Unit 5: Fractional Factorial Experiments at Two Levels

Source : Chapter 5 (sections 5.1 - 5.3, 5.4.1, 5.5, part of 5.6).

- Effect aliasing, resolution, minimum aberration criteria.
- Analysis.
- Techniques for resolving ambiguities in aliased effects.
- Choice of designs, use of design tables.
- Blocking in 2^{k-p} designs.

Leaf Spring Experiment

- y = free height of spring, target = 8.0 inches.
  Goal : get y as close to 8.0 as possible (nominal-the-best problem).
- Five factors at two levels, use a 16-run design with three replicates for each run. It is a 2^{5-1} design, 1/2 fraction of the 2^5 design.

Table 1: Factors and Levels, Leaf Spring Experiment

		Level	
	Factor	$-$	$+$
B.	high heat temperature (°F)	1840	1880
C.	heating time (seconds)	23	25
D.	transfer time (seconds)	10	12
E.	hold down time (seconds)	2	3
Q.	quench oil temperature (°F)	130-150	150-170

Leaf Spring Experiment: Design Matrix and Data

Table 2: Design Matrix and Free Height Data, Leaf Spring Experiment

Factor										
B	C	D	E	Q	Free Height			$\bar{y}_i$	s_i^2	$\ln s_i^2$
−	+	+	−	−	7.78	7.78	7.81	7.7900	0.0003	-8.1117
+	+	+	+	−	8.15	8.18	7.88	8.0700	0.0273	-3.6009
−	−	+	+	−	7.50	7.56	7.50	7.5200	0.0012	-6.7254
+	−	+	−	−	7.59	7.56	7.75	7.6333	0.0104	-4.5627
−	+	−	+	−	7.94	8.00	7.88	7.9400	0.0036	-5.6268
+	+	−	−	−	7.69	8.09	8.06	7.9467	0.0496	-3.0031
−	−	−	−	−	7.56	7.62	7.44	7.5400	0.0084	-4.7795
+	−	−	+	−	7.56	7.81	7.69	7.6867	0.0156	-4.1583
−	+	+	−	+	7.50	7.25	7.12	7.2900	0.0373	-3.2888
+	+	+	+	+	7.88	7.88	7.44	7.7333	0.0645	-2.7406
−	−	+	+	+	7.50	7.56	7.50	7.5200	0.0012	-6.7254
+	−	+	−	+	7.63	7.75	7.56	7.6467	0.0092	-4.6849
−	+	−	+	+	7.32	7.44	7.44	7.4000	0.0048	-5.3391
+	+	−	−	+	7.56	7.69	7.62	7.6233	0.0042	-5.4648
−	−	−	−	+	7.18	7.18	7.25	7.2033	0.0016	-6.4171
+	−	−	+	+	7.81	7.50	7.59	7.6333	0.0254	-3.6717

Why Using Fractional Factorial Designs?

- If a 2^5 design is used for the experiment, its 31 degrees of freedom would be allocated as follows:

	Main	Interactions			
	Effects	2-Factor	3-Factor	4-Factor	5-Factor
#	5	10	10	5	1

- Using effect hierarchy principle, one would argue that 4fi's, 5fi and even 3fi's are not likely to be important. There are 10+5+1 = 16 such effects, half of the total runs! Using a 2^5 design can be wasteful (unless 32 runs cost about the same as 16 runs.)

- Use of a FF design instead of full factorial design is usually done for economic reasons. Since there is **no free lunch** , what **price to pay**? See next.

Effect Aliasing and Defining Relation

- In the design matrix, col E = col $B\times$ col $C\times$ col D. That means,

$$\bar{y}(E+)-\bar{y}(E-)=\bar{y}(BCD+)-\bar{y}(BCD-).$$

  Therefore the design is not capable of distinguishing E from BCD. The main effect E is **aliased** with the interaction BCD. Notationally,

$$E=BCD \quad \text{or} \quad \mathbf{I}=BCDE,$$

  I = column of $+$'s is the identity element in the group of multiplications. (Notice the mathematical similarity between aliasing and confounding. What is the difference?)

- $\mathbf{I}=BCDE$ is the **defining relation** for the 2^{5-1} design. It implies all the 15 effect aliasing relations :

$$B=CDE,\ C=BDE,\ D=BCE,\ E=BCD,$$
$$BC=DE,\ BD=CE,\ BE=CD,$$
$$Q=BCDEQ,\ BQ=CDEQ,\ CQ=BDEQ,\ DQ=BCEQ,$$
$$EQ=BCDQ,\ BCQ=DEQ,\ BDQ=CEQ,\ BEQ=CDQ.$$

Clear Effects

- A main effect or two-factor interaction (2fi) is called **clear** if it is not aliased with any other m.e.'s or 2fi's and **strongly clear** if it is not aliased with any other m.e.'s, 2fi's or 3fi's. Therefore a clear effect is *estimable* under the assumption of negligible 3-factor and higher interactions and a strongly clear effect is *estimable* under the weaker assumption of negligible 4-factor and higher interactions.

