

Using the Euclidean Algorithm

The Decanting Problem is a liquid measuring problem that usually begins with two unmarked decanters with capacities a and b . Usually a and b are integers. The problem is to determine the smallest amount of liquid that can be measured and how such amount can be obtained, by a process of filling, pouring, and dumping. Specifically, there are three actions we can take:

1. fill an empty decanter,
2. dump out a full decanter, and
3. pour from one decanter to the other until either the receiving decanter is full or the poured decanter is empty.

Let's look at an easy one first. Let $a = 3$ and $b = 5$. We can fill the 3 unit decanter twice, and dump the 5 unit decanter once to get 1 unit of liquid. Algebraically, $2 \cdot 3 - 1 \cdot 5 = 1$. Next, suppose the decanters have capacities 5 units and 7 units. A little experimentation leads to the conclusion that 1 unit of water can be obtained by filling the 5 unit decanter 3 times, pouring repeatedly from the 5 unit to the 7 unit decanter and dumping out the 7 unit decanter twice. A finite state diagram is helpful to follow the procedure:

$$\begin{aligned} (0, 0) &\implies (5, 0) \implies (0, 5) \implies (5, 5) \implies (3, 7) \implies (3, 0) \\ &\implies (0, 3) \implies (5, 3) \implies (1, 7) \implies (1, 0), \end{aligned}$$

where the notation (x, y) means the 5-unit container has x units of liquid and the 7-unit container has y units. Notice that the procedure includes 3 fills and 2 dumps, with fills and dumps alternating and separated by 4 pours. An arithmetic equation representing this is

$$3 \cdot 5 - 2 \cdot 7 = 1.$$

Notice that not only does the arithmetic equation follow from the state diagram, the reverse is also true. That is, given the arithmetic equation, it is an easy matter to construct the state diagram.

In the next example, the least amount that can be measured is not 1. Let the decanters have sizes 15 and 99. Before reading on, can you see why it is impossible to obtain exactly one unit of water? An equation can be obtained for any sequence of moves. Such an equation is of the form

$$15x + 99y = z$$

where exactly one of the integers x and y is negative, and z is the amount obtained. Now notice that the left side is a multiple of 3, so the right side must be also. Thus the least positive amount that can be measured is a multiple of 3. One can also reason this as follows: each fill adds a multiple of 3 units of water to the total amount on hand, each pour leaves the total number unchanged, and each dump removes a multiple of three units from the total, so the amount on hand at each stage is a multiple of 3.

In general, when a and b are integers, the least amount that can be measured is the greatest common divisor of the two decanter sizes, and the Euclidean algorithm, as explained below, tells us how to proceed. Suppose $c = GCD(a, b)$. The Euclidean algorithm yields a solution to

$$c = ax + by$$

where x and y are integers exactly one of which is positive and, except in trivial cases, the other is negative. For convenience, we assume x is positive. Then the solution to the decanting problem is to fill the a capacity decanter x times, repeatedly pouring its contents into the b unit decanter. The b unit decanter will be dumped y times, so the total liquid on hand at the end is the difference $ax - by = c$.

Let's look at another specific example. Again we use the Euclidean Algorithm to solve the decanting problem. There are two stages. The first stage is a sequence of divisions. The second is a sequence of 'unwindings'. For this example, let the decanter sizes be $a = 257$ and $b = 341$. Use the division algorithm to get $341 = 1 \cdot 257 + 84$. Then divide 257 by 84 to get $q = 3$ and $r = 5$. That is, $257 = 3 \cdot 84 + 5$. Continue dividing until the dividend is less than the divisor. Thus 84 divided by 5 yields $84 = 16 \cdot 5 + 4$. Finally, divide 5 by 4 to get $5 = 1 \cdot 4 + 1$. This completes the first stage. Now to unwind, start with the final division scheme, writing $1 = 5 - 1 \cdot 4$. Then replace the 4 with $84 - 16 \cdot 5$ to get $1 = 5 - 1(84 - 16 \cdot 5)$. This is equivalent to $1 = 17 \cdot 5 - 1 \cdot 84$. Check this to be sure. Then replace 5 with $257 - 3 \cdot 84$ to get

$$1 = 17 \cdot (257 - 3 \cdot 84) - 1 \cdot 84,$$

i.e., $1 = 17 \cdot 257 - 52 \cdot 84$. Finally, replace 84 with $341 - 257$ to get $1 = 17 \cdot 257 - 52(341 - 257)$, which we can rewrite as

$$1 = 69 \cdot 257 - 52 \cdot 341.$$

Thus, the solution to the decanting problem is to measure out 1 unit of liquid by filling the 257 unit decanter 69 times, repeatedly pouring its contents into the 341 unit decanter, and, in the process, dumping out the 341 unit decanter 52 times.

