1.5 MATHEMATICAL LANGUAGE

The calculus concepts we will explore in this book are simple and powerful, but sometimes subtle. To succeed in calculus you will have to master some techniques, but, more importantly, you will have to understand ideas and be able to work with the ideas in words and pictures -- very clear words and pictures.

You also need to understand some of the common linguistic constructions used in mathematics. In this section we will discuss a few of the most common mathematical phrases, the meaning of these phrases and some of their equivalent forms.

Your calculus teacher is going to use these types of statements, and it is very important that you understand exactly what the teacher means. You have reached the level in mathematics where the precise use of language is important.

Equivalent Statements

Two statements are **equivalent** if they always have the same logical value (a logicl value is either "true" or "fale", that is, if they are both true or are both false. The statements "x = 3" and "x + 2 = 5" are equivalent statements because if one of them is true then so is the other, and if one of them is false then so is the other. The statements "x = 3" and "$x^2 - 4x + 3 = 0$" are not equivalent since x = 1 makes the second statement true but the first one false.

The Logic of "And" and "Or"

> The compound statement "A **and** B are true" is equivalent to "**both of** A and B are true."

If A or if B or if both are false, then the statement "A and B are true" is false. The statement "$x^2 = 4$ and x > 0" is true when x = 2 and is false for every other value of x.

> The compound statement "A **or** B is true" is equivalent to "**at least one** of A or B is true."

If both A and B are false, then the statement "A or B is true" is false. The statement "$x^2 = 4$ or x > 0" is true if x = –2 or x is any positive number. The statement is false when x = –3 and for lots of other values of x.

Practice 1: Which values of x make each statement true?

    (a) "x < 5"   (b) "x + 2 = 6"   (c) "$x^2 - 10x + 24 = 0$"   (d) "(a) and (b)"   (e) "(a) or (c)"

Negation of a Statement

For some simple statements we can construct the negation just by adding the word "not."

Statement	Negation of the Statement
x is equal to 3 ($x = 3$)	x is **not** equal to 3 ($x \neq 3$)
x is less than 5 ($x < 5$)	x is **not** less than 5 ($x \not< 5$)
	x is greater than or equal to 5 ($x \geq 5$)

When the statement contains words such as "all", "no", or "some," then its negation is more complicated.

Statement	Negation of the Statement
All x satisfy A. Every x satisfies A.	At least one x does not satisfy A. There is an x which does not satisfy A. Some x does not satisfy A.
No x satisfies A. Every x does not satisfy A.	At least one x satisfies A. Some x satisfies A.
There is an x which satisfies A. At least one x satisfies A. Some x satisfies A.	No x satisfies A. Every x does not satisfy A.

We can also negate compound statements containing "and" and "or".

Statement	Negation of the Statement
A and B are both true.	At least one of A or B is not true.
A and B and C are all true.	At least one of A or B or C is not true.
A or B is true.	Both A and B are not true.

Practice 2: Write the negation of each of these statements.

(a) $x + 5 \geq 3$ (b) All prime numbers are odd.   (c) $x^2 < 4$

(d) x divides 2 and x divides 3.   (e) No mathematician can sing well.

"If-Then" Statements

The most common and basic structure used in mathematical language is the

"**If** (some hypothesis) **then** (some conclusion)"

sentence. Almost every result in mathematics can be stated using one or more "If … then … " sentences.

"If A then B" means that when the hypothesis A is true, then the conclusion B must also be true.

If the hypothesis is false, then the "If … then … " sentence makes no claim about the truth or falsity of the conclusion — the conclusion may be either true or false.

Even in everyday life you have probably encountered "If ... then ..." statements for a long time. A parent might try to encourage a child with a statement such as " If you clean your room then I will buy you an ice cream cone."

To show that an "If . . . then . . . " statement is not valid (not true), all we need to do is find a **single** example where the hypothesis is true and the conclusion is false. Such an example with a true hypothesis and false conclusion is called a **counterexample** for the "If . . . then . . . " statement. A valid "If . . . then . . . " statement has no counterexample.

> A **counterexample** to the statement "If A then B" is an example in which A is true and B is false.

The only way for the statement " If you clean your room then I will buy you an ice cream cone" to be false is if the child cleaned the room and the parent did not buy the ice cream cone. If the child did not clean the room but the parent still bought the ice cream cone we would say that the statement was true.

The statement "If n is a positive integer, then $n^2 + 5n + 5$ is a prime number" has hypotheses "n is a positive integer" and conclusion "$n^2 + 5n + 5$ is a prime number." This "If ... then" statement is false since replacing n with the number 5 will make the hypothesis true and the conclusion false. The number 5 is a counterexample for the statement. Every invalid "If . . . then . . . " statement has at least one counterexample, and the most convincing way to show that a statement is not valid is to find a counterexample to the statement.