- In the 2^{5-1} design with $\mathbf{I} = BCDE$, which effects are clear and strongly clear?
  Ans: B, C, D, E are clear, Q, BQ, CQ, DQ, EQ are strongly clear.

- Consider the alternative plan 2^{5-1} design with $\mathbf{I} = BCDEQ$. (It is said to have resolution V because the length of the defining word is 5 while the previous plan has resolution IV.) It can be verified that all five main effects are strongly clear and all 10 2fi's are clear. (*Do the derivations*). This is a very good plan because *each* of the 15 degrees of freedom is either clear or strongly clear.

Defining Contrast Subgroup for 2^{k-p} Designs

- A 2^{k-p} design has k factors, 2^{k-p} runs, and it is a 2^{-p}th fraction of the 2^k design. The fraction is defined by p *independent* defining words. The group formed by these p words is called the **defining contrast subgroup**. It has $2^p - 1$ words plus the identity element **I**.

- **Resolution** = shortest wordlength among the $2^p - 1$ words.

- Example: A 2^{6-2} design with $\mathbf{5} = \mathbf{12}$ and $\mathbf{6} = \mathbf{134}$. The two independent defining words are $\mathbf{I} = \mathbf{125}$ and $\mathbf{I} = \mathbf{1346}$. Then $\mathbf{I} = \mathbf{125} \times \mathbf{1346} = \mathbf{23456}$. The defining contrast subgroup = $\{\mathbf{I}, \mathbf{125}, \mathbf{1346}, \mathbf{23456}\}$. The design has resolution III.

Deriving Aliasing Relations for the 2^{6-2} design

- For the same 2^{k-p} design, the defining contrast subgroup is

$$\mathbf{I} = \mathbf{125} = \mathbf{1346} = \mathbf{23456}.$$

All the 15 degrees of freedom (each is a coset in group theory) are identified.

$$\begin{array}{lllllll}
\mathbf{I} & = & \mathbf{125} & = & \mathbf{1346} & = & \mathbf{23456}, \\
\mathbf{1} & = & \mathbf{25} & = & \mathbf{346} & = & \mathbf{123456}, \\
\mathbf{2} & = & \mathbf{15} & = & \mathbf{12346} & = & \mathbf{3456}, \\
\mathbf{3} & = & \mathbf{1235} & = & \mathbf{146} & = & \mathbf{2456}, \\
\mathbf{4} & = & \mathbf{1245} & = & \mathbf{136} & = & \mathbf{2356}, \\
\mathbf{5} & = & \mathbf{12} & = & \mathbf{13456} & = & \mathbf{2346}, \\
\mathbf{6} & = & \mathbf{1256} & = & \mathbf{134} & = & \mathbf{2345}, \\
\mathbf{13} & = & \mathbf{235} & = & \mathbf{46} & = & \mathbf{12456}, \\
\mathbf{14} & = & \mathbf{245} & = & \mathbf{36} & = & \mathbf{12356}, \\
\mathbf{16} & = & \mathbf{256} & = & \mathbf{34} & = & \mathbf{12345}, \\
\mathbf{23} & = & \mathbf{135} & = & \mathbf{1246} & = & \mathbf{456}, \\
\mathbf{24} & = & \mathbf{145} & = & \mathbf{1236} & = & \mathbf{356}, \\
\mathbf{26} & = & \mathbf{156} & = & \mathbf{1234} & = & \mathbf{345}, \\
\mathbf{35} & = & \mathbf{123} & = & \mathbf{1456} & = & \mathbf{246}, \\
\mathbf{45} & = & \mathbf{124} & = & \mathbf{1356} & = & \mathbf{236}, \\
\mathbf{56} & = & \mathbf{126} & = & \mathbf{1345} & = & \mathbf{234}.
\end{array} \tag{1}$$

- It has the clear effects: **3, 4, 6, 23, 24, 26, 35, 45, 56**. It has resolution III.

WordLength Pattern and Resolution

- Define A_i = number of defining words of length i. $W = (A_3, A_4, A_5, \ldots)$ is called the **wordlength pattern**. In this design, W = (1, 1, 1, 0). It is required that $A_2 = 0$. (Why? No main effect is allowed to be aliased with another main effect.)
- **Resolution** = smallest r such that $A_r \geq 1$.
- **Maximum resolution criterion**: For fixed k and p, choose a 2^{k-p} design with maximum resolution.
- **Rules for Resolution IV and V Designs:**

$$
\begin{array}{ll}
(\mathbf{i}) & \textit{In any resolution IV design, the main effects are clear.} \\
(\mathbf{ii}) & \textit{In any resolution V design, the main effects are strongly} \\
 & \textit{clear and the two-factor interactions are clear.} \\
(\mathbf{iii}) & \textit{Among the resolution IV designs with given } k \textit{ and } p, \\
 & \textit{those with the largest number of clear two-factor} \\
 & \textit{interactions are the best.}
\end{array}
\qquad (2)
$$

A Projective Rationale for Resolution

- For a resolution R design, its projection onto any R-1 factors is a full factorial in the R-1 factors. This would allow *effects of all orders among the R-1 factors to be estimable*. (**Caveat**: it assumes the assumption that other factors are inert.) See Figure 5.1 of WH.

