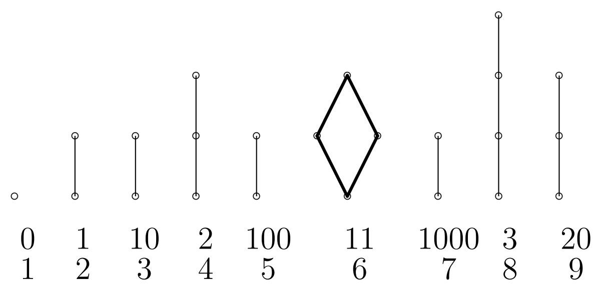
Just the Factors, Ma'am

1 Introduction

The purpose of this note is to find and study a method for determining and counting all the positive integer divisors of a positive integer. Let N be a given positive integer. We say d is a divisor of N and write $d|N$ if N/d is a positive integer. Thus, for example, $2 \mid 6$. Denote by D_N the set of all positive integer divisors of N. For example $D_6 = \{1, 2, 3, 6\}$. There are four parts to this note. In the first part, we count the divisors of a given positive integer N based on its prime factorization. In the second part, we construct all the divisors, and in the third part we discuss the 'geometry' of the D_N. In part four, we discuss applications to contest problems.



Study the figure above. There are two things to work out. One is how does the set of dots and segments represent a number, and the other is how do the digit strings just above the digits 1 through 9 represent those digits. Take some time now before reading more to figure this out.

2 Counting the divisors of N

First consider the example $N = 72$. To find the number of divisors of 72, note that the prime factorization of 72 is given by $72 = 2^3 3^2$. Each divisor d of 72 must be of the form $d = 2^i 3^j$ where $0 \le i \le 3$ and $0 \le j \le 2$. Otherwise, $2^3 3^2/d$ could not be an integer, by the Fundamental Theorem of Arithmetic (the theorem that guarantees the unique factorization into primes of each positive integer). So there are 4 choices for the exponent i and 3 choices for j. Hence there are $4 \cdot 3 = 12$ divisors of 72. Reasoning similarly, we can see that for any integer

$$N = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k},$$

the number of divisors is

$$\Pi_{i=1}^{k}(e_i + 1) = (e_1 + 1)(e_2 + 1) \cdots (e_k + 1).$$

3 Constructing the divisors of N

In part 1 we found the number of members of D_N for any positive integer N. In this part, we seek the list of divisors themselves. Again we start with the prime factorization of N. Suppose $N = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$. If $k = 2$ the listing is straightforward. In this case, we build a table by first listing the powers of p_1 across the top of the table and the powers of p_2 down the side, thus obtaining an $(e_1+1) \cdot (e_2+1)$ matrix of divisors. Again we use $N = 72$ as an example. Notice that each entry in the table is the product of its row label and its column label.

	2^0	2^1	2^2	2^3
3^0	1	2	4	8
3^1	3	6	12	24
3^2	9	18	36	72

What do we do when the number of prime factors of N is more than 2? If $k = 3$ we can construct $e_3 + 1$ matrices of divisors, one for each power of p_3. For example, if $N = 360 = 2^3 3^2 5$, we construct one 3×4 matrix for 5^0 and one for 5^1. The result is a $3 \times 4 \times 2$ matrix of divisors. The two 3×4 matrices are shown below.

5^0	2^0	2^1	2^2	2^3
3^0	1	2	4	8
3^1	3	6	12	24
3^2	9	18	36	72

and

5^1	2^0	2^1	2^2	2^3
3^0	5	10	20	40
3^1	15	30	60	120
3^2	45	90	180	360

For larger values of k we can create multiple copies of the matrix associated with the number $N = p_1^{e_1} p_2^{e_2} \cdots p_{k-1}^{e_{k-1}}$.

4 The geometry of D_N

To investigate the geometry of D_N, we first explore the relation 'divides'. Recall that $a|b$ means that a and b are positive integers for which b/a is an integer. The relation '|' has several important properties, three of which are crucial to our discussion.