A number of other language structures can be translated into the "If ... then ..." form. The statements below all mean the same as "If (A) then (B)" :

"All (A) are (B)."	"Every (A) is (B)."	"Each (A) is (B)."
"Whenever (A), then (B)."	"(B) whenever (A)."	"(A) only if (B)."
"(A) implies (B)."	"(A) $\Rightarrow$ (B)" (the symbol " $\Rightarrow$ " means "implies")	

Practice 3: Restate "**If** (a shape is a square) **then** (the shape is a rectangle)" as many ways as you can.

"If … then … " statements occur hundreds of times in every mathematics book, including this one. It is important that you are able to recognize the various forms of "If … then … " statements and that you are able to distinguish the hypotheses from the conclusions.

Contrapositive Form of an "If … then …" Statement

The statement "If (A) then (B)" means that if the hypothesis A is true, then the conclusion B is guaranteed to be true.

Suppose we know that in a certain town the statement

"If (a building is a church) then (the building is green)"

is a true statement,. What can we validly conclude about a red building? Based on the information we have, we can validly conclude that the red building is "not a church" since every church is green. We can also conclude that a blue building is not a church. In fact, we can conclude that every "not green" building is "not a church." That is, if the conclusion of a valid "If … then … " statement is **false**, then the hypothesis must also be **false**.

> The **contrapositive** form of "If (A) then (B)" is
>
>   "If (negation of B) then (negation of A)"   or "If (B is false) then (A is false)."

> The statement "If (A) then (B)" and its contrapositive "If (not B) then (not A)" are **equivalent**.

What about a green building in this town? The green building may or may not be a church – perhaps every post office is also painted green. Or perhaps every building in town is green, in which case the statement "If (a building is a church) then (the building is green)" is certainly true.

Practice 4:   Write the contrapositive form of each of the following statements.

   (a) If a function is differentiable then it is continuous.   (b) All men are mortal.

   (c) If (x equals 3) then $(x^2 – 5x + 6$ equals 0)   (d) If (2 divides x and 3 divides x) then (6 divides x).

Converse of an "If … then …" Statement

If we switch the hypotheses and the conclusion of an "If A then B" statement we get the converse "If B   then A."

The converse of an "If … then … " statement is a new statement with the hypothesis and conclusion switched: the converse of "If (A) then (B)" is "If (B) then (A)." For example, the converse of "If (a building is a church) then (the building is green)" is "If (a building is green) then (the building is a church)."   The converse of an "If …

then … " statement **is not equivalent** to the original "If … then … " statement. The statement "If x = 2, then $x^2 = 4$" is true, but the converse statement "If $x^2 = 4$, then x = 2" is not true because x = –2 makes the hypothesis of the converse true and the conclusion false.

The **converse** of "If (A) then (B)" is "If (B) then (A)."

The statement "If (A) then (B)" and its converse "If (B) then (A)" are **not equivalent**.

Wrap–up

The precise use of language by mathematicians (and mathematics books) is an attempt to clearly communicate ideas from one person to another, but that requires that both people understand the use and rules of the language. If you don't understand this usage, the communication of the ideas will almost certainly fail.

Problems for Solution

In problems 1 and 2, let A = {1,2,3,4,5}, B = {0,2,4,6}, and C = {–2,–1,0,1,2,3}. Which values of x satisfy each statement.

1. a) x is in A **and** x is in B.   b) x is in A **or** x is in C.   c) x is not in B **and** x is in C.

2. a) x is not in B **or** C.   b) x is in B and C but not in A.   c) x is not in A but is in B or C.

In problems 3 – 5, list or describe all the values of x which make each statement true.

3. a) $x^2 + 3 > 1$ b) $x^3 + 3 > 1$ c) $[\,x\,] \le |\,x\,|$

4. a) $\dfrac{x^2 + 3x}{x} = x + 3$ b) x > 4 and x < 9   c) | x | = 3 and x < 0

5. a) x + 5 = 3 or $x^2 = 9$ b) x + 5 = 3 and $x^2 = 9$ c) | x + 3 | = | x | + 3

In problems 6 – 8, write the **contrapositive** of each statement. If the statement is false, give a **counterexample**.

6. a) If x > 3 then $x^2 > 9$.   b) Every solution of $x^2 - 6x + 8 = 0$ is even.

7. a) If $x^2 + x - 6 = 0$ then x = 2 or x = –3 .   b) All triangles have 3 sides.

8. a) Every polynomial has at least one zero.   b) If I exercise and eat right then I will be healthy.