Minimum Aberration Criterion

- Motivating example: consider the two 2^{7-2} designs:

$$\begin{aligned} d_1: \quad \mathbf{I} &= \mathbf{4567} = \mathbf{12346} = \mathbf{12357}, \\ d_2: \quad \mathbf{I} &= \mathbf{1236} = \mathbf{1457} = \mathbf{234567}. \end{aligned}$$

  Both have resolution IV, but

$$W(d_1) = (0,1,2,0,0) \text{ and } W(d_2) = (0,2,0,1,0).$$

  Which one is better? Intuitively one would argue that d_1 is better because $A_4(d_1) = 1 < A_4(d_2) = 2$. (Why? Effect hierarchy principle.)

- For any two 2^{k-p} designs d_1 and d_2, let r be the smallest integer such that $A_r(d_1) \neq A_r(d_2)$. Then d_1 is said to have *less aberration* than d_2 if $A_r(d_1) < A_r(d_2)$. If there is no design with less aberration than d_1, then d_1 has *minimum aberration*.

- Throughout the book, this is the *major criterion used for selecting fractional factorial designs*. Its theory is covered in a forthcoming book by R. Mukherjee and C. F. J. Wu.

Analysis for Location Effects

- *Same* strategy as in full factorial experiments *except* for the **interpretation and handling of aliased effects**.

- For the location effects (based on $\bar{y}_i$ values), the factorial effects are given in Table 3 and the corresponding half-normal plot in Figure 1. Visually one may judge that Q, B, C, CQ and possibly E, BQ are significant. One can apply the studentized maximum modulus test (see section 3.15, not covered in class) to confirm that Q, B, C are significant at 0.05 level (see pp. 161 and 163).

- The $B \times Q$ and $C \times Q$ plots (Figure 5.3 of WH) show that they are synergystic.

- For illustration, we use the model

$$\begin{aligned} \hat{y} = \quad & 7.6360 + 0.1106 x_B + 0.0519 x_E + 0.0881 x_C - 0.1298 x_Q \\ & + 0.0423 x_B x_Q - 0.0827 x_C x_Q \end{aligned} \tag{3}$$

Factorial Effects, Leaf Spring Experiment

Table 3: Factorial Effects, Leaf Spring Experiment

Effect	$\bar{y}$	$\ln s^2$
B	0.221	1.891
C	0.176	0.569
D	0.029	-0.247
E	0.104	0.216
Q	-0.260	0.280
BQ	0.085	-0.589
CQ	-0.165	0.598
DQ	0.054	1.111
EQ	0.027	0.129
BC	0.017	-0.002
BD	0.020	0.425
CD	-0.035	0.670
BCQ	0.010	-1.089
BDQ	-0.040	-0.432
BEQ	-0.047	0.854

