1, \lambda_2, \lambda_3$ lead to three distinct mutually perpendicular eigenvectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$.

We note that if $\mathbf{r}$ is a solution of (3.68), then $\alpha\mathbf{r}$, where α is any real number, is also a solution. Thus, the eigenvectors are determined only to within a multiplicative constant. By adjusting this constant, we can make the three eigenvectors $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3$ form a right hand orthonormal set $\mathbf{e}_1^*, \mathbf{e}_2^*, \mathbf{e}_3^*$. This calculation leads to $\mathbf{e}_i^* = Q_{ji}\mathbf{e}_j$, where $\mathbf{Q}$ is a proper orthogonal transformation matrix. If we express $\mathbf{A}$ in terms of the eigenvectors $\mathbf{e}_i^*$ as a basis, then $\mathbf{A}$ will have the form

$$\mathbf{A} = \lambda_1\mathbf{e}_1^*\otimes\mathbf{e}_1^* + \lambda_2\mathbf{e}_2^*\otimes\mathbf{e}_2^* + \lambda_3\mathbf{e}_3^*\otimes\mathbf{e}_3^* \tag{3.71}$$

This gives $\mathbf{Ae}_1^* = \lambda_1\mathbf{e}_1^*$, etc. and uniquely defines $\mathbf{A}$ according to the fundamental theorem on linear transformations.

Example 4 *Let the matrix of the components of a tensor be*

$$[\mathbf{A}] = \begin{bmatrix} 5 & -10 & 8 \\ -10 & 2 & 2 \\ 8 & 2 & 11 \end{bmatrix}$$

We calculate $I_{\mathbf{A}} = 18$, $II_{\mathbf{A}} = -81$, and $III_{\mathbf{A}} = -1458$. The characteristic equation

$$\lambda^3 - 18\lambda^2 - 81\lambda - 1458 = 0$$

has the roots $\lambda_1 = -9$, $\lambda_2 = 9$, and $\lambda_3 = 18$. To find the eigenvector $\mathbf{r}_1$, corresponding to the eigenvalue λ_1, we substitute into the equation

$$(A_{ij} - \lambda\delta_{ij})\,x_j = 0$$

3.7 The Polar Decomposition Theorem

Let $\mathbf{F}$ be any second order tensor such that $III_{\mathbf{F}} = \det \mathbf{F} > 0$. The tensor $\mathbf{F}^T\mathbf{F}$ is symmetric. Let the eigenvectors of $\mathbf{F}^T\mathbf{F}$ be $\mathbf{e}_i$, a right hand orthonormal set. We write

$$\mathbf{U}^2 = \mathbf{F}^T\mathbf{F} = \lambda_1^2 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2^2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3^2 \mathbf{e}_3 \otimes \mathbf{e}_3 \tag{3.72}$$

To justify this notation we must show that $\mathbf{F}^T\mathbf{F}$ is positive definite. We have $\det(\mathbf{F}^T\mathbf{F}) = (\det \mathbf{F})^2 = \lambda_1^2\lambda_2^2\lambda_3^2 > 0$. Hence, none of the eigenvalues can be zero. We show next that they are all positive. Consider the set of vectors $\mathbf{Fe}_1, \mathbf{Fe}_2, \mathbf{Fe}_3$. Then

$$\mathbf{Fe}_i \cdot \mathbf{Fe}_j = \mathbf{e}_i \cdot (\mathbf{F}^T\mathbf{Fe}_j) = 0$$

for $i \neq j$. This shows that the set is orthogonal. For $i = j$, we have

$$\mathbf{Fe}_i \cdot \mathbf{Fe}_i = \mathbf{e}_i \cdot (\mathbf{F}^T\mathbf{Fe}_i) = \lambda_i^2 > 0$$

for $i = 1, 2, 3$ (i not summed). Thus, $\mathbf{U}^2$ is positive definite. Now we write

$$\mathbf{Fe}_i = \lambda_i \mathbf{e}_i^* \quad (i \text{ not summed}) \tag{3.73}$$

where $\mathbf{e}_i^*$ is an orthonormal set, and λ_i is the positive square root of the eigenvalues λ_i^2. Finally, we show that the set $\mathbf{e}_i^*$ is the right hand set if $\mathbf{e}_i$ is right hand. From (3.73) it follows that

$$\mathbf{F} = \lambda_1 \mathbf{e}_1^* \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2^* \otimes \mathbf{e}_2 + \lambda_3 \mathbf{e}_3^* \otimes \mathbf{e}_3 \tag{3.74}$$

