Here's a game I like plying with students. I'll write a positive integer on the board that comes from a set S. You can propose other numbers, and I tell you if your proposed number comes from the set. Eventually you will figure out the properties that define the set. Later in our work, we'll count the number of members of the set. For example, I might write 4561 and put a check mark next to it. You could say, what about 567, and I would say 'no'. You might say, what about 7892, and I would say 'no' again. Then you might guess 2345 and I say 'yes'. Eventually, you conclude that I must be thinking about the set of four digit numbers at include at least one 5.

1. Four digit numbers that include the digit 5.

   Solution: 3168. The number of four-digit numbers is $9 \cdot E_4^{10}/10 = 10 \cdot 9 \cdot 10^3 = 9000$ and the number that don't use the digit 5 is $8 \cdot E_4^9 = 8^4 = 5832$ so the number that includes at least one 5 is $9000 - 5832 = 3168$.

2. Four digit numbers that use just the digits 6, 7, and 8.

   Solution: By the product rule, there are $3 \cdot 3 \cdot 3 \cdot 3 = 3^4 = 81$ such numbers.

3. Five digit numbers which don't include the digit 0 and have five different digits.

   Solution: There are 9 choices for the first digit, 8 for the second, 7 for the third, 6 for the fourth, and 5 for the last digit. That product is $9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot = 9!/4! = 3024$.

4. Five digit numbers that have no repeated digits.

   Solution: There are 9 choices for the first digit and 9 choices for each digit after that for a total of $9^5 = 59049$.

5. Numbers (any length) that use all different digits.

   Solution: There are 10 single digit numbers, 81 two-digit numbers, $9 \cdot 9 \cdot 8 = 648$, etc numbers of this type.

6. Three digit numbers for which the sum of the digits is 9.

   Solution: 36. We can code any such number as a string of 'stars and bars'. The bars represents units and the stars separate the units into digits. For example $|| \cdot ||| \cdot ||||$ codes the number 234. It takes 8 bars (we

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have to leave at least one bar at the left end) and 2 stars to code this number. Therefore the number of three digit numbers n with $S(n) = 9$ is $\binom{10}{2} = 45$.

7. Four digit numbers for which the digits increase from left to right, sometimes called *rising numbers*.

   Solution: 126. Each four element set gives rise to such a rising number, and the correspondence is 1-1. In other words each four element subset gives rise to exactly one rising number. There are therefore $\binom{9}{4} = 126$ such numbers.

8. Four digit numbers for which the numbers do not decrease, sometimes called *non-decreasing numbers*.

   Solution: 495. This is similar except that repetition is allowed. Each four element multiset gives rise to such a non-decreasing number, and the correspondence is 1-1. In other words each four element multisubset of $\{1, 2, \ldots, 9\}$ gives rise to exactly one non-decreasing number. There are therefore $Y_4^9 = \binom{9+4-1}{4} = 495$ such numbers.

9. Six digit numbers $a_1a_2a_3a_4a_5a_6$ that use just the digits 1 through 6, each just once and satisfy the requirement $a_1 < a_2 > a_3 < a_4 > a_5 < a_6$. These are called *up-down numbers*.

   Solution: We need to develop some notation to solve this hard problem. Let $U(n)$ denote the number of ways to arrange the numbers from 1 to n in up-down fashion. The number $D(n)$ is the number of ways to arrange these numbers in down-up fashion. Of course $U(n) = D(n)$. Now $U(2) = 1$ and $U(3) = |\{132, 231\}| = 2$, and $U(4) = |\{1423, 1324, 2413, 2314, 3412\}| = 5$. To compute $U(6)$, as yourself where the 6 can go among the digits. It must go in the second, fourth, or sixth place. We'll consider each of the three cases. Reasoning as below, we can see that $U(5) = 16$.

6 in position 2 . There are $\binom{5}{4} = 5$ ways to pick the digits that will go in positions three through six, and then $U(4) = 5$ ways to arrange them. So there are $5 \cdot 5 = 25$ numbers in this case. An example is 463512.

6 in position 4 . There are $\binom{5}{3} = 10$ ways to pick the digits to go in the first three positions, and $U(3) = 2$ ways to arrange them once they are

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selected. The numbers in positions five and six are determined, so there are $10 \cdot 2 = 20$ numbers in this case. An example is 142635.