There is a related problem sometimes called the postage stamp problem. Here we are given an unlimited supply of two denominations of postage, a and b . If a and b are not relatively prime, and say $d > 1$ is the gcd of a and b , then there is no hope of specifying n cents in postage using as and bs unless n is a multiple of d . If a and b are relatively prime, then there is a largest number k that cannot be made. Here we are repeatedly solving the problem $n = ax + by, x, y \geq 0$.

Theorem Suppose m and n are relatively prime positive integers such that $m > n \geq 2$. Then

1. The equation $mx + ny = mn - m - n$ has no solution with $x \geq 0$ and $y \geq 0$.
2. For all $t \geq 1$, the equation $mx + ny = mn - m - n + t$ has a solution with $x \geq 0$ and $y \geq 0$.

Proof

1. Suppose that $mx + ny = mn - m - n$ with $x \geq 0$ and $y \geq 0$. Since $mn - m - n$ is a positive number that is not a multiple of either m or n , both x and y are positive. Next note that $xm = mn - m - n - yn = mn - m - (y + 1)n$ which implies that $m \mid (y + 1)n$. But since m and n are relatively prime, it follows that $m \mid y + 1$. Therefore $y + 1 = cm$ for some $c \geq 1$. It follows that $y = cm - 1$. Likewise there is a $d \geq 1$ such that $x = dn - 1$. Putting all this together, we have $(dn - 1)m + (cm - 1)n = mn - m - n$, and from this it follows that $(c + d)mn = mn$. But this can happen only when $c + d = 1$, a contradiction.
2. We show that for all $t \geq 1$, there exists $x \geq 0$ and $y \geq 0$ such that $mx + ny = mn - m - n + t$. By the Euclidean algorithm, there exists integers \bar{x}, \bar{y} satisfying $m\bar{x} + n\bar{y} = mn - m - n + t$. Therefore, for all integers $c, x = \bar{x} + cn, y = \bar{y} - cn$ also satisfy $mx + ny = mn - m - n + t$. Since $n < m$ we can choose c so that $0 \leq x \leq n - 1$, and $mx + ny = mn - m - n + t$. If $y \geq 0$, we are done. Suppose $y \leq -1$. Then $mx + ny \leq (n - 1)m - n = mn - m - n$ which contradicts the fact that $mx + ny = mn - m - n + t$, where $t \geq 1$. This completes the proof of the theorem.

Problems

For each pair of values s and t below, use repeated division to find $\gcd(s, t)$ the greatest common divisor of s and t and then use the Euclidean Algorithm to solve the equation $\gcd(s, t) = xs + yt$, where x and y are integers. In other words, solve the decanting problems for decanters of sizes s and t .

1. $s = 22$ and $t = 37$

Solution: $\gcd = 1$ and $x = -5$ and $y = 3$.

2. $s = 105$ and $t = 95$

Solution: $\gcd = 5$ and $x = -9$ and $y = 10$.

3. $s = 89$ and $t = 144$

Solution: $\gcd = 1$ and $x = -21$ and $y = 34$. Notice that these are also Fibonacci numbers.

4. Suppose you have decanters of sizes 99 and 105. Find the least amount of liquid that can be measured, show how to measure that amount, and explain in English why you cannot do better.

Solution: You cannot do better than 3 units because each fill and each dump changes the amount of liquid on hand by a multiple of 3. The Euclidean algorithm yields the values $\gcd = 3$ and $3 = 17 \cdot 99 - 16 \cdot 105$. Alternatively, divide both 99 and 105 by 3 to get 33 and 35. Then $1 = 17 \cdot 33 - 16 \cdot 35$ from which it follows that $3 = 17 \cdot 99 - 16 \cdot 105$.

5. Use scissors to cut a large pile of 1×11 inch strips of paper, and another large pile of 1×7 inch strips of paper. Then show how to use these strips to measure objects that are exactly 1, 2, 3, ... inches long.

6. **Dinner Bill Splitting.** Years ago, my neighbors agreed to celebrate our wedding anniversary with my wife and me. The four of us went to a lovely restaurant, enjoyed a fine dinner, and asked for the bill. When it came, we asked that it be split in half. Realizing the waiter's discomfort, we all set to work on the problem. The bill was for an odd amount, so it could not be split perfectly. However, we realized that, except for the penny problem, we could take half the bill by simply reversing the dollars and the cents. In other words, if we double t dollars and s cents, the result differs by 1 cent from s dollars and t cents. We told the waiter about this. He was astounded: "I never knew you could do it that way." Later, over another dinner with

mathematical friends, the question of uniqueness came up, and pretty soon we realized that this number is the only one with this surprising splitting property. What was the amount of the original bill?