1. Reflexivity. For any positive integer a, $a|a$.

2. Antisymmetry. For any pair of positive integers a, b, if $a|b$ and $b|a$, then $a = b$.

3. Transitivity. For any three positive integers, a, b, c, if $a|b$ and $b|c$, then $a|c$.

These properties are easy to prove. The first says that each integer is a divisor of itself; that is, a/a is an integer. The second says that no two different integers can be divisors of one another. This is true since a larger integer can never be a divisor of a smaller one. The third property follows from the arithmetic $b/a \cdot c/b = c/a$ together with the property that the product of two positive integers is a positive integer. Any set S with a relation $\preceq$ defined on it that satisfies all three of the properties above is called a *partially ordered set*, or a *poset*. A branch of discrete mathematics studies the properties of posets, $(S, \preceq)$.

Each finite poset has a unique directed graph representation. This pictorial representation is what we mean by the geometry of D_N. To construct the directed graph of a poset $(S, \preceq)$, draw a vertex (dot) for each member of S. Then connect two vertices a and b with a directed edge (an arrow) if $a \preceq b$. Of course, in our case D_N this means we connect a to b if $a|b$. The case D_6 is easy to draw:

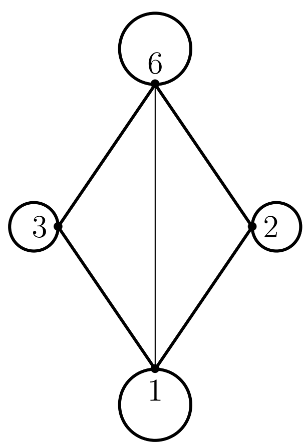


Fig. 1 The digraph of D_6

The circles at each of the four vertices are called *loops*. They are included as directed edges because each number is a divisor of itself. The reader should imagine that all the non-loop edges are upwardly directed. The directed edge from 1 to 6 indicates that $1|6$. But since we know that the vertices of D_6 satisfy all three properties required of a poset, we can leave off both (a) the loops, which are implied by the reflexive property, and (b) the edges that are implied by the transitivity condition. The 'slimmed down' representation, called the Hasse diagram, is much easier to understand. It captures all the essential information without cluttering up the scene. The Hasse diagram of D_6 is shown below.

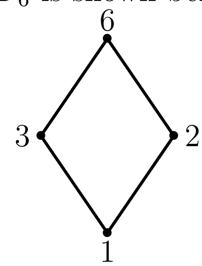


Fig. 2, the Hasse diagram of D_6

In general, the Hasse diagram for D_N has only those non-loop edges which are not implied by transitivity, that is, those edges from a to b for which b is a prime number multiple of a. The Hasse diagrams of D_{72}, D_{30}, and D_{60} are shown below.

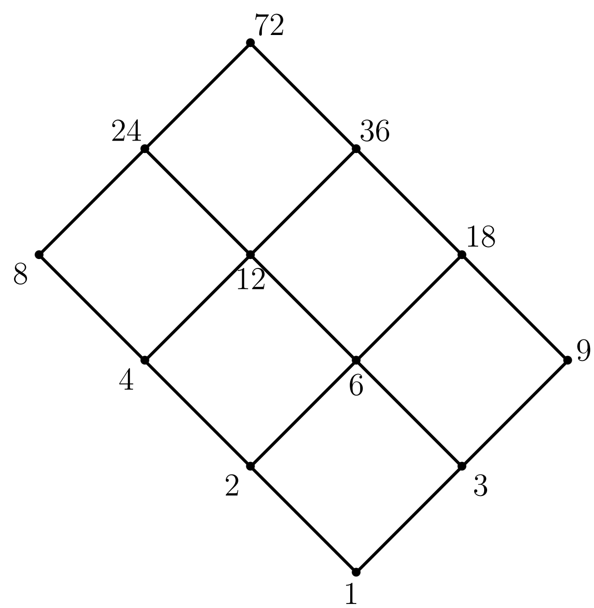


Fig. 3 The Hasse diagram of D_{72}

Notice that each prime divisor of 30 can be considered a direction, multiplication by 2 moves us to the left ($\nwarrow$), by 3 moves us upward ($\uparrow$) and, by 5 moves us to the right ($\nearrow$). Also note that if a and b are divisors of 30 then $a|b$ if and only if there is a sequence of upwardly directed edges starting at a and ending at b. For example, $1|30$ and $(1,3),(3,15),(15,30)$ are all directed edges in the digraph of D_{30}. On the other hand, we say 2 and 15 are incomparable because neither divides the other, and indeed there is no upwardly directed sequence of edges from either one to the other.