6 in position 6 . There are $U(5) = 16$ ways to arrange the numbers from 1 to 5 in up-down fashion. An example is given by 152436.

Taking the sum of the three cases, we have $25 + 20 + 16 = 61$ up-down arrangements of the digits 1 through 6. IE, $U(6) = 61$. Amazingly, the value of the 6th derivative of the secant function at zero is 61 also, and this is not a coincidence. See
`http://www.math.uncc.edu/ hbreiter/m6105/snakes.pdf`
for details.

We're going to start with some easier problems. For convenience we are discussing numbers that can be built without the digit 0. In other words, we can use only digits from $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. These items are counted in the chart below.

1. How many four-digit number are there?

2. How many four-digit numbers have four different digits?

3. How many four-element subsets does the nine-element set $D = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$ have? Alternatively, how many four-digit increasing numbers are there?

4. How many how many four-digit non-decreasing numbers are there?

Throughout we use both the notations $\binom{n}{r}$ and C^n_r for the number $\frac{n!}{(n-r)!r!}$. We are ready to discuss the general idea of counting samples taken from a population of objects. In doing such sampling we are allowed to make a distinction between the order in which the objects became a part of the sample or not. We are also allowed to sample with replacement or not. This leads to four different types of samples. If we count as different two samples that have the same elements but in a different order, we call these *arrangements*, and if we don't distinguish on this basis, we call the samples *selections*. Let's classify each of the counting problems above using these two questions.

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	order matters arrangements()	order does not matter selections$\{\}$
with repetitions	Exponations $E_r^n = n^r$ all $9^4 = 6561$ four-digit numbers	Yahtzee Rolls $Y_r^n = C_r^{n+r-1} = \binom{n+r-1}{r}$ all $\binom{9+4-1}{4} = 495$ four-digit nondecreasing numbers
without repetitions	Permutations $P_r^n = \frac{n!}{(n-r)!}$ all $P_4^9 = 9!/5! = 3024$ four-digit numbers with four different digits	Combinations $C_r^n = \frac{n!}{(n-r)!r!} = \binom{n}{r}$ all $\binom{9}{4} = 126$ four-digit increasing numbers

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1. (1998 state math contest) There are 8 girls and 6 boys in the Math Club at Central High School. The Club needs to form a delegation to send to a conference, and the delegation must contain exactly two girls and two boys. How many delegations that can be formed?

   Solution: 420. The two girls can be picked in $\binom{8}{2} = 28$ ways and the two boys in $\binom{6}{2} = 15$ ways. The product is $28 \cdot 15 = 420$.

2. (1998 state math contest) Given n a positive integer, a plus or minus sign is assigned randomly to each of the integers $1, 2, \ldots, n$. Let $P(n)$ be the probability that the sum of the n signed numbers is positive. What is the value of $[P(1) + P(2) + \cdots + P(6)]$?

   Solution: 45/16. Note that $P(1) = P(2) = P(5) = P(6) = 1/2$ because the probability that the signed sum is zero must be 0 in each of those cases, and getting a positive sum is just as likely as getting a negative sum. In the other cases, $P(3)$ and $P(4)$, a signed sum of 0 is possible: $-1 - 1 + 3 = 1 + 2 - 3 = 0$, so the probability that the signed sum is positive is $P(3) = \frac{1}{2}(1 - \frac{2}{8}) = \frac{3}{8}$. In the $P(4)$ case, $-1+2+3-4 = 1-2-3+4 = 0$, so $P(4) = \frac{1}{2}(1 - \frac{2}{16}) = \frac{7}{16}$. Thus the sum is $\frac{45}{16}$.

3. (1999 state math contest) A box contains b red, $2b$ white and $3b$ blue balls, where b is a positive integer. Three balls are selected at random and without replacement from the box. Let $p(b)$ denote the probability that no two of the selected balls have the same color. Is there a value of b for which $p(b) = 1/6$?

   Solution: No. Note that
   $p(b) = 1-$the probability that all three colors are different $= 1 - \frac{b \cdot 2b \cdot 3b}{\binom{6b}{3}} =$ $1 - \frac{6b^3 \cdot 6}{6b \cdot (6b-1)(6b-2)} = 1 - \frac{6b^2}{(6b-1)(6b-2)}$. Setting this quantity equal to 1/6 and massaging get the quadratic $144b^2 - 90b + 10$ which has a discriminant that is not a perfect square. Thus, there is no b for which $p(b) = 1/6$.

