WELCOME to CALCULUS

Calculus is the mathematics of CHANGE and almost everything in our world is changing.

Calculus is among the most important and useful developments of human thought, and, even though it is over 300 years old, it is still considered the beginning and cornerstone of modern mathematics. It is a wonderful and beautiful and useful set of ideas and techniques

You will see the fundamental ideas of this course over and over again in future courses in mathematics, the sciences (physical, biological and social) as well as in economics, engineering and others.

But calculus is an intellectual step up from your previous mathematics courses. Many of the ideas are more carefully defined, and they have both a functional and a graphical meaning Some of the algorithms are more complicated, and in many cases you will need to decide on the appropriate algorithm to use. And there is a huge variety of applications, too many to be able to discuss each one in class in detail.

What this means for you, the student.

Probably more than in your previous mathematics classes you need

   * to think about the concepts as well as the techniques,

   * to think about the patterns as well as the individual steps,

   * to think about the meaning of the concepts and techniques in the context of particular applications,

   * to think about how the ideas and techniques apply to functions given by graphs and tables as well as by formulas, and

   * to spend enough time (1 to 2 hours each day) doing problems to sort out the concepts and to master the techniques and to get better and more efficient with the algebra skills that are vital to success.

Sometimes all this mental stretching can seem overwhelming, but stick with it (and do lots of problems). It can even become fun.

So welcome to calculus.

WELCOME TO CALCULUS

Calculus was first developed more than three hundred years ago by Sir Isaac Newton and Gottfried Leibniz to help them describe and understand the rules governing the motion of planets and moons. Since then, thousands of other men and women have refined the basic ideas of calculus, developed new techniques to make the calculations easier, and found ways to apply calculus to problems besides planetary motion. Perhaps most importantly, they have used calculus to help understand a wide variety of physical, biological, economic and social phenomena and to describe and solve problems in those areas.

The discovery, development, and application of calculus is a great intellectual achievement, and now you have the opportunity to share in that achievement. You should feel exhilarated. You may also be somewhat concerned, a common reaction of students just beginning to study calculus. You need to be concerned enough to work to master calculus and confident enough to keep going when you don't understand something at first.

Part of the beauty of calculus is that it is based on a few very simple ideas. Part of the power of calculus is that these simple ideas can help us understand, describe, and solve problems in a variety of fields. This book tries to emphasize both the beauty and the power.

In Section 0.1 (Preview) we will look at the main ideas which will continue throughout the book: the problems of tangent lines and areas. We will also consider a process that underlies both of those problems, the limiting process of approximating a solution and then getting better and better approximations until we finally get an exact solution.

Sections 0.2 (Lines), 0.3 (Functions), and 0.4 (Combinations of Functions) contain review material which you need to recall before we begin calculus. The emphasis in these sections is on material and skills you will need to succeed in calculus. You should have worked with most of this material in previous courses, but the emphasis and use of the material in these sections may be different than in those courses.

Section 0.5 (Mathematical Language) discusses a few key mathematical phrases. It considers their use and meaning and some of their equivalent forms. It will be difficult to understand the meaning and subtleties of calculus if you don't understand how these phrases are used and what they mean.

1.1   PREVIEW OF CALCULUS

Two Basic Problems

Beginning calculus can be thought of as an attempt, a historically successful attempt, to solve two fundamental problems. In this section we will start to examine geometric forms of those two problems and some fairly simple ways to attempt to solve them. At first, the problems themselves may not appear very interesting or useful, and

the methods for solving them may seem crude, but these simple problems and methods have led to one of the most beautiful, powerful, and useful creations in mathematics: Calculus.

First Problem: Finding the Slope of a Tangent Line

Suppose we have the graph of a function y = f(x), and we want to find the equation of the line which is **tangent** to the graph at a particular point P on the graph (Fig. 1). (We will give a precise definition of **tangent** in Section 1.0. For now, think of the tangent line as the line which touches the curve at the point P and stays close to the graph of y = f(x) near P.) We know that the point P is on the tangent line, so if the x–coordinate of P is x = a, then the y–coordinate of P must be y = f(a) and P = (a, f(a)). The only other information we need to find the equation of the tangent line is its slope, m_{tan}, and that is where the difficulty arises. In algebra, we needed
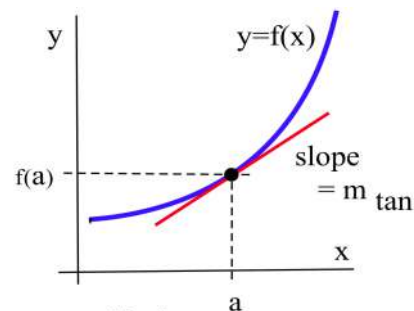
two points in order to determine a slope, and so far we only have the point P. Lets simply pick a second point, say Q, on the graph of y = f(x). If the x–coordinate of Q is b, then the y–coordinate is f(b), so Q = (b, f(b)). The slope of the line through P and Q is

$$m_{PQ} = \frac{\text{rise}}{\text{run}} = \frac{f(b) - f(a)}{b - a} \quad .$$

If we drew the graph of y = f(x) on a wall, put nails at the points P and Q on the graph, and laid a straightedge on the nails, then the straightedge would have slope m_{PQ} (Fig. 2). However, the slope m_{PQ} can be very different from the value we want, the slope m_{tan} of the tangent line. The key idea is that if the point Q is **close to** the point P, then the slope m_{PQ} is **close to** the slope we want, m_{tan}. Physically, if we slide the

nail at Q along the graph towards the fixed point P, then the slope, $m_{PQ} = \frac{f(b) - f(a)}{b - a}$, of the straightedge gets closer and closer to the slope, m_{tan} , of the tangent line. If the value of b is very close to a, then

the point Q is very close to P, and the value of m_{PQ} is very close to the value of m_{tan}. Rather than defacing walls with graphs and nails, we can calculate

$$m_{PQ} = \frac{f(b) - f(a)}{b - a}$$

and examine the values of m_{PQ} as b gets closer and closer to a. We say that m_{tan} is the limiting value of m_{PQ} as b gets very close to a, and we write