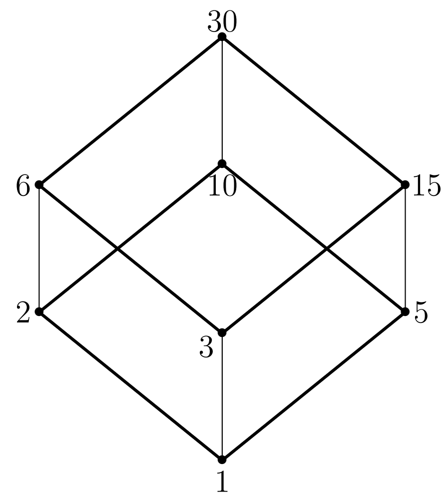


Fig. 4 The Hasse diagram of D_{30}

What would the divisors of 60 look like if we build such a diagram for them? Try to construct it before you look at D_{60}.

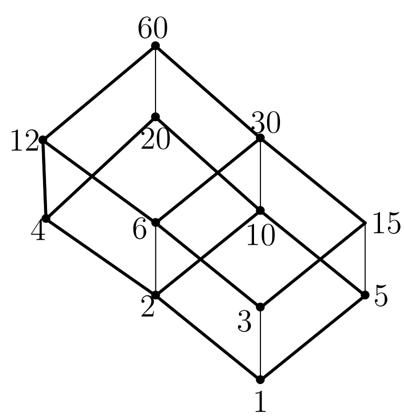


Fig. 5 The Hasse diagram of D_{60}

Consider the same (lattice/Hasse) diagram for the divisors of 210. We can draw this in several ways. The first one (Fig. 6a) places each divisor of 210 at a level determined by its number of prime divisors. The second one (Fig. 6b) emphasizes the 'degrees of freedom'. These two diagrams are representations of a four dimensional cube, not surprising since the Hasse diagram for D_{30} is a three-dimensional cube. A mathematical way to say the two digraphs are the same is to say they are *isomorphic*. This means that they have the same number of vertices and the same number of edges and that a correspondence between the vertices also serves as a correspondence between the edges. Note that the digraphs in 6a and 6b have the required number of vertices (16) and the required number of edges (32). Can you find an N such that the Hasse diagram of D_N is a representation of a five-dimensional cube? Such a digraph must have $2^5 = 32$ vertices, and $2 \cdot 32 + 16 = 80$ edges.