Half-normal Plot of Location Effects, Leaf Spring Experiment

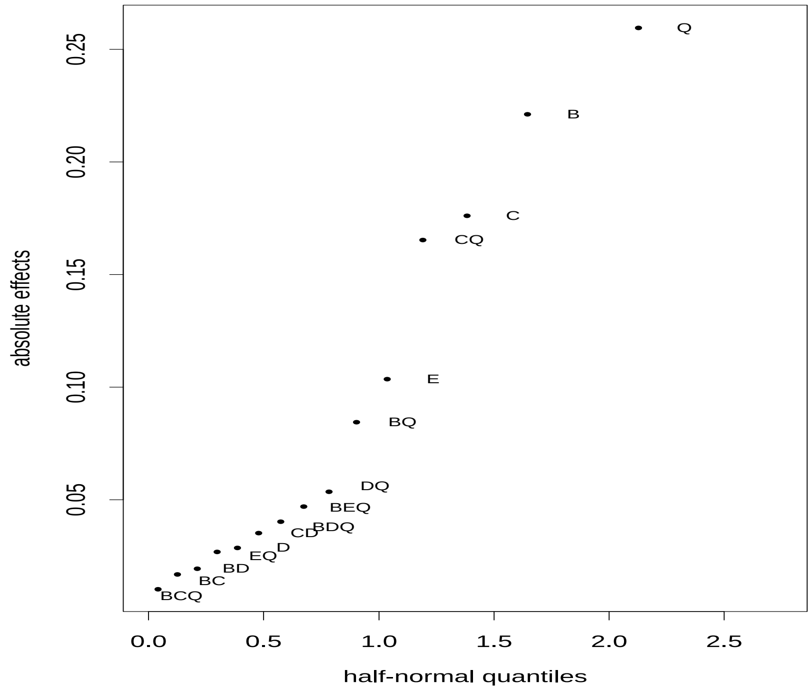

Figure 1: Half-Normal Plot of Location Effects, Leaf Spring Experiment

Interaction Plots

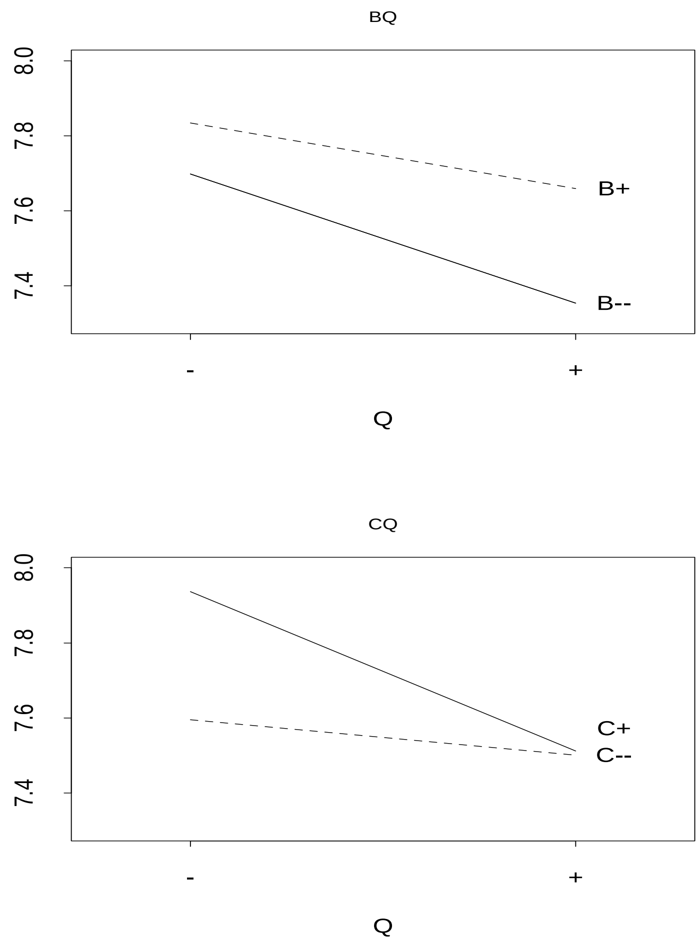

Figure 2: $B \times Q$ and $C \times Q$ interaction plots, Leaf Spring Experiment

Analysis for Dispersion Effects

- For the dispersion effects (based on $z_i = \ln s_i^2$ values), the half-normal plot is given in Figure 2. Visually only effect B stands out. This is confirmed by applying the studentized maximum modulus test. For illustration, we will include B, DQ, BCQ in the following model,

$$\ln \hat{\sigma}^2 = -4.9313 + 0.9455 x_B + 0.5556 x_D x_Q - 0.5445 x_B x_C x_Q. \tag{4}$$