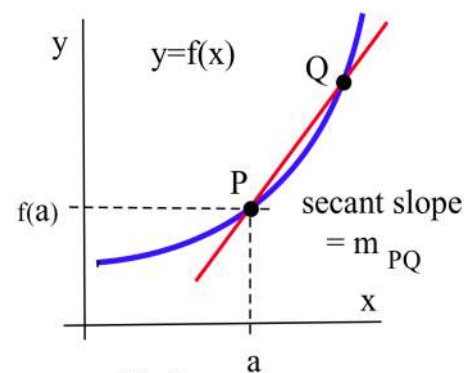$$m_{tan} = \lim_{b \to a} \frac{f(b) - f(a)}{b - a} \quad .$$

The slope m_{tan} of the tangent line is called the **derivative** of the function $f(x)$ at the point P, and this part of calculus is called **differential calculus**. Chapters 2 and 3 begin differential calculus.

The slope of the tangent line to the graph of a function will tell us important information about the function and will allow us to solve problems such as:
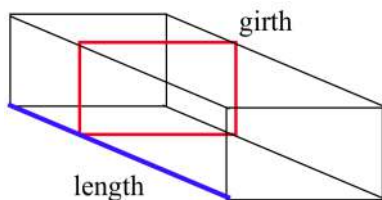

Fig. 3

"The US Post Office requires that the length plus the girth (Fig. 3) of a package not exceed 84 inches. What is the largest volume which can be mailed in a rectangular box?"

An oil tanker was leaking oil, and a 4 inch thick oil slick had formed. When first measured, the slick had a radius 200 feet and the radius was increasing at a rate of 3 feet per hour. At that time, how fast was the oil leaking from the tanker?

Derivatives will even help us solve such "traditional" mathematical problems as finding solutions of equations like $x^2 = 2 + \sin(x)$ and $x^9 + 5x^5 + x^3 + 3 = 0$.

Second Problem: Finding the Area of a Shape

Suppose we need to find the area of a leaf (Fig. 4) as part of a study of how much energy a plant gets from sunlight. One method for finding the area would be to trace the shape of the leaf onto a piece of paper and then divide the region into "easy" shapes such as rectangles and triangles whose areas we could calculate. We could add all of the "easy" areas together to get the area of the leaf.

Fig. 4

A modification of this method would be to trace the shape onto a piece of graph paper and then count the number of squares completely inside the edge of the leaf to get a lower estimate of the area and count the number of squares that touch the leaf to get an upper estimate of the area. If we repeat this process with smaller squares, we have to do more counting and adding, but our estimates are closer together and closer to the actual area of the leaf. (This area can also be approximated using a sheet of paper, scissors and an accurate scale. How?)
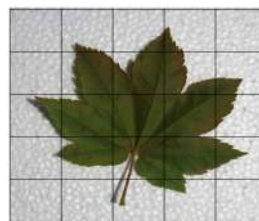
Each square is 1 sq. cm
totally inside = 1
partially inside = 18
$1 \le$ number ≤ 19
1 sq.cm $\le$ area $\le$ 19 sq.cm
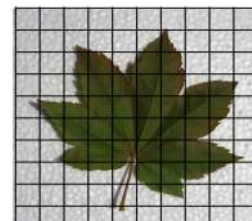
Each square is 1/4 sq/ cm
totally inside = 16
partially inside = 34
$16 \le$ number ≤ 50
4 sq.cm $\le$ area $\le$ 12.5 sq.cm

We can calculate the area A between the graph of a function $y = f(x)$ and the x–axis (Fig. 5) by using similar methods. We can divide the area into strips of width w and determine the lower and upper values of $y = f(x)$ on each strip. Then we can approximate the area of each rectangle and add all of the little areas together to get A_w, an approximation of the exact area. The key idea is that if w is small, then the rectangles are narrow, and the approximate area A_w is very

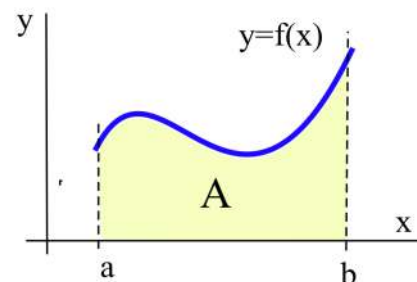Fig. 5: What is the area of region A?

close to the actual area A. If we take narrower and narrower rectangles,
the approximate areas get closer and closer to the actual area: $A = \underset{w \to 0}{\text{limit}}\ A_w$.

Fig. 5: What is the area of region A?

The process we used is the basis for a technique called **integration**, and this
part of calculus is called **integral calculus**. Integral calculus and integration
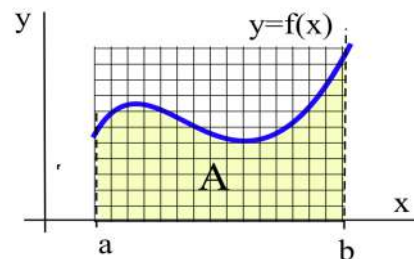will begin in Chapter 4.

The process of taking the limit of a sum of "little" quantities will give us important information about the function
and will also allow us to solve problems such as:

"Find the length of the graph of $y = \sin(x)$ over one period (from $x = 0$ to $x = 2\pi$)."