4. (2009 Mathcounts) Four points A, B, C and D on one line segment are jointed by line segments to each of five points E, F, G, H, and I on a second line segment. What is the maximum number of points **interior** to the angle belonging to two of these twenty segments.

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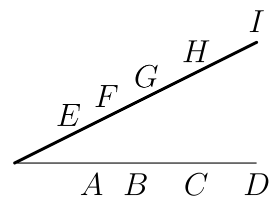

Solution: Each point of intersection is determined by a taking a pair from the set $\{A, B, C, D\}$ and a pair from the set $\{E, F, G, H, I\}$. There are $\binom{4}{2} \cdot \binom{5}{2} = 6 \cdot 10 = 60$ such points.

5. (Mathcounts 2009) How many three-digit numbers can be built from the digits in the list $2, 3, 5, 5, 5, 6, 6$?

   Solution: 43.

6. A *falling* number is an integer whose decimal representation has the property that each digit except the units digit is larger than the one to its right. For example, 96520 is a falling number but 89642 is not. How many five-digit falling numbers are there? How many n-digit falling numbers are there, for $n = 1, 2, 3, 4, 5, 6, 7, 8,$ and 9? What is the total number of falling numbers of all sizes?

   Solution: Each set of k digits can be arranged in exactly one way to form a falling number. Thus, there are just as many k-digit falling numbers as there are k-member subsets of $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. That is, $\binom{10}{1}\binom{10}{2}, \binom{10}{3}, \binom{10}{4}$,
   $\binom{10}{5}, ..., \binom{10}{10}$. Adding all these numbers together gives $2^{10} - 1 = 1023$, since we don't count numbers with zero digits.

7. Cyprian writes down the middle number in each of the $\binom{9}{5} = 126$ five-element subsets of $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Then he adds all these numbers together. What sum does he get?

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Solution: He gets $3 \cdot \binom{6}{2} + 4 \cdot \binom{3}{2} \cdot \binom{5}{2} + 5 \cdot \binom{4}{2} \cdot \binom{4}{2} + 6 \cdot \binom{5}{2} \cdot \binom{3}{2} + 7 \cdot \binom{6}{2} \cdot \binom{2}{2} = 45 + 120 + 180 + 180 + 105 = 630$. Alternatively, there is a 1-1 correspondence between the five-element subsets with middle value k and those with middle value $10 - k$, so the average value of the middle number is 5. Since there are $\binom{9}{5} = 126$ five-element subsets, the sum Cyprian gets is $5 \cdot 126 = 630$. An example of the 1-1 correspondence is given below: $\{1, 3, 6, 7, 8\} \Leftrightarrow \{2, 3, 4, 7, 9\} = \{10-8, 10-7, 10-6, 10-3, 10-1\}$.

8. Counting sums of subset members.

   (a) How many numbers can be expressed as a sum of two or more distinct members of the set $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$?

   Solution: The sums are not distinct. However the numbers we can get are consecutive integers. The smallest is $1 + 2 = 3$ and the largest is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$, so there are $45 - 2 = 43$ such numbers.

   (b) How many integers can be expressed as a sum of two or more different members of the set $\{0, 1, 2, 4, 8, 16, 32\}$?

   Solution: The 0 in the set means that we should count the individual elements among the sums. Every number from 1 and 63 has a unique representation (its binary representation) as a sum of powers of 2. The powers of 2 themselves can be represented as a sum because of the 0 in the set. Thus the answer is 63.

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(c) How many numbers can be expressed as a sum of four distinct members of the set $\{17, 21, 25, 29, 33, 37, 41\}$?

Solution: Each member of $\{17, 21, 25, 29, 33, 37, 41\}$ is one more than a multiple of four. Therefore, any sum of four of them is a multiple of 4. The smallest such number is $17 + 21 + 25 + 29 = 92$ and the largest is $29 + 33 + 37 + 41 = 140$ and all the multiples of 4 between them are obtainable. There are $\frac{140-92}{4} + 1 = \frac{48}{4} + 1 = 13$ such numbers.

Alternatively, we transform the problem into a simpler one. Because $17 = 4 \cdot 4 + 1, 21 = 4 \cdot 5 + 1, \ldots, 41 = 4 \cdot 10 + 1$, it makes sense to set up a correspondence between the set of sums R and the set of numbers generated by $\{4, 5, 6, 7, 8, 9, 10\}$ OR between R and those numbers generated by $\{-3, -2, -1, 0, 1, 2, 3\}$. The set R in this case is just $\{-6, -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5, 6\}$, which has 13 members.