Half-normal Plot of Dispersion Effects, Leaf Spring Experiment

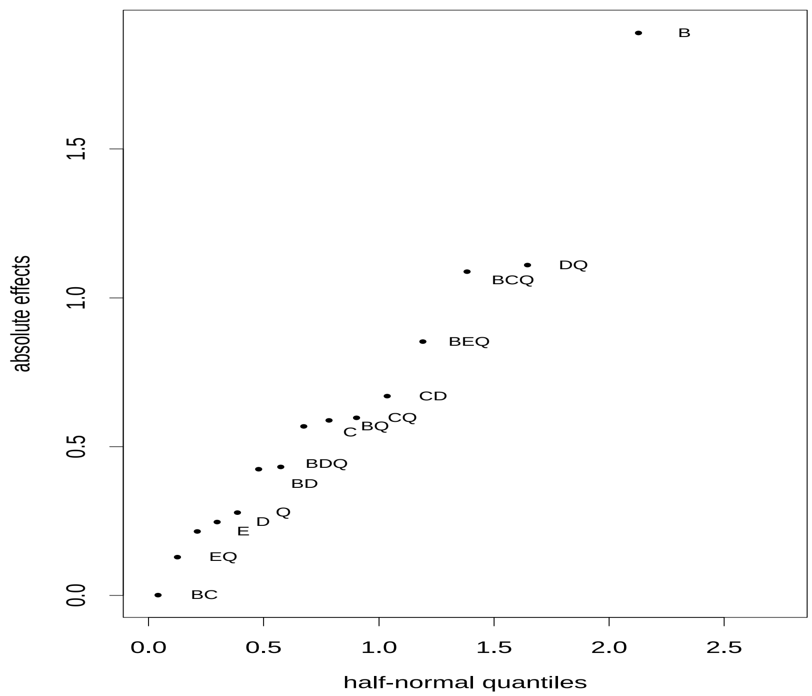

Figure 3: Half-Normal Plot of Dispersion Effects, Leaf Spring Experiment

Two-Step Procedure for Optimization

- Step 1: To minimize s^2 (or $\ln s^2$) based on eq. (4), choose $B = -$. Based on the $D \times Q$ plot (Figure 3), choose the combination with the lowest value, $D = +$, $Q = -$. With $B = -$ and $Q = -$, choose $C = +$ to attain the minimum in the $B \times C \times Q$ interaction plot (Figure 4). Another confirmation: they lead to $x_B = -, x_D x_Q = -$ and $x_B x_C x_Q = +$ in the model (4), which make each of the last three terms negative.

- Step 2: With $BCDQ = (-,+,+,-)$,

$$
\begin{aligned}
\hat{y} &= 7.6360 + 0.1106(-1) + 0.0519 x_E + 0.0881(+1) - 0.1298(-1) \\
&\quad +0.0423(-1)(-1) - 0.0827(+1)(-1) \\
&= 7.8683 + 0.0519 x_E .
\end{aligned}
$$

By solving $\hat{y} = 8.0$, $x_E = 2.54$.

Warning: This is way outside the experimental range for factor E. Such a value may not make physical sense and the predicted variance value for this setting may be too optimistic and not substantiated.

Interaction Plots for Dispersion Effects

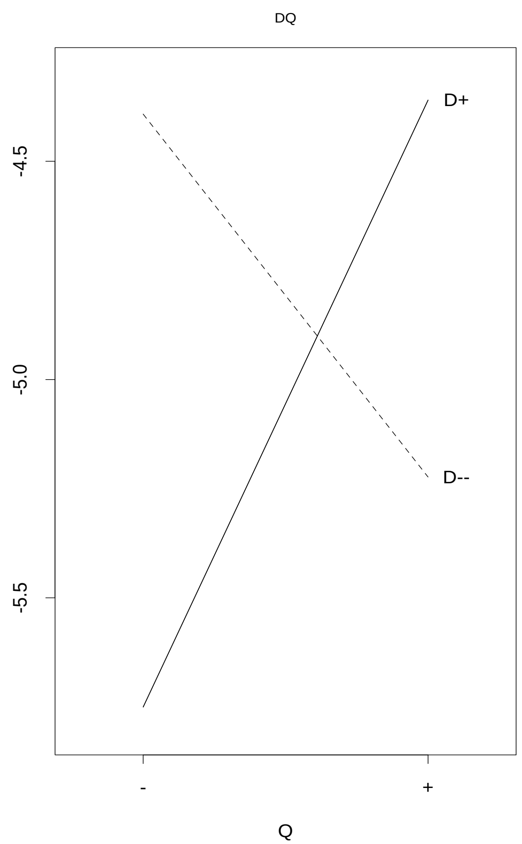

Figure 4 : $D \times Q$ Interaction Plot

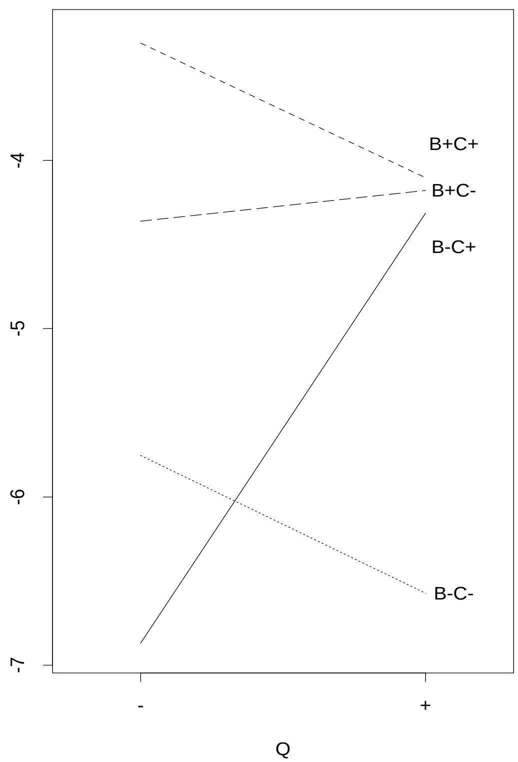

Figure 5 : $B \times C \times Q$ Interaction Plot

Techniques for Resolving Ambiguities in Aliased Effects

- Among the three factorial effects that feature in model (4), B is clear and DQ is strongly clear.
- However, the term $x_B x_C x_Q$ is aliased with $x_D x_E x_Q$ (See bottom of page 5). The following three techniques can be used to resolve the ambiguities.
- *Subject matter knowledge* may suggest some effects in the alias set are not likely to be significant (or does not have a good physical interpretation).
- Or use *effect hierarchy principle* to *assume away* some higher order effects.
- Or use a **follow-up experiment** to **de-alias** these effects. Three methods are given in section 5.4 of WH. Two are considered here.