"Find the volume of a torus ("doughnut") of radius 1 inch
which has a hole of radius 2 inches. (Fig. 6)"

"A car starts at rest and has an acceleration of
$5 + 3\sin(t)$ feet per second per second in the northerly
direction at time t seconds. Where will the car be, relative
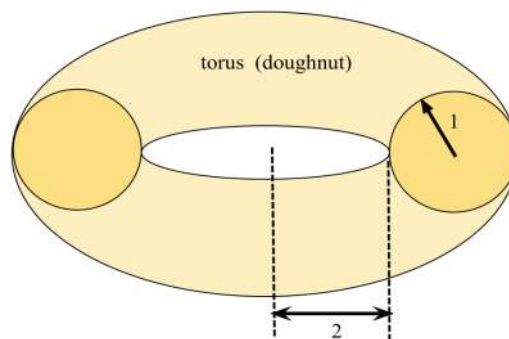to its starting position, after 100 seconds?"

Fig.6:   What is the volume of the torus?

A Unifying Process: Limits

We used a similar processes to "solve" both the tangent line problem and the area problem. First, we found a way
to get an approximate solution, and then we found a way to improve our approximation. Finally, we asked what
would happen if we continued improving our approximations "forever", that is, we "took a limit." For the tangent
line problem, we let the point Q get closer and closer and closer to P, the limit as b approached a. In the area
problem, we let the widths of the rectangles get smaller and smaller, the limit as w approached 0. Limiting
processes underlie derivatives, integrals, and several other fundamental topics in calculus, and we will examine
limits and their properties in Chapter 1.

Two Sides Of The Same Coin: Differentiation and Integration

Just as the set–up of each of the two basic problems involved a limiting process, the solutions to the two problems
are also related. The process of differentiation for solving the tangent line problem and the process of integration
for solving the area problem turn out to be "opposites" of each other: each process undoes the effect of the other
process. The Fundamental Theorem of Calculus in Chapter 4 will show how this "opposite" effect works.

Extensions of the Main Problems

The first 5 chapters present the two key ideas of calculus, show "easy" ways to calculate derivatives and integrals, and examine some of their applications. And there is more. In later chapters, new functions will be examined and ways to calculate their derivatives and integrals will be found. The approximation ideas will be extended to use "easy" functions, such as polynomials, to approximate the values of "hard" functions such as $\sin(x)$ and e^x . And the notions of "tangent lines" and "areas" will be extended to 3–dimensional space as "tangent planes" and "volumes".

> **Success in calculus will require time and effort on your part, but such a beautiful and powerful field is worth that time and effort.**

PROBLEMS

(Solutions to odd numbered problems are given at the back of the book.)

Problems 1 – 4 involve estimating slopes of tangent lines.

1)   Sketch the lines tangent to the curve shown in Fig. 7 at
     x = 1, 2 and 3. Estimate the slope of each of the tangent
     lines you drew.

Fig.7
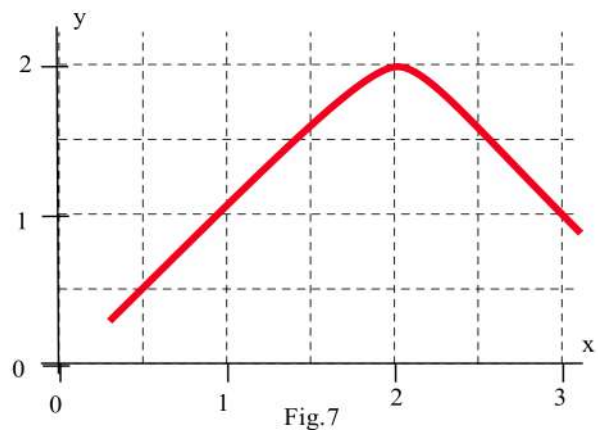
2)   Fig. 8 shows the weight of a "typical" child from age 0 to
     age 24 months. (Each of your answers should have the
     units "kilograms per month.")
     (a)   What was the average rate of weight gain from
           month 0 to month 24?
     (b)   What was the average weight gain from month
           9 to month 12? from month 12 to month 15?
     (c)   Approximately how fast was the child gaining
           weight at age 12 months?   at age 3 months?

Fig.8
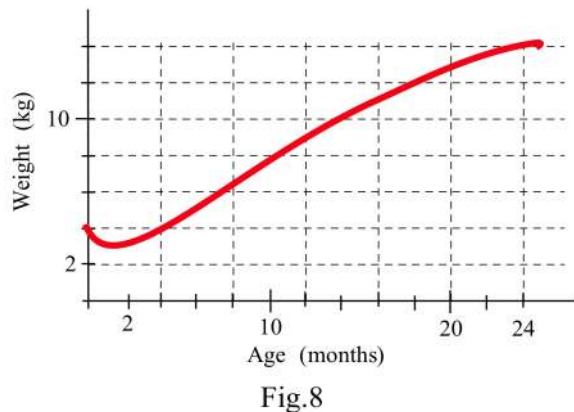
3)   Fig. 9 shows the temperature of a cup of coffee during a ten minute period. (Each of your answers in (a) – (c) should have the units "degrees per minute.")

(a)   What was the average rate of cooling from minute 0 to minute 10?

(b)   What was the average rate of cooling from minute 7 to minute 8? from minute 8 to minute 9?

(c)   What was the rate of cooling at minute 8?   at minute 2?

(d)   When was the cold milk added to the coffee?