In problems 9 – 11, write the **contrapositive** of each statement. If necessary, first write the original statement in the "If . . . then . . . " form.

9. a)   If your car is properly tuned, it will get at least 24 miles per gallon.

    b)   You can have dessert if you eat your vegetables.

10. a)   A well–prepared student will miss less than 15 points.

    b)   I feel good when I jog.

11. a)   If you love your country, you will vote for me.

    b)   If guns are outlawed then only outlaws will have guns.

In problems 12 – 15, write the **negation** of each statement.

12. a)   It is raining.   b)   Some equations have solutions.   c)   f(x) and g(x) are polynomials.

13. a)   f(x) or g(x) is positive.   b)   x is positive.   c)   8 is a prime number.

14. a)   Some months have 6 Mondays.   b)   All quadratic equations have solutions.

    c)   The absolute value of a number is positive.

15. a)   For all numbers a and b, $|a + b| = |a| + |b|$.   b)   All snakes are poisonous.

    c)   No dog can climb trees.

16. Write an "If . . . then . . . " statement which is true but whose converse is false.

17. Write an "If . . . then . . . " statement which is true and whose converse is true.

18. Write an "If . . . then . . . " statement which is false and whose converse is false.

In problems 19 – 22, state whether each statement is true or false. If the statement is false, give a counterexample.

19. a)   If a and b are real numbers then $(a + b)^2 = a^2 + b^2$.

    b)   If $a > b$ then $a^2 > b^2$.   c)   If $a > b$ then $a^3 > b^3$.

20. a)   For all real numbers a and b, $|a + b| = |a| + |b|$

    b)   For all real numbers a and b, $[a] + [b] \le [a + b]$ ([ ] represents the greatest integer function.)

    c)   If f(x) and g(x) are linear functions then f(g(x)) is a linear function.

21. a)   If f(x) and g(x) are linear functions then f(x) + g(x) is a linear function.

    b)   If f(x) and g(x) are linear functions then f(x)g(x) is a linear function.

    c)   If x divides 6 then x divides 30.

22. a)   If x divides 50 then x divides 10.   b)   If x divides yz then x divides y or z.

    c)   If x divides a^2 then x divides a.

In problems 23 – 26, rewrite each statement as an "If … then … " statement and state whether it is true or false. If

the statement is false, give a counterexample.

23. a)   The sum of two prime numbers is a prime.   b)   The sum of two prime numbers is never a prime.

    c)   Every prime number is odd.   d)   Every prime number is even.

24. a) Every square has 4 sides.
    b) All 4–sided polygons are squares.
    c) Every triangle has 2 equal sides.
    d) Every 4–sided polygon with equal sides is a square.

25. a) Every solution of x+5=9 is odd.
    b) Every 3–sided polygon with equal sides is a triangle.
    c) Every calculus student studies hard.
    d) All (real number) solutions of $x^2 - 5x + 6 = 0$ are even.

26. a) Every straight line intersects the x–axis.
    b) Every (real number) solution of $x^2 + 3 = 0$ is even.
    c) All birds can fly.
    d) No mammal can fly.

Section 1.5   **PRACTICE Answers**

Practice 1:   (a) All values of x less than 5.   (b) x = 4

(c) Both x = 4 and x = 6.   (d) x = 4

(e) x = 6 and all x less than 5.

Practice 2:   (a) x + 5 < 3 .

(b) At least one prime number is even.

There is an even prime number.

(c) $x^2 \geq 4$.

(d) x does not divide 2 or x does not divide 3.

(e) At least one mathematician can sing well.

There is a mathematician who can sing well.

Practice 3:   Here are several ways to restate "**If** (a shape is a square) **then** (the shape is a rectangle)."

All squares are rectangles.

Every square is a rectangle.

Each square is a rectangle.

Whenever a shape is a square, then it is a rectangle.

A shape is a rectangle whenever it is a square.

A shape is a square only if it is a rectangle.

A shape is a square implies that it is a rectangle.

Being a square implies being a rectangle.

Practice 4:   (a)   statement   "If a function is differentiable then it is continuous."

contrapositive   "If a function is not continuous then it is not differentiable."

(b)   statement   "All men are mortal."

contrapositves   "All immortals are not men."

"If a thing is not mortal then it is not human."

(c)   statement   "If (x equals 3) then $(x^2 – 5x + 6$ equals 0)."

contrapositive   "If $(x^2 – 5x + 6$ does not equal 0) then (x does not equal 3)."

(d)   statement   "If (2 divides x and 3 divides x) then (6 divides x)."

contrapositive   "If (6 does not divide x) then (2 does not divide x **or** 3 does not divide x)."