Method of Adding Orthogonal Runs

Because of aliasing, the model in (4) contains B, DQ, BCQ and DEQ, where $BCQ = DEQ$ are aliased. Suppose we can have 4 additional runs (run no. 17-20 in Table 4). The criterion is to choose those 4 runs to be *orthogonal* for the two columns BCQ and DEQ (i.e., to *de-alias* BCQ and DEQ). Referring to Table 4, we follow the steps:

1. Choose $(++--)$ and $(+-+-)$ for BCQ and DEQ (these two vectors are orthogonal).

2. Choose $(+--+)$ for B (This vector is orthogonal to the previous two; B has the largest effect and needs to be determined before DQ.)

3. Choice of DQ is arbitrary (we have used up three orthogonal vectors), choose $(-++-)$ for DQ.

Method of Adding Orthogonal Runs (contd.)

4. $E = DQ \times DEQ = (-++-) \times (+-+-) = (--++)$.

5. Arbitrarily choose $(+-+-)$ for Q, then $C = BCQ \times B \times Q = (++++)$.

6. Introduce a blocking variable "block" to represent the possible effect due to time difference in conducting the first 16 runs and the follow-up 4 runs; use block = $-$ for number 1-16, block = $+$ for number 17-20.

Note: The regression model for Table 4 should include $x_B, x_D x_Q, x_B x_C x_Q, x_D x_E x_Q$ and x_{bl} (blocking variable). See (5.17) of WH.

Augmented Model Matrix and Design Matrix

Table 4: Augmented Design Matrix and Model Matrix, Leaf Spring Experiment

Run	B	C	D	E	Q	Block	BCQ	DEQ	DQ
1	$-$	$+$	$+$	$-$	$-$	$-$	$+$	$+$	$-$
2	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$-$	$-$
3	$-$	$-$	$+$	$+$	$-$	$-$	$-$	$-$	$-$
4	$+$	$-$	$+$	$-$	$-$	$-$	$+$	$+$	$-$
5	$-$	$+$	$-$	$+$	$-$	$-$	$+$	$+$	$+$
6	$+$	$+$	$-$	$-$	$-$	$-$	$-$	$-$	$+$
7	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$+$
8	$+$	$-$	$-$	$+$	$-$	$-$	$+$	$+$	$+$
9	$-$	$+$	$+$	$-$	$+$	$-$	$-$	$-$	$+$
10	$+$	$+$	$+$	$+$	$+$	$-$	$+$	$+$	$+$
11	$-$	$-$	$+$	$+$	$+$	$-$	$+$	$+$	$+$
12	$+$	$-$	$+$	$-$	$+$	$-$	$-$	$-$	$+$
13	$-$	$+$	$-$	$+$	$+$	$-$	$-$	$-$	$-$
14	$+$	$+$	$-$	$-$	$+$	$-$	$+$	$+$	$-$
15	$-$	$-$	$-$	$-$	$+$	$-$	$+$	$+$	$-$
16	$+$	$-$	$-$	$+$	$+$	$-$	$-$	$-$	$-$
17	$+$	$+$	$-$	$-$	$+$	$+$	$+$	$+$	$-$
18	$-$	$+$	$-$	$-$	$-$	$+$	$+$	$-$	$+$
19	$-$	$+$	$+$	$+$	$+$	$+$	$-$	$+$	$+$
20	$+$	$+$	$+$	$+$	$-$	$+$	$-$	$-$	$-$

Fold-over Technique

- Suppose the original experiment is based on a 2^{7-4}_{III} design with generators

$$d_1: \ \ \mathbf{4} = \mathbf{12}, \ \ \mathbf{5} = \mathbf{13}, \ \ \mathbf{6} = \mathbf{23}, \ \ \mathbf{7} = \mathbf{123}.$$

  None of its main effects are clear.

- To de-alias them, we can choose another 8 runs (no. 9-16 in Table 5) with **reversed** signs for each of the 7 factors. This follow-up design d_2 has the generators

$$d_2: \mathbf{4} = -\mathbf{12}, \mathbf{5} = -\mathbf{13}, \mathbf{6} = -\mathbf{23}, \mathbf{7} = \mathbf{123}$$

  With the extra degrees of freedom, we can introduce a new factor **8** for run number 1-8, and **-8** for run number 9-16. See Table 5.