(d) How many numbers can be expressed as a sum of two or more distinct members of the set $\{17, 21, 25, 29, 33, 37, 41\}$?

Solution: Let A, B, C, D, E, F denote the sets of numbers that can be written as a sum of $2, 3, 4, 5, 6, 7$ members of the set respectively. We counted C above. Counting the others yields $|A| = 11, |B| = 13, |C| = 13, |D| = 11, |E| = 7$ and $|F| = 1$. But what about the overlap? There is none! Why? So the total is $11 + 13 + 13 + 11 + 7 + 1 = 56$.

(e) How many integers can be expressed as a sum of two or more distinct elements of the set $\{1, -3, 9, -27, 81, -243\}$?

Solution: An equivalent question is How many numbers have base -3 representations that have at most six digits all of which are 1's and 0's? The answer to the second question is 7 more than the answer to the first one however, because of the requirement that we must add two or more of the powers of -3. There are $2^6 = 64$ such six-digit numbers, so the answer to our question is $64 - 7 = 57$.

9. How many of the first 242 positive integers are expressible as a sum of three or fewer members of the set $\{3^0, 3^1, 3^2, 3^3, 3^4\}$ if we are allowed to use the same power more than once? For example, $5 = 3 + 1 + 1$ can be represented, but 8 cannot. Hint: think about the ternary

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representations.

Solution: The number of powers of 3 used is just the sum of the ternary digits. We are looking for the number of integers whose ternary representations have a sum of digits of 3 or less. We consider three types, those with sum of digits 1, 2 and 3. There are just 5 with sum of digits 1: $10000, 1000, 100, 10, 1$. There are $\binom{5}{2} + \binom{5}{1} = 10 + 5 = 15$ numbers with sum of digits 2. There are two types, those with two 1's and those with one 2. There are $\binom{5}{3} + P_2^5 = 10 + 20 = 30$ of the third type for a total of $5 + 15 + 30 = 50$ numbers altogether.

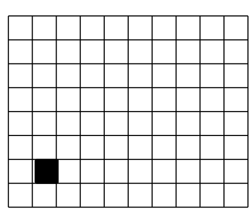
10. John has 2 pennies, 3 nickels, 2 dimes, 3 quarters, and 8 dollars. For how many different amounts can John make an exact purchase (with no change required)?

    Solution: We'll count achievable amounts less than \$1, and multiply by 9, then add in the 9 values $9.00, 9.01, 9.02, 9.05, 9.06, 9.07, 9.10, 9.11, 9.12$. They are $\{0, 1, 2, 5, 6, 7, 10, 11, 12, \ldots\}$, exactly three of every five consecutive values. So, counting 0, there are $\frac{3}{5}(100) = 60$ such values. Hence there are $9 \times 60 - 1 = 539$, since we don't count the value 0. Now adding in the nine uncounted values, we get $539 + 9 = 548$.

11. How many positive integers less than 1000 have an odd number of positive integer divisors?

    Solution: An integer has an odd number of divisors precisely when it is a perfect square, and there are 31 of these in the range $1, 2, 3, \ldots, 1000$.

12. (2004 AMC 10 and extensions) An 8×10 grid of squares with one shaded square is given.

    (a) How many different squares are bounded by the gridlines?

    Solution: This is just $10\cdot8+9\cdot7+8\cdot6+7\cdot5+6\cdot4+5\cdot3+4\cdot2+3\cdot1 = 276$.

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(b) How many different rectangles are bounded by the gridlines?

Solution: The idea we discussed in class of choosing the top and bottom boundaries in $\binom{9}{2} = 36$ ways and the left and right boundaries in $\binom{11}{2} = 55$ ways for a total of $36 \cdot 55 = 1980$.

(c) $(*)$How many different squares bounded by the gridlines contain the shaded square?

Solution: The answer is 27. Counting by size and starting with the 1×1, we have $1 + 4 + 4 + 4 + 4 + 4 + 4 + 2 = 27$.

(d) How many different rectangles bounded by the gridlines contain the shaded square?

Solution: There are just two ways to pick the lower boundary and two ways to pick the left boundary. There are 9 ways to pick the right boundary and 7 ways to pick the top boundary. Their product is 252.