If we define a proper orthogonal transformation by

$$\mathbf{R} = \mathbf{e}_1^* \otimes \mathbf{e}_1 + \mathbf{e}_2^* \otimes \mathbf{e}_2 + \mathbf{e}_3^* \otimes \mathbf{e}_3 \tag{3.75}$$

and use

$$\mathbf{U} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2 + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3 \tag{3.76}$$

from (3.72), then

$$\mathbf{F} = \mathbf{RU} \tag{3.77}$$

where $\mathbf{U}$ is a positive definite symmetric tensor. From (3.74), we have

$$\mathbf{F}^T = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1^* + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2^* + \lambda_3 \mathbf{e}_3 \otimes \mathbf{e}_3^*$$

Let

$$\begin{aligned} \mathbf{V}^2 = \mathbf{FF}^T &= \lambda_1^2 \mathbf{e}_1^* \otimes \mathbf{e}_1^* + \lambda_2^2 \mathbf{e}_2^* \otimes \mathbf{e}_2^* + \lambda_3^2 \mathbf{e}_3^* \otimes \mathbf{e}_3^* \\ \mathbf{V} &= \lambda_1 \mathbf{e}_1^* \otimes \mathbf{e}_1^* + \lambda_2 \mathbf{e}_2^* \otimes \mathbf{e}_2^* + \lambda_3 \mathbf{e}_3^* \otimes \mathbf{e}_3^* \end{aligned} \tag{3.78}$$

Now we can represent $\mathbf{F}$ in the form

$$\mathbf{F} = \mathbf{VR} \tag{3.79}$$

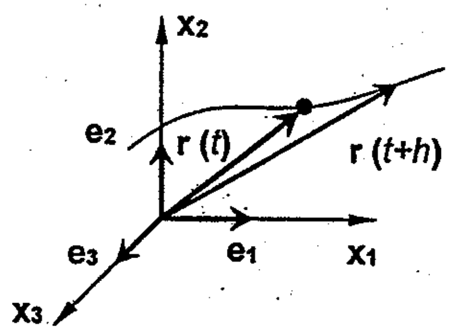

Figure 4.1.The trajectory of a particle and the postion vectors $\mathbf{r}(t)$ and $\mathbf{r}(t+h)$ in a rectangular basis $\mathbf{e}_i$.

4.1.2 Product Rule for Functions

Often, we need to compute the time derivative of the product of two functions. Examples of these products include products of a scalar and a vector, inner products, vector products and tensor products. We define a general product of two functions $\mathbf{f}(t)$, $\mathbf{g}(t)$ by $\mathbf{h}(t) = \pi(\mathbf{f}(t), \mathbf{g}(t))$. The functions $\mathbf{f}(t)$ and $\mathbf{g}(t)$ must have a common domain of definition. Then, the derivative of the product $\mathbf{h}(t)$ with respect to time is

$$\dot{\mathbf{h}}(t) = \pi\left(\dot{\mathbf{f}}(t), \mathbf{g}(t)\right) + \pi(\mathbf{f}(t), \dot{\mathbf{g}}(t))$$

This is analogous to the product rule for scalar functions of a single variable. The product rule allows us to write the following relations.

$$\begin{aligned}(\phi\mathbf{v})^{\cdot} &= \left(\dot{\phi}\mathbf{v}\right) + (\phi\dot{\mathbf{v}}) \\ (\mathbf{u}\cdot\mathbf{v})^{\cdot} &= (\dot{\mathbf{u}}\cdot\mathbf{v}) + (\mathbf{u}\cdot\dot{\mathbf{v}}) \\ (\mathbf{T}\mathbf{v})^{\cdot} &= \left(\dot{\mathbf{T}}\mathbf{v}\right) + (\mathbf{T}\dot{\mathbf{v}}) \\ (\mathbf{T}\mathbf{S})^{\cdot} &= \left(\dot{\mathbf{T}}\mathbf{S}\right) + \left(\mathbf{T}\dot{\mathbf{S}}\right)\end{aligned}$$