- The combined design $d_1 + d_2$ is a 2^{8-4}_{IV} design and thus all main effects are clear. (Its defining contrast subgroup is on p.227 of WH).

Augmented Design Matrix Using Fold-over Technique

Table 5: Augmented Design Matrix Using Fold-Over Technique

				d_1				
Run	**1**	**2**	**3**	**4=12**	**5=13**	**6=23**	**7=123**	**8**
1	−	−	−	+	+	+	−	+
2	−	−	+	+	−	−	+	+
3	−	+	−	−	+	−	+	+
4	−	+	+	−	−	+	−	+
5	+	−	−	−	−	+	+	+
6	+	−	+	−	+	−	−	+
7	+	+	−	+	−	−	−	+
8	+	+	+	+	+	+	+	+
				d_2				
Run	**-1**	**-2**	**-3**	**-4**	**-5**	**-6**	**-7**	**-8**
9	+	+	+	−	−	−	+	−
10	+	+	−	−	+	+	−	−
11	+	−	+	+	−	+	−	−
12	+	−	−	+	+	−	+	−
13	−	+	+	+	+	−	−	−
14	−	+	−	+	−	+	+	−
15	−	−	+	−	+	+	+	−
16	−	−	−	−	−	−	−	−

Fold-over Technique: Version Two

- Suppose one factor, say **5**, is very important. We want to de-alias **5** and all 2fi's involving **5**.

- Choose, instead, the following 2^{7-4}_{III} design

$$d_3 : \mathbf{4} = \mathbf{12}, \mathbf{5} = -\mathbf{13}, \mathbf{6} = \mathbf{23}, \mathbf{7} = \mathbf{123}.$$

  Then the combined design $d_1 + d_3$ is a 2^{7-3}_{III} design with the generators

$$d' : \mathbf{4} = \mathbf{12}, \mathbf{6} = \mathbf{23}, \mathbf{7} = \mathbf{123}. \tag{5}$$

  Since **5** does not appear in (5), **5** is strongly clear and all 2fi's involving **5** are clear. However, other main effects are not clear.

- Choice between d_2 and d_3 depends on the priority given to the effects.

Critique of Fold-over Technique

- Fold-over technique is not an efficient technique. It requires doubling of the run size and can only de-alias a *specific* set of effects. In practice, after analyzing the first experiment, a set of effects will emerge and need to be de-aliased. It will usually require much *fewer* runs to de-alias a few effects.
- A more efficient technique that does not have these deficiencies is the optimum design approach given in Section 5.4.2.

Use of Design Tables

- Tables are given in Appendix 5A. Minimum aberration (MA) designs are given in the tables. If two designs are given for same k and p, the first is an MA design and the second is better in having a larger number of clear effects. Two tables are given on next pages.

- In Table 7, the first 2^{9-4} design has MA and 8 clear 2fi's. The second 2^{9-4} design is the second best according to the MA criterion but has 15 clear 2fi's. Details on p. 234 of WH. Using Rule (iii) on page 9 in (2), the second design is better because both have resolution IV.

- It is not uncommon to find a design with slightly worse aberration but more clear effects. Thus **the number of clear effects** should be used as a *supplementary criterion* to the MA criterion.

Table 6: 16-Run 2^{k-p} FFD $(k-p=4)$

(k is the number of factors and F&R is the fraction and resolution.)

k	F&R	Design Generators	Clear Effects
5	2_{V}^{5-1}	$5=1234$	all five main effects, all 10 2fi's
6	2_{IV}^{6-2}	$5=123,\ 6=124$	all six main effects
6^{*}	2_{III}^{6-2}	$5=12,\ 6=134$	3, 4, 6, 23, 24, 26, 35, 45, 56
7	2_{IV}^{7-3}	$5=123,\ 6=124,\ 7=134$	all seven main effects
8	2_{IV}^{8-4}	$5=123,\ 6=124,\ 7=134,\ 8=234$	all eight main effects
9	2_{III}^{9-5}	$5=123,\ 6=124,\ 7=134,\ 8=234,\ 9=1234$	none
10	2_{III}^{10-6}	$5=123,\ 6=124,\ 7=134,\ 8=234,\ 9=1234,\ t_0=34$	none
11	2_{III}^{11-7}	$5=123,\ 6=124,\ 7=134,\ 8=234,\ 9=1234,\ t_0=34,\ t_1=24$	none
12	2_{III}^{12-8}	$5=123,\ 6=124,\ 7=134,\ 8=234,\ 9=1234,\ t_0=34,\ t_1=24, t_2=14$	none
13	2_{III}^{13-9}	$5=123,\ 6=124,\ 7=134,\ 8=234,\ 9=1234,\ t_0=34,\ t_1=24, t_2=14,\ t_3=23$	none
14	2_{III}^{14-10}	$5=123,\ 6=124,\ 7=134,\ 8=234,\ 9=1234,\ t_0=34,\ t_1=24, t_2=14,\ t_3=23,\ t_4=13$	none
15	2_{III}^{15-11}	$5=123,\ 6=124,\ 7=134,\ 8=234,\ 9=1234,\ t_0=34,\ t_1=24, t_2=14,\ t_3=23,\ t_4=13,\ t_5=12$	none