Solution: First, we'll state the problem in a more precise way. Twice t dollars and s cents differs by just one cent from s dollars and t cents. Find s and t . In terms of cents,

$$|100s + t - 2(100t + s)| = 1.$$

This is equivalent to $|98s - 199t| = 1$ which can be interpreted as the two equations $98s - 199t = 1$ or $98s - 199t = -1$. Imagine that we have two decanters with capacities $a = 199$ and $b = 98$. Notice that $GCD(199, 98) = 1$. As we did above, we can use the Euclidean algorithm to find numbers x and y satisfying $199x + 98y = 1$ where exactly one of the numbers x, y is negative. We do this by dividing repeatedly. First, 98 into 199 yields $199 = 2 \cdot 98 + 3$, Then 3 into 98 yields $98 = 32 \cdot 3 + 2$ and finally we can write $1 = 3 - 2$. Next we go to the unwinding stage.

$$\begin{aligned} 1 &= 3 - 2 \\ &= 3 - (98 - 32 \cdot 3) \\ &= 3 - 98 + 32 \cdot 3 \\ &= 33 \cdot 3 - 1 \cdot 98 \\ &= 33(199 - 2 \cdot 98) - 98 \\ &= 33 \cdot 199 - 66 \cdot 98 - 98 \\ &= 33 \cdot 199 - 67 \cdot 98. \end{aligned}$$

Thus, we have the values $s = 67$ and $t = 33$. Indeed, $2 \cdot 33.67 - 67.33 = 0.01$.

7. Chicken McNuggets can be purchased in quantities of 6, 9, and 20 pieces. You can buy exactly 15 pieces by purchasing a 6 and a 9, but you can't buy exactly 10 McNuggets. What is the largest number of McNuggets that can NOT be purchased?

Solution: 43. Both 6 and 9 are multiples of 3, and 20 is 2 bigger than a multiple of 3, ie. $20 \equiv 2 \pmod{3}$. This means that in order to solve $n = 6a + 9b + 20c$ for $n \equiv 1 \pmod{3}$, we must use at least two 20's, and when $n \equiv 2 \pmod{3}$, we must use at least one 20. Consider the number 43. It is one bigger than a multiple of 3, so we must use two 20's, and this clearly

cannot work. On the other hand,

$$\begin{aligned} 44 &= 4 \cdot 6 && + 1 \cdot 20 \\ 45 &= && 5 \cdot 9 \\ 46 &= 1 \cdot 6 && + 2 \cdot 20 \\ 47 &= && 3 \cdot 9 + 1 \cdot 20 \\ 48 &= 8 \cdot 6 \\ 49 &= && 1 \cdot 9 + 2 \cdot 20 \end{aligned}$$

Since we have shown how to construct 6 consecutive integers, it is clear that we can construct all the larger integers. So 43 is the largest non-obtainable number of McNuggets.

See

<http://www.mathnerds.com/mathnerds/best/mcnuggets/solution.aspx>

8. Does the equation $399x + 703y = 114$ have an integer solution in x and y ?

Solution: Yes, using the Euclidean algorithm we find that $\gcd(399, 703) = 19$ and $399(-7) + 703(4) = 19$. Since $114 = 6 \cdot 19$, $399(-42) + 703(24) = 114$.

9. Does the equation $399x + 703y = 115$ have an integer solution in x and y ?

Solution: No. Suppose there is an integer solution in x and y . Since $19|399$ and $19|703$, 19 must divide $399x + 703y$. But $115 = 1 \pmod{19}$, so there cannot be an integer solution.

10. For m and n integers, characterize those integers k for which the equation $mx + ny = k$ has integer solutions in x and y .

Solution: For integers m and n , the equation $mx + ny = k$ has integer solutions in x and y if and only if $\gcd(m, n)|k$. Proof: Suppose there are integers x and y such that $mx + ny = k$. Since $\gcd(m, n)|m$ and $\gcd(m, n)|n$, it follows that $\gcd(m, n)|(mx + ny)$, and thus $\gcd(m, n)|k$.

11. **The Subtraction Game** In the Subtraction Game, two players start with some positive integers written on a board. The first player must find a pair of numbers whose positive difference is not already written on the board. Then he writes this new number on the board. At each stage, the next player finds a positive difference between two numbers on the board that is not already written on the board and writes it on the board. The first player who cannot

find a new positive difference loses. For each of the sets of numbers listed below, decide how many numbers will be on the board at the end of the game. Use this information to state whether you would like to move first or not (in order to win).

- (a) 101, 102, 103

Solution: The greatest common divisor of the three numbers is 1, and there are $103 \div 1 = 103$ numbers written on the board at the end of the game, $103 - 3 = 100$ of which are new, so the second player wins.

- (b) 3105 and 4104

Solution: The greatest common divisor of $3105 = 3^2 \cdot 5^2 \cdot 23$ and $4104 = 2^3 \cdot 3^2 \cdot 19$ is 9, and there are $4104 \div 9 = 456$ multiples of 9 written on the board at the end of the game, $456 - 2 = 454$ of which are new, so the second player wins.

- (c) 21, 24, 81, 87

Solution: The greatest common divisor of the four numbers is 3, and there are $87 \div 3 = 29$ multiples of 3 written on the board at the end of the game, $29 - 4 = 25$ of which are new, so the first player wins.