13. How many squares in the plane have two or more vertices in the set $S = (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (2, 0)$?

Solution: One way to do this is to determine the number of each size. By possible areas of the squares are $1/2, 1, 2, 5/2, 4, 5$ and 8 and there are, respectively, $6, 7, 6, 2, 3, 4,$ and 2 of these for a total of 30. Alternatively, there are $\binom{6}{2} = 15$ pairs of points, each of which could give rise to as many as 3 squares. Since one square has all four of its vertices in the grid, this overcounts by 5, and there are 5 squares with three vertices in the grid, each of which leads to overcounts by 2 for a total of $5 + 10 = 15$. Thus there are $45 - 15 = 30$ squares. There is yet another method: start with two points and count the squares: 3. Add a third point and count the new squares, then add a fourth point and count the new ones.

14. Numbers with a given digit sum.

(a) How many numbers in the set $\{100, 101, 102, \ldots, 999\}$ have a sum of digits equal to 9?

Solution: Insert two dividers into a string of 8 counters to code such a three digit number. For example $|| \diamond ||| \diamond |||$ is a coding for 333 and $\diamond \diamond ||||||||$ codes 108. There are $\binom{10}{2} = 45$ ways to code such a number.

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(b) How many four digit numbers have a sum of digits 9?

Solution: For four digit numbers, we get $\binom{11}{3} = 165$.

(c) How many integers less than one million have a sum of digits equal to 9?

Solution: These number are all at most 6 digits, so insert 5 dividers in a string of 9 vertical bars. For example $\diamond\diamond|||\diamond||\diamond\diamond||||$ represents the number 3204. There are $\binom{14}{9} = 2002$ ways to pick the 9 positions from the 14 locations.

15. (2008 state math contest) Two coins are removed randomly and without replacement from a box containing 3 nickels, 2 dimes, and 1 coin of value 0. What is the probability that one of the removed coins is worth five cents more than the other?

    Solution: There are six coins which we can label as $N_1, N_2, N_3, D_1, D_2, Z$. There are $\binom{6}{2} = 15$ ways to choose a pair of these coins, and $\binom{3}{1}\binom{2}{1} + \binom{3}{1}\binom{1}{1} = 6 + 3$ ways to be successful, so the probability is $9/15 = 3/5$.

16. (2008 state math contest) A *palindrome* on the alphabet $\{H, T\}$ is a sequence of h's and T's which reads the same from left to right as from right to left. Thus $HTH, HTTH, HTHTH$ and $HTHHTH$ are palindromes of lengths 3,4,5, and 6 respectively. Let $P(n)$ denote the number of palindromes of length n over $\{H, T\}$. For how many values of n is $1000 < P(n) < 10000$?

    Solution: 8. The number of palindromes of length n is given by $2^{\lfloor \frac{n+1}{2} \rfloor}$, so $P(n)$ satisfies the inequality for $n = 19, 20, \ldots, 26$.

17. An urn contains marbles of four colors: red, white, blue, and green. When four marbles are drawn without replacement, the following events are equally likely:

    (a) the selection of four red marbles;

    (b) the selection of one white and three red marbles;

    (c) the selection of one white, one blue, and two red marbles; and

    (d) the selection of one marble of each color.

    What is the smallest number of marbles that the urn could contain?

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Solution: 21. The hypothesis of equally likely events can be expressed as

$$\frac{\binom{r}{4}}{\binom{n}{4}} = \frac{\binom{r}{3}\binom{w}{1}}{\binom{n}{4}} = \frac{\binom{r}{2}\binom{w}{1}\binom{b}{1}}{\binom{n}{4}} = \frac{\binom{r}{1}\binom{w}{1}\binom{b}{1}\binom{g}{1}}{\binom{n}{4}}$$

where r, w, b, and g denote the number of red, white, blue, and green marbles, respectively, and $n = r + w + b + g$. Eliminating common terms and solving for r in terms of w, b, and g, we get

$$r - 3 = 4w, \qquad r - 2 = 3b, \quad \text{and} \quad r - 1 = 2g.$$

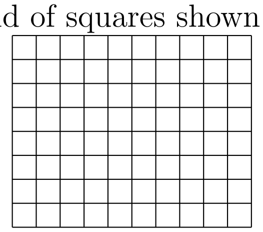
The smallest r for which w, b, and g are all positive integers is $r = 11$, with corresponding values $w = 2$, $b = 3$, and $g = 5$. So the smallest total number of marbles is $11 + 2 + 3 + 5 = 21$.