4.2 Scalar, Vector, and Tensor Fields

We now consider scalars that are functions of the coordinates $\{x_i\}$. In rectangular coordinate systems, the coordinates of a point are also the components of a position vector. Hence, we denote a point (x_1, x_2, x_3) by $\mathbf{r}$ or $\mathbf{x}$. Thus, if ϕ is a scalar-valued function of $\{x_i\}$, we write $\phi(\mathbf{r})$ or $\phi(\mathbf{x})$. The most general linear function has the form $\phi(\mathbf{x}) = \alpha_i x_i$. In this case there is no restriction on the domain of $\mathbf{x}$. All values of $\mathbf{x}$ are admissible. However, if ϕ is not a

Comparing these expressions shows that

$$Df(\mathbf{v})[\mathbf{h}] = \text{grad} f(\mathbf{v})[\mathbf{h}] = 2\mathbf{v} \cdot \mathbf{h}$$

We now express the derivative $Df(\mathbf{x})[\mathbf{u}]$ in terms of the components u_i of $\mathbf{u}$ and the coordinates x_i of $\mathbf{x}$ in a Cartesian basis. From the linearity of $Df(\mathbf{x})[\mathbf{u}]$ in $\mathbf{u}$, we have

$$Df(\mathbf{x})[\mathbf{u}] = Df(\mathbf{x})[u_i \mathbf{e}_i] = u_i Df(\mathbf{x})[\mathbf{e}_i]$$

By definition, this last term is

$$Df(\mathbf{x})[\mathbf{e}_i] = \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{e}_i) - f(\mathbf{x})}{h}$$

We recall that $f(\mathbf{x})$ means $f(x_1, x_2, x_3)$. Consider the case for $i = 1$,

$$f(\mathbf{x} + h\mathbf{e}_1) = f(x_1 + h, x_2, x_3)$$

and

$$\begin{aligned} \lim_{h \to 0} \frac{f(\mathbf{x} + h\mathbf{e}_1) - f(\mathbf{x})}{h} &= \lim_{h \to 0} \frac{f(x_1 + h, x_2, x_3) - f(x_1, x_2, x_3)}{h} \\ &= \frac{\partial f}{\partial x_1} \end{aligned}$$

This is the definition of the partial derivative in the x_1 direction. We repeat this for the remaining directions and find that

$$Df(\mathbf{x})[\mathbf{e}_i] = \frac{\partial f}{\partial x_i} = f_{,i}$$

This is the usual set of partial derivatives.

We have introduced a new notation. We have employed a comma to denote partial differentiation with respect to a coordinate direction. This notation allows us to write indicial expressions involving differentiation compactly.

Now we have

$$Df(\mathbf{x})[\mathbf{u}] = u_i Df(\mathbf{x})[\mathbf{e}_i] = u_i \frac{\partial f}{\partial x_i} = f_{,i} u_i$$

This result with our definition of the gradient gives

$$Df(\mathbf{x})[\mathbf{u}] = \text{grad} f(\mathbf{x}) \cdot \mathbf{u} = \nabla f(\mathbf{x}) \cdot \mathbf{u}$$

and we have in indicial form

$$Df(\mathbf{x})[\mathbf{u}] = u_i \frac{\partial f}{\partial x_i} = f_{,i} u_i$$

This suggests that grad $f(\mathbf{x})$ is the vector $\frac{\partial f}{\partial x_i}\mathbf{e}_i$ in an orthonormal basis.

We note that the components of the gradient are grad $f(\mathbf{x}) \cdot \mathbf{e}_i = \nabla f(\mathbf{x}) \cdot \mathbf{e}_i = Df(\mathbf{x})[\mathbf{e}_i]$. These are just the directional derivatives of $f(\mathbf{x})$ in the directions of the coordinate bases.

Equation (4.84) is an example of the functional notation and the equivalent scalar product and matrix operations that we have been using.

4.4.1 Divergence of a Vector Field

In many applications, parts of the derivative of a vector field are significant. We will define the divergence of a vector field, a scalar field, and curl of the vector field, a vector field, using the gradient. Many texts define these derivatives through the operations that are used to calculate them. Here, we have presented a single derivative of a vector field. Our hope is to make it clear that a vector field has only one derivative. The definition of this derivative is consistent with our idea of a derivative for scalar valued functions.