Temperature of a cup of coffee

Fig. 9
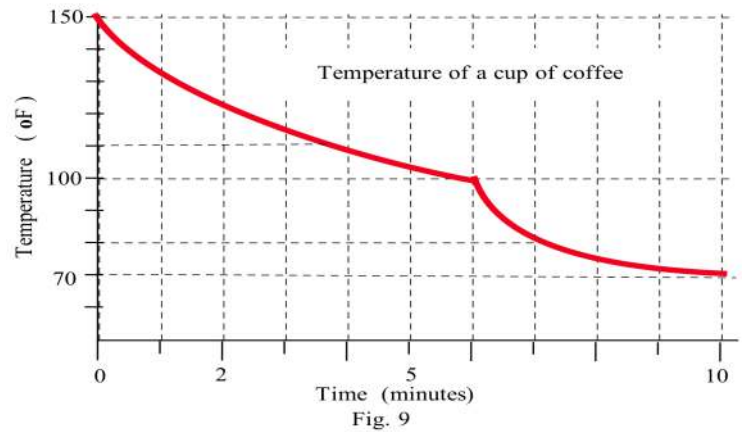
4)   Describe a method for determining the slope at the middle of a steep hill on campus

(a)   using a ruler, a long piece of string, a glass of water and a loaf of bread.

(b)   using a protractor, a piece of string and a helium–filled balloon.

Problems 5 and 6 involve approximating areas.

5)   Approximate the area of the leaf in Fig. 4 .

6)   Fig. 10 shows temperatures during the month of November.

(a)   Approximate the shaded area between the temperature curve and the 65° line from Nov. 15 to Nov. 25.

(b)   The area of the "rectangle" is (base)(height) so what are the units of your answer in part (a)?

(c)   Approximate the shaded area between the temperature curve and the 65° line from Nov. 5 to Nov. 30.

(d)   Who might use or care about these results?

November date

Fig. 10
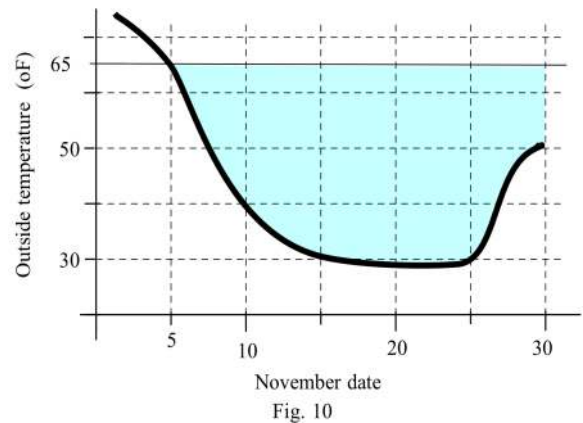
7)   Describe a method for determining the volume of a standard incandescent light bulb using a ruler, a tin coffee can, a scale, and a jug of wine.

1.2 LINES IN THE PLANE

The first graphs and functions you encountered in algebra were straight lines and their equations. These lines were easy to graph, and the equations were easy to evaluate and to solve. They described a variety of physical, biological and business phenomena such as **d = rt** relating the distance d traveled to the rate r and time t of travel, and $C = \dfrac{5}{9}(F - 32)$ for converting the temperature in Fahrenheit degrees (F) to Celsius (C).

The first part of calculus, differential calculus, will deal with the ideas and techniques and applications of tangent **lines** to the graphs of functions, so it is important that you understand the graphs and properties and equations of straight lines.

The Real Number Line

The real numbers (consisting of all integers, fractions, rational and irrational numbers) can be represented as a line, called the **real number line** (Fig. 1). Once we have selected a starting location, called the **origin**, a positive direction (usually up or to the right), and unit of length, then every number can be located as a point on the number line. If we move from a point x = a to point x = b on the line (Fig. 2), then we will have moved an **increment** of b – a. This increment is denoted by the symbol **Δx** (read "delta x").
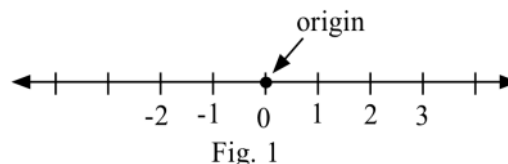
The Greek capital letter delta, Δ, will appear often in the future and will represent the "change" in something. If b is larger than a, then we will have moved in the positive direction, and Δx = b – a will be positive. If b is smaller than a, then Δx = b – a will be negative and we will have moved in the negative direction. Finally, if Δx = b – a is zero, then a=b and we did not move at all.
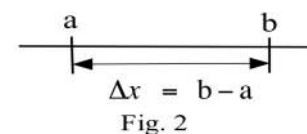
We can also use the Δ notation and absolute values to write the **distance** that we have moved. On the number line, the distance from x = a to x = b is

$$\textbf{dist(a,b)} \; = \; \begin{cases} b - a & \text{if } b \geq a \\ a - b & \text{if } b < a \end{cases} \quad \text{or simply,} \quad \text{dist(a,b)} \; = \; |\, b - a \,| = |\, \Delta x \,| = \sqrt{(\Delta x)^2} \;.$$

The **midpoint** of the segment from x = a to x = b is the point $M = \dfrac{a + b}{2}$ on the number line.

Example 1: Find the length and midpoint of the interval from x = –3 to x = 6.

Solution: Dist(–3,6) = | 6 – (–3) | = | 9 | = 9. The midpoint is at $\dfrac{(-3) + (6)}{2}$ = 3/2 .

Practice 1: Find the length and midpoint of the interval from x = –7 to x = –2.

(Note: Solutions to Practice Problems are given at the end of each section, after the Problems.)