18. Look at the $m \times n$ multiplication table below. What is the sum of the mn entries in the table?

$\times$	1	2	3	4	5	6	7	8	9	$\dots$	n
1	1	2	3								n
2	2	4	6								$2n$
3	3	6	9								$3n$
$\vdots$											
m	m	$2m$	$3m$								mn

Solution: The answer is $(1+2+\cdots+n)(1+2+\cdots+m) = \frac{n(n+1)}{2} \cdot \frac{m(m+1)}{2}$.

19. Consider the $m \times n$ grid of squares shown below.



How many rectangles are bounded by the gridlines?

Solution: See the problem above.

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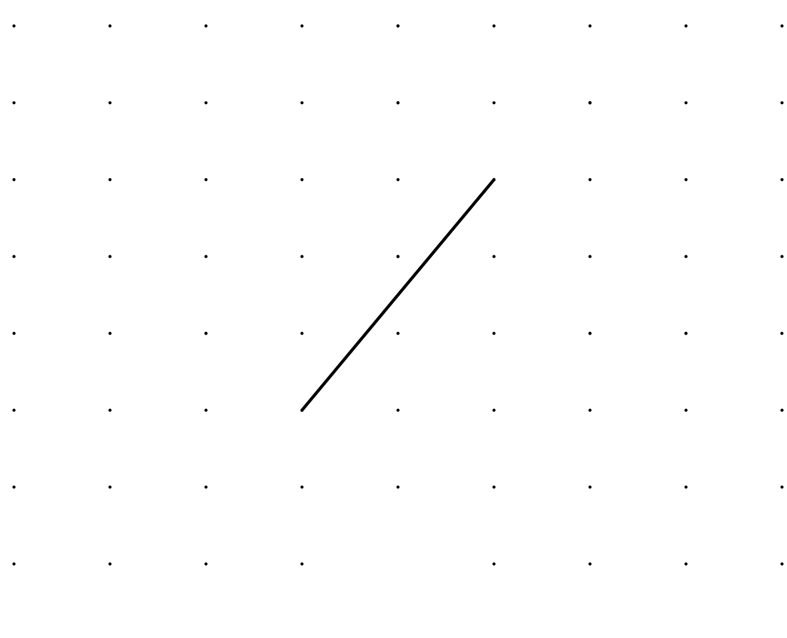
20. How many circles in the plane contain at least three of the points $(0,0),(0,1),\ (0,2),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2)$?

    Solution: There are $\binom{9}{3} = 84$ three element subsets, all of which give rise to a circle except those which are co- linear, of which there are 8. But some circles get counted four times, namely those which contain four of the nine points. So the problem is to count the four element subsets that belong to a circle. There are six such sets that are vertices of a square, and four more that are vertices of a non-square rectangle. Then there are four more that are isomorphic with $\{(0,0),(1,0),(2,1),(2,2)\}$. For each of these $4+6+4=14$ sets, we must subtract three circles that have been counted 4 times. Therefore there are $84-8-(3\cdot 14)=34$ such circles. Alternatively, note that there are 8 circles with centers in the region $x>0, y>0$, thinking now that the nine points have been translated so that the middle one is the origin. Thus there are $8\cdot 4+2(\text{circles with center at the origin}) = 34$.

21. Consider the 9×9 grid of lattice points shown below. Points P and Q are given. How many points R in the grid are there for which triangle PQR is isosceles?



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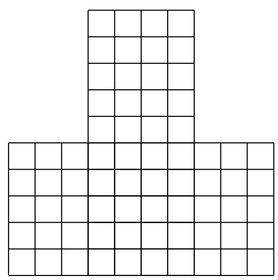
Next, let $P = (a, b)$ be a lattice point. Find necessary and sufficient conditions on P so that the set of points Q for which triangle PQO has integer area, where O is the origin, is finite.

22. How many subsets of $S = \{1, 2, 3, 4, 5, 6, 7, 8\}$ have either 5, 6 or 7 as their largest element?

    Solution: These are sets that don't include 8 but have at least one of the set $T = \{5, 6, 7\}$. There are 7 non-empty subsets of T, and $2^4 = 16$ subsets of $\{1, 2, 3, 4\}$, so there are $7 \cdot 16 = 112$ subsets of S satisfying the required conditions.