Since $D\mathbf{f}(\mathbf{x})$ or $\nabla\mathbf{f}$ is a tensor of order two, it will have the three scalar invariants that we have already discussed, Section (3.4). In the present case, these quantities are functions of $\mathbf{x}$. One invariant of particular interest is the first invariant, the trace. This is called the divergence of the vector field and is written as.

$$\operatorname{div}\mathbf{f} = \operatorname{tr}(\nabla\mathbf{f})$$

The divergence is a scalar field. The component representation of the divergence is

$$\operatorname{div}\mathbf{f} = \operatorname{tr}\left(\frac{\partial f_i}{\partial x_j}\mathbf{e}_i \otimes \mathbf{e}_j\right) = f_{i,i}$$

This is a contraction of the tensor field $\nabla\mathbf{f}$. We will give an alternate way to obtain this expression using operator notation in a later section.

The divergence is of significance in fluid mechanics. When the vector field is the velocity, a physical interpretation is possible. The divergence of the velocity field for a steady incompressible flow is a measure of the source intensity of the flow at a point. There are no sources at a point if $\operatorname{div}\mathbf{v} = 0$.

4.4.2 Curl of a Vector Field

The curl of a vector $\mathbf{f}$ is a unique vector field. It is denoted by $\operatorname{curl}\mathbf{f}$. The curl is defined by

$$\operatorname{curl}\mathbf{f} \times \mathbf{a} = \left(\nabla\mathbf{f} - \nabla\mathbf{f}^T\right)\mathbf{a}$$

for every vector $\mathbf{a}$. The curl is the axial vector of the skew-symmetric tensor $\nabla\mathbf{f} - \nabla\mathbf{f}^T$.

The curl is often denoted by $\nabla \times \mathbf{f}$. This notation arises from the operator notation that will be discussed later. It is sometimes referred to as vector invariant of the vector field $\mathbf{f}$.

Like the divergence, the curl also has a physical interpretation. Again, when the vector field is the velocity in a fluid, the curl of the velocity field represents the circulation of the fluid. The curl is a measure of the rate of angular rotation in the neighborhood of the point $\mathbf{x}$.

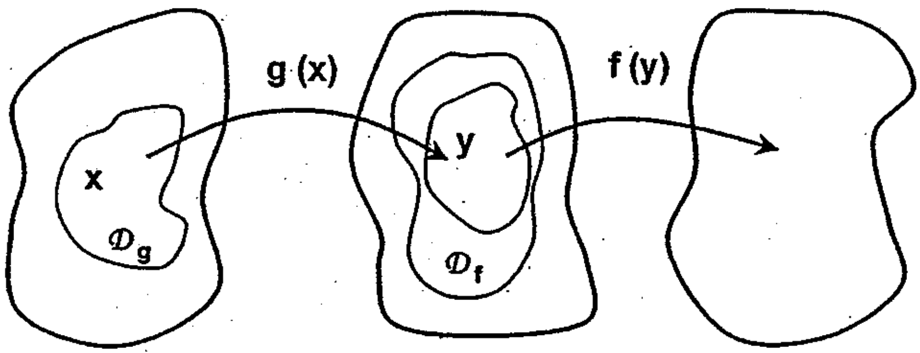

Figure 4.2.The composition of functions $\mathbf{h} = \mathbf{f} \circ \mathbf{g}$ showing that the range of $\mathbf{g}$ lies in the domain $\mathcal{D}_\mathbf{f}$ of $\mathbf{f}$.

4.8 Chain Rule

Another result that is frequently used in the calculus of tensors is the chain rule for derivatives. The chain rule involves the composition of functions. Let $\mathbf{f}$ and $\mathbf{g}$ be two functions where the range of $\mathbf{g}$ is contained in the domain of $\mathbf{f}$. The composition of the functions is written

$$\mathbf{h} = \mathbf{f} \circ \mathbf{g}$$

This is graphically represented in Figure (4.2). We have encountered this notion in calculus. Suppose that $f(x) = \sqrt{x}$ and $g(x) = x^2 + 1$, then $(f \circ g)(x) = f(g(x)) = \sqrt{x^2+1}$.