The Cartesian Plane

A real number **plane** (Fig. 3) is determined by two perpendicular number lines, called the **coordinate axes**, which intersect at a point, called the **origin of the plane** or simply the origin. Each point P in the plane can be described by an **ordered pair** (x,y) of numbers which specify how far, and in which directions, we must move from the origin to reach the point P. The point $P = (x,y)$ can then be located in the plane by starting at the origin and moving x units horizontally and then y units vertically. Similarly, each point in the plane can be labeled with the ordered pair (x,y) which directs us how to reach that point from the origin. In this book, a point in the plane will be labeled either with a name, say P, or with an ordered pair (x,y), or with both $P = (x,y)$. This coordinate system is called the **rectangular coordinate system** or the **Cartesian coordinate system** after Rene Descartes, and the resulting plane is called the Cartesian Plane. The coordinate axes divide the plane into four **quadrants** which are labeled quadrants I, II, III and IV as in Fig. 4   We will often call the horizontal axes the **x-axis** and the vertical axis the **y-axis** and then refer to the plane as the **xy-plane**. This choice of x and y as labels for the axes is simply a common choice, and we will sometimes prefer to use different labels and even different units of measure on the two axes.
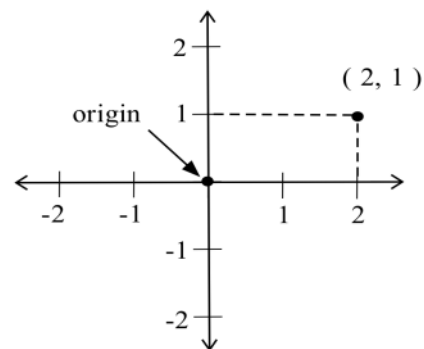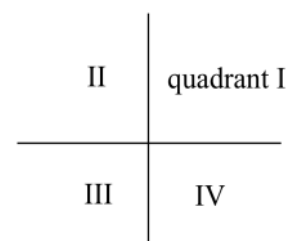
Increments and Distance Between Points In The Plane

If we move from a point $P = (x_1,y_1)$ to a point $Q = (x_2,y_2)$ in the plane, then we will have two **increments** or changes to consider. The increment in the x or horizontal direction is $x_2 - x_1$ which is denoted by $\Delta x = x_2 - x_1$. The increment in the y or vertical direction is $\Delta y = y_2 - y_1$. These increments are shown in Fig. 5 . Δx does not represent Δ times x, it represents the difference in the x coordinates: $\Delta x = x_2 - x_1$.
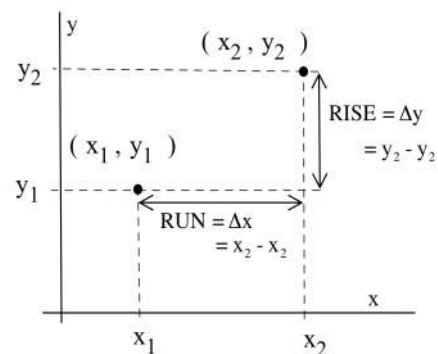
The distance between the points $P = (x_1,y_1)$ and $Q = (x_2,y_2)$ is simply an application of the Pythagorean formula for right triangles, and

$$\text{dist}(P,Q) = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}\ .$$

The **midpoint** M of the line segment joining P and Q is $M = \left(\dfrac{x_1 + x_2}{2} , \dfrac{y_1 + y_2}{2} \right).$

Example 2: Find an equation describing the points $P = (x,y)$ which are equidistant from $Q = (2,3)$ and

$R = (5,–1)$. (Fig. 6)

Solution: The points $P=(x,y)$ must satisfy $\text{dist}(P,Q) = \text{dist}(P,R)$ so

$$\sqrt{(x–2)^2+(y–3)^2} = \sqrt{(x–5)^2+(y–(–1))^2} .$$

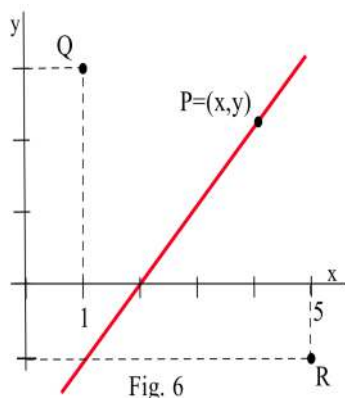By squaring each side we get $(x–2)^2+(y–3)^2 = (x–5)^2+(y+1)^2$.

Then $x^2 – 4x + 4 + y^2 – 6y + 9 = x^2 – 10x + 25 + y^2 + 2y + 1$

so $–4x – 6y + 13 = –10x + 2y + 26$ and $y = .75x – 1.625$, a straight line.

Every point on the line $y = .75x – 1.625$ is equally distant from Q and R.

Practice 2: Find an equation describing all points $P = (x,y)$ equidistant from $Q = (1,–4)$ and $R = (0,–3)$.

A **circle** with radius r and center at the point $C = (a,b)$ consists of all points $P = (x,y)$ which are at a distance of r from the center C: the points P which satisfy $\text{dist}(P,C) = r$.

Example 3: Find the equation of a circle with radius $r = 4$ and center $C = (5,–3)$. (Fig. 7)

Solution: A circle is the set of points $P=(x,y)$ which are at a fixed
distance r from the center point C, so this circle will be the
set of points $P=(x,y)$ which are at a distance of 4 units from
the point $C = (5,–3)$. P will be on this circle if $\text{dist}(P,C) = 4$.
Using the distance formula and simplifying,

$$\sqrt{(x–5)^2 + (y+3)^2} = 4 \quad \text{so} \quad (x–5)^2 + (y+3)^2 = 16 \quad \text{or}$$

$x^2 – 10x + 25 + y^2 + 6y + 9 = 16.$
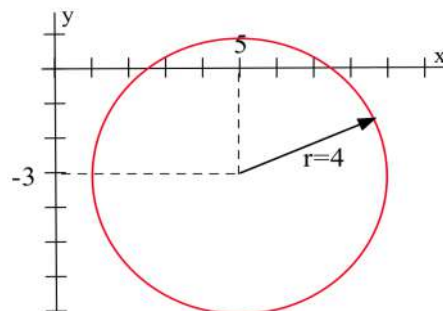
Practice 3: Find the equation of a circle with radius $r = 5$ and center $C = (–2,6)$.