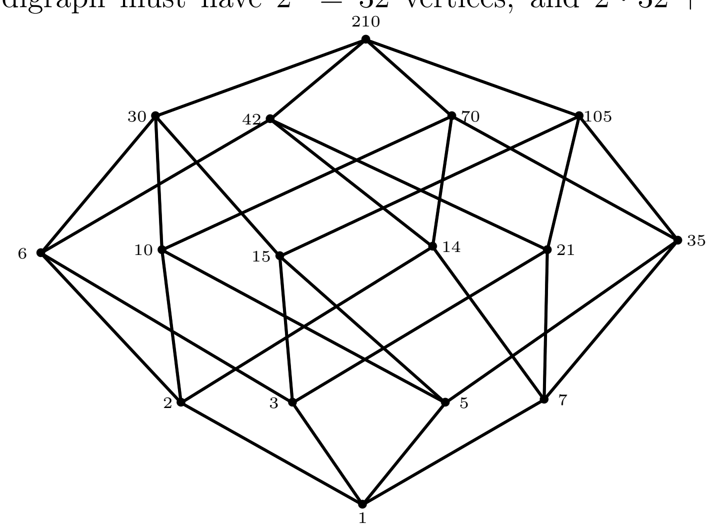


Fig 6a. The Hasse diagram of D_{210}

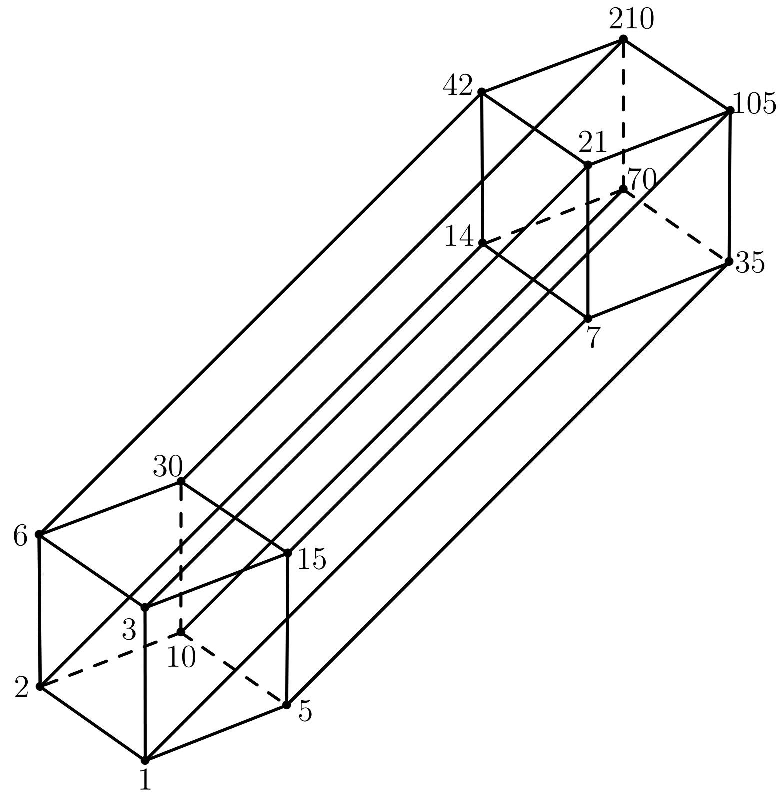


Fig 6b. The Hasse diagram of D_{210}

How can the geometry help us do number theory? One way to use the geometry is

in the calculation of the GCF and LCM of two members of D_N. Note that each element d of D_N generates a downward 'cone' of divisors and an upward cone of multiples. We can denote these cones by $F(d)$ and $M(d)$ respectively. Then the $GCF(d, e) = \max\{F(d) \cap F(e)\}$ and $LCM(d, e) = \min\{M(d) \cap M(e)\}$.

5 Problems On Divisors

Some of the following problems come from MathCounts and the American Mathematics Competitions.

1. Find the number of three digit divisors of 3600.

   Solution: 15. Construct three 3×5 matrices of divisors. The first matrix has all the divisors of the form $2^i 3^j$, the second all divisors of the form $2^i 3^j 5^1$ and the third all the divisors of the form $2^i 3^j 5^2$. There is one three digit divisor in the first matrix, 144, five in the second, and nine in the third for a total of $1 + 5 + 9 = 15$.

2. How many positive integers less than 50 have an odd number of positive integer divisors?

   Solution: A number has an odd number of divisors if and only if it is a perfect square. Therefore, there are exactly seven such numbers, $1, 4, 9, 16, 25, 36$, and 49.

3. (The Locker Problem) A high school with 1000 lockers and 1000 students tries the following experiment. All lockers are initially closed. Then student number 1 opens all the lockers. Then student number 2 closes the even numbered lockers. Then student number 3 changes the status of all the lockers numbered with multiples of 3. This continues with each student changing the status of all the lockers which are numbered by multiples of his or her number. Which lockers are closed after all the 1000 students have done their jobs?

   Solution: Build a table for the first 20 lockers, and notice that the lockers that end up open are those numbered $1, 4, 9$, and 16. This looks like the squares. Think about what it takes to make a locker end up open. It takes an odd number of changes, which means an odd number of divisors. We know how to count the number of divisors of a number $N = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$. N has $D_N = (e_1 + 1)(e_2 + 1) \cdots (e_n + 1)$ divisors. So the issue is how can this last number be odd. It's odd if each factor $e_i + 1$ is odd, which means all the e_i have to be even. This is true precisely when N is a perfect square. So the lockers that are closed at the end are just the non-square numbered lockers.

4. If N is the cube of a positive integer, which of the following could be the number of positive integer divisors of N?

   (A) 200 **(B)** 201 **(C)** 202 **(D)** 203 **(E)** 204

   Solution: C. The number N must be of the form $N = p^{3e} q^{3f} \cdots$. So the number of divisors must be of the form $D = (3e+1)(3f+1) \cdots$. Such a number must be one greater than a multiple of 3. The number $N = (2^{67})^3 = 2^{201}$ has exactly 202 divisors.