Let $\mathbf{g}$ be differentiable at $\mathbf{x}$ and let $\mathbf{f}$ be differentiable at $\mathbf{y} = \mathbf{g}(\mathbf{x})$. The composition

$$\mathbf{h} = \mathbf{f} \circ \mathbf{g}$$

is differentiable at $\mathbf{x}$ and the derivative is

$$\begin{aligned} D\mathbf{h}(\mathbf{x})[\mathbf{u}] &= D\mathbf{f}(\mathbf{y}) \circ D\mathbf{g}(\mathbf{x})[\mathbf{u}] \\ &= D\mathbf{f}(\mathbf{g}(\mathbf{x}))[D\mathbf{g}(\mathbf{x})[\mathbf{u}]] \end{aligned}$$

We are more familiar with this when $\mathbf{g}$ and, hence, $\mathbf{h}$ is a function of a real variable t. Then

$$D\mathbf{h}(t)[\alpha] = \alpha\dot{\mathbf{h}}(t) \quad \text{and} \quad D\mathbf{g}(t)[\alpha] = \alpha\dot{\mathbf{g}}(t)$$

and we have

$$\frac{d}{dt}\mathbf{h}(t) = \frac{d}{dt}\mathbf{f}(\mathbf{g}(t)) = D\mathbf{f}(\mathbf{g}(t))\,\dot{\mathbf{g}}(t)$$

Example 10 *Let $f(x) = \sin x$ and let $g(t) = \theta(t)$. The composite function $h(t)$ is*

$$h(t) = (f \circ g)(t) = f(g(t)) = \sin(g(t)) = \sin(\theta(t))$$

This is interpreted as

$$\nabla(\mathbf{u}\times\mathbf{v}) = \nabla\mathbf{u}\times\mathbf{v} - \nabla\mathbf{v}\times\mathbf{u}$$

Using the gradient operator, we have

$$\begin{aligned}\nabla(\mathbf{u}\times\mathbf{v}) &= \mathbf{e}_i\frac{\partial}{\partial x_i}(\mathbf{u}\times\mathbf{v}) = \left(\mathbf{e}_i\frac{\partial \mathbf{u}}{\partial x_i}\times\mathbf{v}\right) + \left(\mathbf{u}\times\mathbf{e}_i\frac{\partial \mathbf{v}}{\partial x_i}\right)\\ &= \nabla\mathbf{u}\times\mathbf{v} + \mathbf{u}\times\nabla\mathbf{v} = \nabla\mathbf{u}\times\mathbf{v} - \nabla\mathbf{v}\times\mathbf{u}\end{aligned}$$

We also use the definition of the derivative and the product rules to obtain the derivative.

$$\begin{aligned}D(\mathbf{u}\times\mathbf{v})[\mathbf{h}] &= D\mathbf{u}[\mathbf{h}]\times\mathbf{v} + \mathbf{u}\times D\mathbf{v}[\mathbf{h}]\\ &= \nabla\mathbf{u}[\mathbf{h}]\times\mathbf{v} + \mathbf{u}\times\nabla\mathbf{v}[\mathbf{h}]\\ &= -\mathbf{v}\times\nabla\mathbf{u}[\mathbf{h}] + \mathbf{u}\times\nabla\mathbf{v}[\mathbf{h}]\end{aligned}$$

Example 17 *Let $\mathbf{v}(\mathbf{x})$ be a continuously differentiable vector field. In this case, partial differentiation is commutative and we have*

$$\begin{aligned}\operatorname{curl}\operatorname{grad}\mathbf{v} &= \nabla\times\nabla\mathbf{v} = \left(\frac{\partial}{\partial x_i}\mathbf{e}_i\right)\times\left(\frac{\partial}{\partial x_j}\mathbf{e}_j\right)\mathbf{v}\\ &= \frac{\partial^2\mathbf{v}}{\partial x_i\partial x_j}\varepsilon_{ijk}\mathbf{e}_k = 0\end{aligned}$$

and

$$\begin{aligned}\operatorname{div}\operatorname{curl}\mathbf{v} &= \nabla\cdot\nabla\times\mathbf{v} = \left(\frac{\partial}{\partial x_i}\mathbf{e}_i\right)\cdot\left(\frac{\partial}{\partial x_j}\mathbf{e}_j\right)\times v_k\mathbf{e}_k\\ &= \frac{\partial^2 v_k}{\partial x_i\partial x_j}\varepsilon_{ijk} = \varepsilon_{ijk}v_{k,ij}\end{aligned}$$

Example 18 *Let $\phi(\mathbf{x})$ and $\mathbf{v}(\mathbf{x})$ be a scalar and vector field both of sufficient smoothness so that all the derivatives exist. The Laplacian of the scalar field is a scalar field defined by*

$$\Delta\phi = \nabla^2\phi = \operatorname{div}(\operatorname{grad}\phi) = \nabla\cdot(\nabla\phi) = \phi_{,ii}$$