The Slope Between Points In The Plane

In one dimension on the number line, our only choice was to move in the positive direction (so the x–values were increasing) or in the negative direction. In two dimensions in the plane, we can move in infinitely many directions and a precise means of describing direction is needed. The **slope** of the line segment joining $P = (x_1,y_1)$ to $Q = (x_2,y_2)$, is

$$m = \{ \text{slope from P to Q} \} = \frac{\text{rise}}{\text{run}} = \frac{y_2-y_1}{x_2-x_1} = \frac{\Delta y}{\Delta x} .$$

In Fig. 8, the slope of a line measures how fast we rise or fall as we move from left to right along the line. It measures the rate of change of the y-coordinate with respect to changes in the x-coordinate. Most of our work will occur in 2 dimensions, and **slope** will be a very useful concept which will appear often.

If P and Q have the same x coordinate, then $x_1 = x_2$ and

$\Delta x = 0$. The line from P to Q is **vertical** and the slope

$m = \Delta y/\Delta x$ is **undefined** because $\Delta x = 0$. If P and Q

have the same y coordinate, then $y_1 = y_2$ and

$\Delta y = 0$, so the line is **horizontal** and the slope is

$m = \Delta y/\Delta x = 0/\Delta x = 0$ (assuming $\Delta x \neq 0$).

Fig. 8
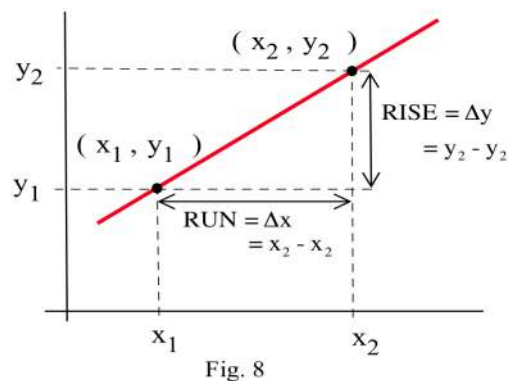
Practice 4:   For P = (–3,2) and Q = (5,–14), find Δx, Δy , and the slope of the line segment from P to Q.

If the coordinates of P or Q contain variables, then the slope m is still given by $\Delta y/\Delta x$, but we will need to use algebra to evaluate and simplify m.

Example 4:   Find the slope of the line segment from P = (1,3) to

Q = (1+h, 3 + 2h). (Fig. 9)

Solution: $y_1 = 3$ and $y_2 = 3 + 2h$ so $\Delta y = (3 + 2h) - (3) = 2h$. $x_1 = 1$ and

$x_2 = 1 + h$ so $\Delta x = (1 + h) - (1) = h$, and the slope is $m = \dfrac{\Delta y}{\Delta x} = \dfrac{2h}{h} = 2$.

In this example, the value of m is the constant 2 and does not depend on

the value of h.

Fig. 9

Practice 5:   Find the slope and midpoint of the line segment from

P = (2,–3) to Q = (2 + h, –3 + 5h).

Example 5:   Find the slope between the points $P = (x, x^2 + x)$ and

$Q = (a, a^2 + a)$ for $a \neq x$.

Solution: $y_1 = x^2+x$ and $y_2 = a^2+a$ so $\Delta y = (a^2 + a) - (x^2 + x)$. $x_1 = x$ and $x_2 = a$ so $\Delta x = a-x$ and

the slope is $m = \dfrac{\Delta y}{\Delta x} = \dfrac{(a^2+a) - (x^2+x)}{a-x} = \dfrac{a^2 - x^2 + a - x}{a - x}$

$= \dfrac{(a-x)\,(a+x) + (a-x)}{a - x} = \dfrac{(a-x) \cdot \{(a+x) + 1\}}{a - x} = (a + x) + 1.$

In this example, the value of m depends on the values of both a and x.
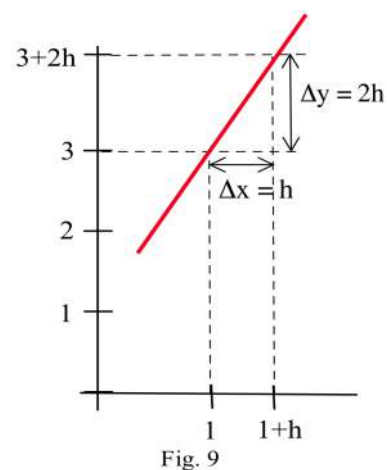
Practice 6:   Find the slope between $P = (x, 3x^2 + 5x)$ $Q = (a, 3a^2 + 5a)$ for $a \neq x$.

In application problems it is important to read the information and the questions very carefully. Including the units of measurement of the variables can help you avoid "silly" answers.

Example 6:   In 1970 the population of Houston was 1,233,535 and in 1980 it was 1,595,138. Find the slope of the line through the points (1970, 1233535) and (1980, 1595138).

Solution: $m = \dfrac{\Delta y}{\Delta x} = \dfrac{1595138 - 1233535}{1980 - 1970} = \dfrac{361603}{10} = 36{,}160.3$

But 36,160.3 is just a number which may or may not have any meaning to you. If we include the units of measurement along with the numbers we will get a more meaningful result:

$$m = \frac{\Delta y}{\Delta x} = \frac{1595138 \text{ people} - 1233535 \text{ people}}{\text{year } 1980 - \text{year } 1970}$$

$$= \frac{361603 \text{ people}}{10 \text{ years}} = 36{,}160.3 \text{ people/year}$$

which says that during the decade from 1970 to 1980 the population of Houston grew at an average rate of 36,160 people per year.

If the x–unit is time in hours and the y-unit is distance in kilometers, then m is $\dfrac{\Delta y \text{ kilometers}}{\Delta x \text{ hours}}$, so the units for m are kilometers/hour ("kilometers per hour"), a measure of velocity, the rate of change of distance with respect to time. If the x-unit is the number of employees at a bicycle factory and the y-unit is the number of bicycles manufactured, then m is $\dfrac{\Delta y \text{ bicycles}}{\Delta x \text{ employees}}$, and the units for a are bicycles/employee ("bicycles per employee"), a measure of the rate of production per employee.