Table 7: 32 Run 2^{k-p} FFD ($k-p=5$, $6 \leq k \leq 11$)

(k is the number of factors and F&R is the fraction and resolution.)

k	F&R	Design Generators	Clear Effects
6	2^{6-1}_{VI}	$6=12345$	all six main effects, all 15 2fi's
7	2^{7-2}_{IV}	$6=123$, $7=1245$	all seven main effects, 14, 15, 17, 24, 25, 27, 34, 35, 37, 45, 46, 47, 56, 57, 67
8	2^{8-3}_{IV}	$6=123$, $7=124$, $8=1345$	all eight main effects, 15, 18, 25, 28, 35, 38, 45, 48, 56, 57, 58, 68, 78
9	2^{9-4}_{IV}	$6=123$, $7=124$, $8=125$, $9=1345$	all nine main effects, 19, 29, 39, 49, 59, 69, 79, 89
9	2^{9-4}_{IV}	$6=123$, $7=124$, $8=134$, $9=2345$	all nine main effects, 15, 19, 25, 29, 35, 39, 45, 49, 56, 57, 58, 59, 69, 79, 89
10	2^{10-5}_{IV}	$6=123$, $7=124$, $8=125$, $9=1345$, $t_0=2345$	all 10 main effects
10	2^{10-5}_{III}	$6=12$, $7=134$, $8=135$, $9=145$, $t_0=345$	3, 4, 5, 7, 8, 9, t_0, 23, 24, 25, 27, 28, 29, $2t_0$, 36, 46, 56, 67, 68, 69, $6t_0$
11	2^{11-6}_{IV}	$6=123$, $7=124$, $8=134$, $9=125$, $t_0=135$, $t_1=145$	all 11 main effects
11	2^{11-6}_{III}	$6=12$, $7=13$, $8=234$, $9=235$, $t_0=245$, $t_1=1345$	4, 5, 8, 9, t_0, t_1, 14, 15, 18, 19, $1t_0$, $1t_1$

Choice of Fractions and Avoidance of Specific Combinations

- A 2^{k-p} design has 2^p choices. In general, use randomization to choose one of them. For example, the 2^{7-3} design has 8 choices $4 = \pm 12, 5 = \pm 13, 6 = \pm 23$. Randomly choose the signs.
- If specific combinations (e.g., $(+++)$ for high pressure, high temperature, high concentration) are deemed undesirable or even disastrous, they can be avoided by choosing a fraction that does not contain them. Example on p.237 of WH.

Blocking in FF Designs

Example: Arrange the 2^{6-2} design in four $(= 2^2)$ blocks with

$$\mathbf{I} = \mathbf{1235} = \mathbf{1246} = \mathbf{3456}.$$

Suppose we choose

$$\mathbf{B_1} = \mathbf{134}, \mathbf{B_2} = \mathbf{234}\ , \mathbf{B_1 B_2} = \mathbf{12}.$$

Then

$$\begin{aligned} \mathbf{B_1} &= \mathbf{134} = \mathbf{245} = \mathbf{236} = \mathbf{156}, \\ \mathbf{B_2} &= \mathbf{234} = \mathbf{145} = \mathbf{136} = \mathbf{256}, \\ \mathbf{B_1 B_2} &= \mathbf{12} = \mathbf{35} = \mathbf{46} = \mathbf{123456}; \end{aligned}$$

i.e., these effects are confounded with block effects and cannot be used for estimation. Among the remaining 12 degrees of freedom, six are main effects and the rest are

$$\begin{array}{ccccccc} \mathbf{13} & = & \mathbf{25} & = & \mathbf{2346} & = & \mathbf{1456}, \\ \mathbf{14} & = & \mathbf{26} & = & \mathbf{2345} & = & \mathbf{1356}, \\ \mathbf{15} & = & \mathbf{23} & = & \mathbf{2456} & = & \mathbf{1346}, \\ \mathbf{16} & = & \mathbf{24} & = & \mathbf{2356} & = & \mathbf{1345}, \\ \mathbf{34} & = & \mathbf{56} & = & \mathbf{1245} & = & \mathbf{1236}, \\ \mathbf{36} & = & \mathbf{45} & = & \mathbf{1256} & = & \mathbf{1234}. \end{array}$$

Use of Design Tables for Blocking

- FF designs in blocks are given in Appendix 5B. You only need to learn how to use the tables and interpret the results. Theory or criterion used in choosing designs are not required.

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