For the vector field, the Laplacian is a vector field given by

$$\Delta\mathbf{v} = \nabla^2\mathbf{v} = \operatorname{div}(\operatorname{grad}\mathbf{v}) = \nabla\cdot(\nabla\mathbf{v}) = v_{j,ii}$$

The fields are said to be harmonic when

$$\begin{aligned}\Delta\phi &= 0\\ \Delta\mathbf{v} &= 0\end{aligned}$$

Example 19 *The derivative of the determinant is often required in continuum mechanics. We can obtain this by the following calculation. The determinant has the representation*

$$\det(\mathbf{S}-\lambda\mathbf{I}) = -\lambda^3 + \lambda^2 I_{\mathbf{S}} - \lambda II_{\mathbf{S}} + III_{\mathbf{S}}$$

Chapter 5
Tensors in Physical Theories

Introduction

Ideally, one would like to construct mathematical theories of physical phenomena such as rigid body motion, elasticity, fluid dynamics, thermodynamics and electrodynamics in terms of fundamental equations that have the same form for all possible coordinate systems and all observers. There is a distinction here between "all allowable coordinate systems" and "all observers". We will consider this distinction in the subsequent sections and show that the equations of Newtonian mechanics and, hence, the fundamental balance equations of continuum mechanics are not valid for all observers.

5.1 Tensor Equations

If we are able to express a physical law as a tensor equation in some given coordinate system, then it follows from the rules for the transformation of a tensor that this same tensor form will hold for all allowable coordinate systems. Suppose, for example, that we have the tensor equation $T_{ji,j} + u_i = v_i$ for a coordinate system $\{x_i\}$. If we multiply each term in the above equation by $\left|\frac{\partial x}{\partial y}\right| \frac{\partial x_i}{\partial y_j}$, we obtain the equation $T^*_{ji,i} + u^*_i = v^*_i$ for the new coordinate system y_i. We say that these two equations have the same form. What we have done is to show explicitly what is implied by the coordinate free or symbolic form of the equation, $\operatorname{div}\mathbf{T}^T + \mathbf{u} = \mathbf{v}$. This coordinate free form is useful, but it does not show the detail of representation that is often necessary in computation.
An important question is how do we know whether some indexed set representing a physical quantity is a tensor? The answer to the latter question is that it is an essential part of any well developed physical theory to postulate that certain fundamental quantities are tensors. We will illustrate this later. Any physical theory that does not have such postulates can be said to be in an unsatisfactory state of development. Knowing that certain quantities are tensors, we are then able to prove that other derived quantities are tensors by using the techniques previously developed. There are other quantities, independent of any physical theories, that are natural tensors associated with the geometry of the space and the coordinate systems used. There are also quantities that are simply tensors according to the abstract mathematical definition or coordinate transformation test.

Similarly, the velocity of p measured by O* is

$$\mathbf{v}_p^* = \dot{\mathbf{r}}^* = \dot{x}_k^* \mathbf{e}_k^* = v_i^* \mathbf{e}_i^*$$

Now, differentiating (5.98) with respect to time t, we obtain

$$\begin{aligned} \dot{\mathbf{x}}^* &= \dot{\mathbf{Q}}(t)\,\mathbf{x} + \mathbf{Q}(t)\,\dot{\mathbf{x}} + \dot{\mathbf{c}}(t) \\ \mathbf{v}_p^* &= \mathbf{Q}(t)\,\mathbf{v}_p + \dot{\mathbf{Q}}(t)\,\mathbf{x} + \dot{\mathbf{c}}(t) \end{aligned}$$

or since $\mathbf{x} = \mathbf{Q}^T(t)\,[\mathbf{x}^* - \mathbf{c}(t)]$

$$\begin{aligned} \mathbf{v}_p^* &= \mathbf{Q}(t)\,\mathbf{v}_p + \dot{\mathbf{Q}}(t)\,\mathbf{Q}^T(t)\,[\mathbf{x}^* - \mathbf{c}(t)] + \dot{\mathbf{c}}(t) \\ &= \mathbf{Q}(t)\,\mathbf{v}_p + \boldsymbol{\Omega}^*(t)\,[\mathbf{x}^* - \mathbf{c}(t)] + \dot{\mathbf{c}}(t) \end{aligned} \tag{5.100}$$