EQUATIONS OF LINES

Every line has the property that the slope of the segment between any two points on the line is the same, and this constant slope property of straight lines leads to ways of finding equations to represent nonvertical lines.

Point–Slope Equation

In calculus, we will usually know a point on the line and the slope of the line so the point–slope form will be the easiest to apply, and the other forms of equations for straight lines can be derived from the point–slope form.

If L is a nonvertical line through a known point $P = (x_1, y_1)$ with a known slope m (Fig. 10), then the equation of the line L is

slope = m

(x , y)

(x_1 , y_1)

y_1

equation of the line is
$y_1 - y_2 = m(x_1 - x_2)$

x_1
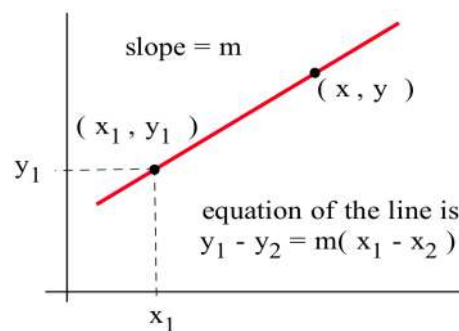
Fig. 10

> **Point-Slope:** $y - y_1 = m(x - x_1).$

Example 7: Find the equation of the line through $(2,-3)$ with slope 5.

Solution: The solution is simply a matter of knowing and using the point–slope
formula. $m = 5$, $y_1 = -3$ and $x_1 = 2$ so $y - (-3) = 5(x - 2)$. This
equation simplifies to $y = 5x - 13$ (Fig. 11).

The equation of a **vertical line** through a point $P = (a,b)$ is $x = a$. The only
points $Q = (x,y)$ on the vertical line through the point P have the same x–
coordinate as P.
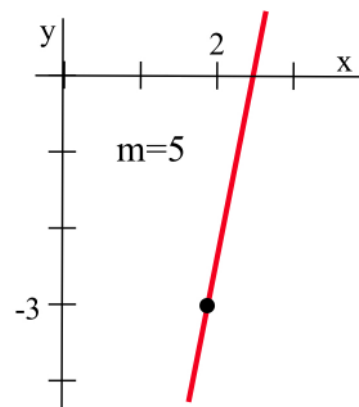
Fig. 11

Two–Point and Slope–Intercept Equations

If two points $P = (x_1,y_1)$ and $Q = (x_2,y_2)$ are on the line L, then we can calculate the slope between them and
use the first point and the point–slope equation to get the equation of L:

> **Two Points:** $y - y_1 = m(x - x_1)$ where $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Once we have the slope m, it does not matter whether we use P or Q as the point. Either choice will give the
same simplified equation for the line.

It is common practice to rewrite the equation of the line in the form $y = mx + b$, the **slope-intercept** form
of the line. The line $y = \mathbf{m}x + \mathbf{b}$ has slope **m** and crosses the y-axis at the point $(0, \mathbf{b})$.

Practice 7: Use the $\Delta y / \Delta x$ definition of slope to calculate the slope of the line $y = mx + b$.

The point-slope and the two-point formulas are usually more useful for finding the equation of a line, but the
slope-intercept form is usually the most useful form for an answer because it allows us to easily picture the
graph of the line and to quickly calculate y-values.

Angles Between Lines

The **angle of inclination** of a line with the x-axis is the smallest
angle θ which the line makes with the positive x-axis as measured
from the x-axis counterclockwise to the line (Fig. 12). Since the slope
$m = \Delta y / \Delta x$ and since $\tan(\theta) = $ opposite/adjacent, we have that

$$m = \tan(\theta) .$$

The slope of the line is the tangent of the angle of inclination of the line.

Fig. 12

Parallel and Perpendicular Lines

Two parallel lines L_1 and L_2 make equal angles with the
x-axis so their angles of inclination will be equal (Fig. 13) and
so will their slopes. Similarly, if their slopes m_1 and m_2 are equal,

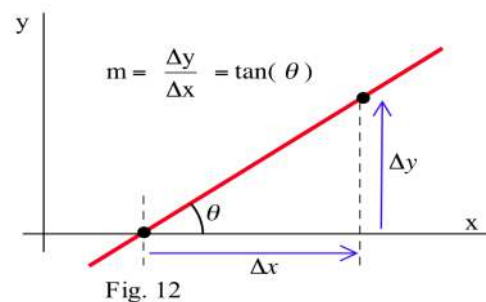

Fig. 13

then the equations of the lines will always differ by a constant:

$$y_1 - y_2 = \{m_1 x + b_1\} - \{m_2 x + b_2\}$$
$$= (m_1 - m_2)x + (b_1 - b_2)$$
$$= b_1 - b_2$$

which is a constant so the lines will be parallel. These two ideas
can be combined into a single statement:

> **Two nonvertical lines L_1 and L_2 with slopes m_1 and m_2 are parallel**
>
> **if and only if $m_1 = m_2$.**
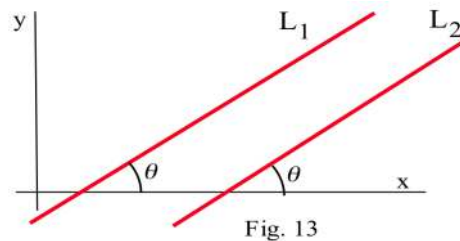
Practice 8:   Find the equation of the line in Fig. 14 which contains
the point (–2,3) and is parallel to the line $3x + 5y = 17$.

If two lines are perpendicular and neither line is vertical, the situation
is a bit more complicated (Fig. 15).