5. Let
$$N = 69^5 + 5 \cdot 69^4 + 10 \cdot 69^3 + 10 \cdot 69^2 + 5 \cdot 69 + 1.$$
How many positive integers are factors of N?

(A) 3 **(B)** 5 **(C)** 69 **(D)** 125 **(E)** 216

Solution: The expression given is the binomial expansion of $N = (69+1)^5 = 70^5 = 2^5 \cdot 5^5 \cdot 7^5$. So N has $(5+1)(5+1)(5+1) = 216$ divisors.

6. A teacher rolls four dice and announces both the sum S and the product P. Students then try to determine the four dice values a, b, c, and d. Find an ordered pair (S, P) for which there is more than one set of possible values.

Solution: If $(S, P) = (12, 48)$, then $\{a, b, c, d\}$ could be either $\{1, 3, 4, 4\}$ or $\{2, 2, 2, 6\}$. Note: strictly speaking, these are not sets, but multisets because multiple membership is allowed.

7. How many of the positive integer divisors of $N = 2^3 \cdot 3^3 \cdot 5^3 \cdot 7^3 \cdot 11^3$ have exactly 12 positive integer divisors?

Solution: Let n be a divisor of N with 12 divisors. Then $n = p^3q^2$ where p and q belong to $\{2, 3, 5, 7, 11\}$ or $n = p^2qr$ where $p, q, r \in \{2, 3, 5, 7, 11\}$. There are $5 \cdot 4 = 20$ of the former and $5 \cdot \binom{4}{2} = 30$ of the later. So the answer is $20 + 30 = 50$.

8. How many ordered pairs (x, y) of positive integers satisfy
$$xy + x + y = 199?$$

Solution: Add one to both sides and factor to get $(x+1)(y+1) = 200 = 2^3 5^2$. which has $4 \cdot 3$ factors, one of which is 1. Neither $x + 1$ not $y + 1$ can be 1, so there are 10 good choices for $x + 1$.

9. How many positive integers less than 400 have exactly 6 positive integer divisors?

   Solution: Having exactly 6 divisors means that the number N is of the form p^5 or p^2q where p and q are different prime numbers. Only $2^5 = 32$ and $3^5 = 243$ are of the first type. If $p = 2$, then q could be any prime in the range $3, 5, \dots 97$ of which there are 24 primes. If $p = 3$, then q could be 2 or any prime in the range $5, \dots 43$ and there are 13 of these. If $p = 5$, then q could be 2 or 3 or any prime in the range $7, \dots 13$. If $p = 7$ then $q = 2, 3$, or 5. There are two values of q for $p = 11$ and one value of q for $p = 13$. Tally these to get $2 + 24 + 13 + 5 + 3 + 2 + 1 = 50$.

10. The product of four distinct positive integers, a, b, c, and d is 8!. The numbers also satisfy

$$ab + a + b = 391 \quad (1)$$
$$bc + b + c = 199. \quad (2)$$

   What is d?

   Solution: Add 1 to both sides and factor to get

$$(a + 1)(b + 1) = 392 = 2^3 7^2 \quad (3)$$
$$(b + 1)(c + 1) = 200 = 2^3 5^2. \quad (4)$$

   Build a factor table for 392.

	2^0	2^1	2^2	2^3
7^0	1	2	4	8
7^1	7	14	28	56
7^2	49	98	196	392

   Thus it follows that $(a+1, b+1)$ is one of the following pairs: $(1, 392), (2, 196), (4, 98), (8, 49), (7, 56), (14, 28), (392, 1), (196, 2), (98, 4), (49, 8), (56, 7), (28, 14)$. Each of these lead to dead ends except $(49, 8)$ and $(8, 49)$. For example, $a+1 = 7, b + 1 = 56$ leads to $a = 6$ and $b = 55 = 5 \cdot 11$, which is not a factor of 8!. Next note that $b = 48$ cannot work because $b + 1 = 49$ is not a factor of 200. Therefore $a = 48$ and $b = 7$. This implies that $c + 1 = 200/(b + 1) = 25$ and $c = 24$. Finally, $d = 5$. Thus , $a = 48, b = 7, c = 24$ and $d = 5$.