23. How many rectangular regions are bounded by the gridlines of the figure below?



    Solution: 1225. Let A denote the set of rectangular subregions of the 5×10 rectangle at the base, and let B denote the rectangular subregions of the 10×4 rectangle down the middle. Let $|S|$ denote the number of elements of the set S. Then $|A \cup B| = |A| + |B| - |A \cap B| = \binom{6}{2}\binom{11}{2} + \binom{11}{2}\binom{5}{2} - \binom{6}{2}\binom{5}{2} = 15 \cdot 55 + 55 \cdot 10 = 15 \cdot 10 = 1225$.

24. (Mathcounts 2010, Target Round) Seven identical red cards and three identical black cards are laid down in a row on a table. How many distinguishable arrangements are possible if no two black cards are allowed to be adjacent to each other?

    Solution: Consider the arrangement without black cards:

    $$\_R\_R\_R\_R\_R\_R\_R\_.$$

    Now three black cards can be inserted among the red cards by selecting three blanks. This can be done in $\binom{8}{3} = 56$ ways.

25. (old USAMO problem) In a math contest, three problems, A, B, and C were posed. Among the participants there were 25 who solved at least

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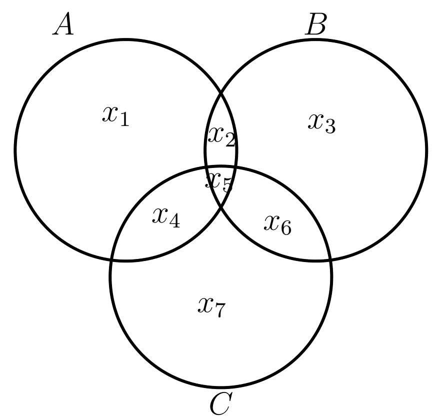
one problem. Of all the participants who did not solve problem A, the number who solved problem B was twice the number who solved C. The number who solved only problem A was one more than the number who solved A and at least one other problem. Of all participants who solved just one problem, half did not solve problem A. How many solved only problem B?

Solution: Consider the venn diagram below:



Letting $x_1, \ldots, x_7$ denote the number of students in each of the regions depicted, it follows that

$$\begin{array}{cccccccccc} x_1 & +x_2 & +x_3 & +x_4 & +x_5 & +x_6 & +x_7 & = & 25 \\ & & +x_3 & & & -x_6 & -2x_7 & = & 0 \\ x_1 & -x_2 & & -x_4 & -x_5 & & & = & 1 \\ x_1 & & -x_3 & & & & -x_7 & = & 0 \end{array}$$

Subtracting equation 4 from the sum of the first three yields

$$x_1 + 3x_3 = 26.$$

Only the pairs $(x_1, x_3) = (2, 8), (5, 7), (8, 6), (11, 5), (14, 5), (17, 3), (20, 2), (23, 1), (26, 0)$. However, since $1 + x_2 + x_4 + x_5 = x_3 + x_7 = x_1$, it follows that $x_1 < 11$. On the other hand, $x_1 > x_3$. This means that only the pair $(8, 6)$ can work. Indeed, $x_1 = 8, x_3 = 6, x_7 = 2, x_6 = 2$, and $x_2 + x_4 + x_5 = 7$.

26. In a chess tournament, the number of boy participants is double the number of girl participants. Every two participants play exactly one game against each other. At the end of the tournament, no games were

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drawn. The ratio between the number of wins by the girls and the number of wins by the boys is 7:5. How many boys were there in the tournament?

27. Find the number of positive integer triples (x, y, z) satisfying $xy^2z^3 = 1,000,000$.

    Solution: 49. Let $x = 2^i5^j$, $y = 2^k5^l$, and $z = 2^s5^t$. Writing the equation in terms of the primes 2 and 5, we have $xy^2z^3 = 2^{i+2k+3s}5^{j+2l+3t} = 2^65^6$. Thus $i + 2k + 3s = 6$ and $j + 2l + 3t = 6$. Since each of these has seven solutions, there are 49 solutions.

28. For how many different subsets of the set $S = \{2, 3, 4, 6, 15, 20, 30\}$ is the sum of the elements at least 50?

    Solution: By considering cases, we get $1 + 32 + 8 + 4 = 45$ subsets that have sum at least 50 and 83 that have smaller sums counting the empty set.

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