This is the velocity transformation formula relating the velocity $\mathbf{v}_p$ observed by O and the velocity $\mathbf{v}_p^*$ observed by O*. The quantity $\mathbf{Q}(t)\,\mathbf{v}_p$ is the velocity of p as seen by the observer O*. The remaining terms are the result of the relative motion of the frames.
The accelerations of point p observed by O and O* are

$$\begin{aligned} \mathbf{a}_p &= \ddot{\mathbf{r}} = \ddot{x}_i \mathbf{e}_i = a_i \mathbf{e}_i \\ \mathbf{a}_p^* &= \ddot{\mathbf{r}}^* = \ddot{x}_i^* \mathbf{e}_i^* = a_i^* \mathbf{e}_i^* \end{aligned}$$

respectively. Differentiating the first expression for $\dot{\mathbf{x}}^*$ once more, we obtain

$$\begin{aligned} \ddot{\mathbf{x}}^* &= \mathbf{Q}(t)\,\ddot{\mathbf{x}} + 2\dot{\mathbf{Q}}(t)\,\dot{\mathbf{x}} + \ddot{\mathbf{Q}}(t)\,\mathbf{x} + \ddot{\mathbf{c}}(t) \\ \mathbf{a}_p^* &= \mathbf{Q}(t)\,\mathbf{a}_p + 2\dot{\mathbf{Q}}(t)\,\mathbf{v}_p + \ddot{\mathbf{Q}}(t)\,\mathbf{x} + \ddot{\mathbf{c}}(t) \end{aligned}$$

From $(5.97)_2$ we obtain (Here, we suppress the arguments for brevity.)

$$\dot{\mathbf{Q}} = \boldsymbol{\Omega}^*\mathbf{Q}$$

Then

$$\begin{aligned} \ddot{\mathbf{Q}} &= \dot{\boldsymbol{\Omega}}^*\mathbf{Q} + \boldsymbol{\Omega}^*\dot{\mathbf{Q}} \\ &= \dot{\boldsymbol{\Omega}}^*\mathbf{Q} + \boldsymbol{\Omega}^*\boldsymbol{\Omega}^*\mathbf{Q} = \dot{\boldsymbol{\Omega}}^*\mathbf{Q} + \boldsymbol{\Omega}^{*2}\mathbf{Q} \end{aligned}$$

Also, from

$$\mathbf{x} = \mathbf{Q}^T(\mathbf{x}^* - \mathbf{c})$$

we conclude that

$$\ddot{\mathbf{Q}}\mathbf{x} = \dot{\boldsymbol{\Omega}}^*(\mathbf{x}^* - \mathbf{c}) + \boldsymbol{\Omega}^{*2}(\mathbf{x}^* - \mathbf{c})$$

Using these results in the expression for $\mathbf{a}_p^*$, we have

$$\mathbf{a}_p^* = \mathbf{Q}\mathbf{a}_p + 2\dot{\mathbf{Q}}\mathbf{v}_p + \dot{\boldsymbol{\Omega}}^*(\mathbf{x}^* - \mathbf{c}) + \boldsymbol{\Omega}^{*2}(\mathbf{x}^* - \mathbf{c}) + \ddot{\mathbf{c}}$$

or

$$\mathbf{a}_p^* = \mathbf{Q}\mathbf{a}_p + 2\boldsymbol{\Omega}^*\mathbf{Q}\mathbf{v}_p + \dot{\boldsymbol{\Omega}}^*(\mathbf{x}^* - \mathbf{c}) + \boldsymbol{\Omega}^{*2}(\mathbf{x}^* - \mathbf{c}) + \ddot{\mathbf{c}}$$

Bibliography

[1] Bowen, Ray M., *Introduction to Continuum Mechanics for Engineers*, Plenum Press, New York, 1989.

[2] Dunholter, Russell, *Lecture Notes, Theory and Practice of Tensor Analysis*, Department of Engineering Analysis, College of Engineering, University of Cincinnati, 1969.

[3] Gurtin, Morton E., *An Introduction to Continuum Mechanics*, Academic Press, New York, 1981.

[4] Knowles, James K., *Linear Vector Spaces and Cartesian Tensors*, Oxford University Press, Inc., Oxford, 1998.

[5] Malvern, Lawrence J., *Introduction to the Mechanics of a Continuous Media*, Prentice Hall. Englewood Cliffs, 1969.

[6] Simmonds, James G., *A Brief on Tensor Analysis, Second Edition*, Springer-Verlag, New York, 1994.