11. How many multiples of 30 have exactly 30 divisors?

   Solution: I am grateful to Howard Groves (Uniter Kingdom) for this problem. A number $N = 2^i 3^j 5^k \cdot M$ is a multiple of 30 if each of i, j, k are at least 1.

Since 30 has exactly three different prime factors, it follows that $M = 1$, and that $(i + 1)(j + 1)(k + 1) = 30$. Thus, $\{i, j, k\} = \{1, 2, 4\}$, and there are exactly 6 ways to assign the values to the exponents. The numbers are $720, 1200, 7500, 1620, 4050$, and 11250.

12. Find the number of odd divisors of 13!.

    Solution: The prime factorization of 13! is $2^{10}3^5 5^2 \cdot 7 \cdot 11 \cdot 13$. Each odd factor of 13! is a product of odd prime factors, and there are $6 \cdot 3 \cdot 2 \cdot 2 \cdot 2 = 144$ ways to choose the five exponents.

13. Find the smallest positive integer with exactly 60 divisors.

14. (2008 Purple Comet) Find the smallest positive integer the product of whose digits is 9!.

15. (1996 AHSME) If n is a positive integer such that $2n$ has 28 positive divisors and $3n$ has 30 positive divisors, then how many positive divisors does $6n$ have?

    Solution: 35. Let $2^{e_1}3^{e_2}5^{e_3}\cdots$ be the prime factorization of n. Then the number of positive divisors of n is $(e_1+1)(e_2+1)(e_3+1)\cdots$. In view of the given information, we have

    $$28 = (e_1 + 2)(e_2 + 1)P$$

    and

    $$30 = (e_1 + 1)(e_2 + 2)P,$$

    where $P = (e_3+1)(e_4+1)\cdots$. Subtracting the first equation from the second, we obtain $2 = (e_1 - e_2)P$, so either $e_1 - e_2 = 1$ and $P = 2$, or $e_1 - e_2 = 2$ and $P = 1$. The first case yields $14 = (e_1+2)e_1$ and $(e_1+1)^2 = 15$; since e_1 is a nonnegative integer, this is impossible. In the second case, $e_2 = e_1 - 2$ and $30 = (e_1+1)e_1$, from which we find $e_1 = 5$ and $e_2 = 3$. Thus $n = 2^5 3^3$, so $6n = 2^6 3^4$ has $(6+1)(4+1) = 35$ positive divisors.

16. Find the sum of all the divisors of $N = 2^2 \cdot 3^3 \cdot 5^4$.

    Solution: Consider the product $(2^0+2^1+2^2)(3^0+3^1+3^2+3^3)(5^0+5^1+5^2+5^3+5^4)$. Notice that each divisor of N appears exactly once in this expanded product. This sum can be thought of as the volume of a $(2^0 + 2^1 + 2^2) \times (3^0 + 3^1 + 3^2 + 3^3) \times (5^0 + 5^1 + 5^2 + 5^3 + 5^4)$ box. The volume of the box is $7 \cdot 40 \cdot 776 = 217280$.

17. (2010 Mathcounts Competition, Target 7) Find the product P of all the divisors of 6^3, and express your answer in the form 6^t, for some integer t.

Solution: Build the 4×4 matrix of divisors of 6^3, and notice that they can be paired up so that each has 6^3 as their product. There are 8 such pairs, so $P = (6^3)^8 = 6^{24}$. So $t = 24$.

18. Find the largest 10-digit number with distinct digits that is divisible by 11.

    Solution: We want to keep the digits in decreasing order as much as possible. Trying $98765xyuvw$ works because we can arrange for $9+7+5+y+v = 28$, while $8+6+x+u+w = 17$ works. More later.

19. (This was problem 25 on the Sprint Round of the National MATHCOUNTS competition in 2011.) Each number in the set $\{5, 9, 10, 13, 14, 18, 20, 21, 25, 29\}$ was obtained by adding two number from the set $\{a, b, c, d, e\}$ where $a < b < c < d < e$. What is the value of c?

    Solution: $c = 7$. The sum of all 10 numbers is 4 times the sum $a+b+c+d+e$. Why? Now $a + b = 5$ and $d + e = 29$, so $c = (164 \div 4) - 29 - 5 = 41 - 34 = 7$.

20. (This was problem 8 on the Target Round of the National MATHCOUNTS competition in 2011.) How many positive integers less than 2011 cannot be expressed as the difference of the squares of two positive integers?

    Solution: The answer is 505. If $n = a^2 - b^2 = (a - b)(a + b)$, then $a - b$ and $a + b$ have the same parity, so n is either odd or a multiple of 4. So we want to count all the numbers of the form $4k + 1$, $k = 0, 1, \dots, 502$ together with two values 2 and 4, for a total of 505.

21. Consider the number $N = 63000 = 2^3 \cdot 3^2 \cdot 5^3 \cdot 7$.

    (a) How many divisors does N have?
    Solution: Use the formula developed here: $\tau = (3+1)(2+1)(3+1)(1+1) = 96$ divisors.

    (b) How many single digit divisors does N have?

    (c) How many two-digit divisors does N have?

    (d) How many three-digit divisors does N have?

    (e) How many four-digit divisors does N have?

    (f) How many five-digit divisors does N have?

    Solution: The answers are respectively $9, 25, 33, 23$ and 6. You can get these numbers by building the six 4×4 matrices of divisors. Use the basic matrix of divisors of 1000 a model.

	2^0	2^1	2^2	2^3
5^0	1	2	4	8
5^1	5	10	20	40
5^2	25	50	100	200
5^3	125	250	500	1000

. The other five matrices are obtained from this one by multiplying by 3, by 9, by 7, by 21, and by 63.

22. The integer N has both 12 and 15 as divisors. Also, N has exactly 16 divisors. What is N?

    Solution: In order to have both 12 and 15 as divisors, N must be of the form $2^2 \cdot 3 \cdot 5 \cdot d$ for some positive integer d. Since the number of divisors of N is 16, d must be even. But $2^3 \cdot 3 \cdot 5$ has exactly 16 divisors, so $d = 2$ and $N = 120$.

23. How many multiples of 30 have exactly 3 prime factors and exactly 60 factors.

    Solution: This requires some casework. We're considering numbers of the form $N = 2^i 3^j 5^k$ which satisfy $(i+1)(j+1)(k+1) = 60$, $i, j, k \geq 1$. There are 21 such numbers.

24. How many whole numbers n satisfying $100 \leq n \leq 1000$ have the same number of odd divisors as even divisors?

    Solution: Such a number is even but not a multiple of 4, so it follows that we want all the numbers that are congruent to 2 modulo 4, and there are $900/4 = 225$ of them.

25. What is the sum of the digits in the decimal representation of $N = \sqrt{25^{64} \cdot 64^{25}}$?

    Solution: The number is $N = \sqrt{5^{128} \cdot 2^{150}} = 5^{64} \cdot 2^{75} = 10^{64} \cdot 2048$, the sum of whose digits is 14.

26. What is the fewest number of factors in the product 27! that have to be removed so that the product of the remaining numbers is a perfect square?

    Solution: Factor 27! to get $27! = 2^{23} \cdot 3^{13} \cdot 5^6 \cdot 7^3 \cdot 11^2 \cdot 17 \cdot 19 \cdot 23$. The largest square divisor of 27! is $2^{22} \cdot 3^{12} \cdot 5^6 \cdot 7^2 \cdot 11^2$, so each of the primes $2, 3, 7, 17, 19, 23$ must be removed. It takes five factors to do this. We must remove each of $17, 19, 23$, and then we have $2, 3, 7$ to remove. We cannot do this with just one factor, so it takes two, either 6 and 7 or 21 and 2.

27. Euclid asks his friends to guess the value of a positive integer n that he has chosen. Archimedes guesses that n is a multiple of 10. Euler guesses that n is a multiple of 12. Fermat guesses that n is a multiple of 15. Gauss guesses that n is a multiple of 18. Hilbert guesses that n is a multiple of 30. Exactly two of the guesses are correct. Which persons guessed correctly?

    Solution: Hilbert must be wrong because if $30|n$ then so do 10 and 15, which would mean that three of them were right. Now $2|n$ because all but on of the numbers are multiples of 2, and $3|n$ for the same reason. But 5 does not divide n because, if it did, then $30|n$ which contradicts our first observation. Thus Euler and Gauss are the two correct mathematicians.