Fig. 14

Assume L_1 and L_2 are two nonvertical lines that intersect at the
origin (for simplicity) and that $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ are
points away from the origin on L_1 and L_2 , respectively. Then the
slopes of L_1 and L_2 will be $m_1 = y_1/x_1$ and $m_2 = y_2/x_2$. The line
connecting P and Q forms the third side of the triangle OPQ , and this will
be a right triangle if and only if L_1 and L_2 are perpendicular. In particular,
L_1 and L_2 are perpendicular if and only if the triangle OPQ satisfies the
Pythagorean theorem:
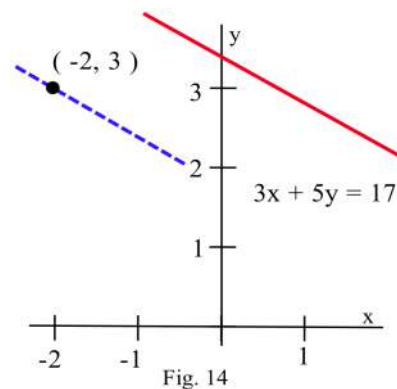
$\{\text{dist}(O,P)\}^2 + \{\text{dist}(O,Q)\}^2 = \{\text{dist}(P,Q)\}^2$ or

$(x_1 - 0)^2 + (y_1 - 0)^2 + (x_2 - 0)^2 + (y_2 - 0)^2$

$\quad = (x_1 - x_2)^2 + (y_1 - y_2)^2$.

By squaring and simplifying, this last equation reduces to

$0 = -2x_1x_2 - 2y_1y_2$ so $y_2/x_2 = -x_1/y_1$ and

$m_2 = y_2/x_2 = -x_1/y_1 = -\dfrac{1}{(y_1/x_1)} = -\dfrac{1}{m_1}$.

We have just proved the following result:

> **Two nonvertical lines L_1 and L_2 with slopes m_1 and m_2 are perpendicular if
> and only if their slopes are negative reciprocals of each other: $m_2 = -\dfrac{1}{m_1}$.**

Practice 9:   Find the line which goes through the point $(2,-5)$ and is perpendicular to the line $3y - 7x = 2$.

Example 8: Find the distance (the shortest distance) from the point $(1,8)$ to the line L: $3y - x = 3$.

Solution: This is a sophisticated problem which requires several steps to solve.
First we need a picture of the problem (Fig. 16). We will find the line L*
through the point $(1,8)$ and perpendicular to L. Then we will find the point
P where L and L* intersect, and, finally, we will find the distance from P
to $(1,8)$.

(i)   L has slope $1/3$ so L* has slope $m = -\dfrac{1}{1/3} = -3$, and

  L* has the equation $y - 8 = -3(x - 1)$ which simplifies

  to $y = -3x + 11$.

(ii) We can find the point of intersection of L and L* by replacing the y
in the equation for L with the y from L* so

  $3(-3x + 11) - x = 3$. Then $x = 3$ so $y = -3x + 11 = -3(3) + 11 = 2$,
  so L and L* intersect at $P = (3,2)$.

(iii) Finally, the distance from L to $(1,8)$ is just the distance from the point $(1,8)$ to the
point $P = (3,2)$ which is $\sqrt{(1 - 3)^2 + (8 - 2)^2} = \sqrt{40} \approx 6.325$.
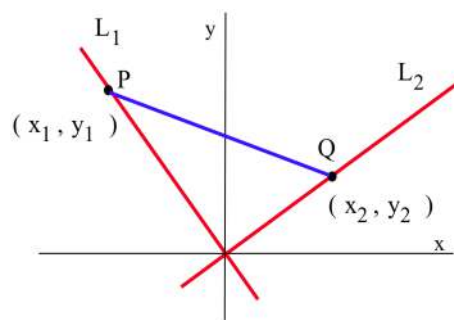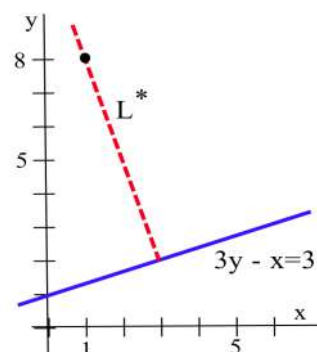
Angle Formed by Intersecting Lines

If two lines which are not perpendicular intersect at a point and neither line is vertical, then we can use some geometry and trigonometry to determine the angles formed by the intersection of the lines (Fig. 17). Since θ_2 is an exterior angle of the triangle ABC, θ_2 is equal to the sum of the two opposite interior angles so $\theta_2 = \theta_1 + \theta$ and $\theta = \theta_2 - \theta_1$. Then, from trigonometry,
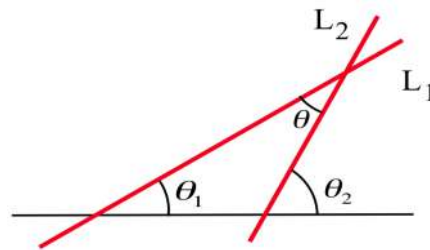
Fig. 17

$$\tan(\theta) = \tan(\theta_2 - \theta_1) = \frac{\tan(\theta_2) - \tan(\theta_1)}{1 + \tan(\theta_2)\tan(\theta_1)} = \frac{m_2 - m_1}{1 + m_2 m_1} \ .$$
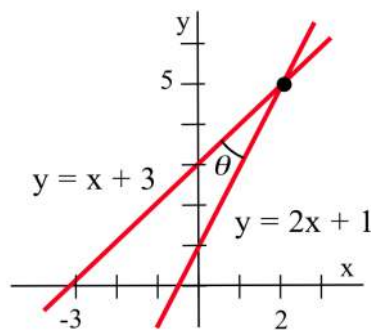
Fig. 18

The inverse tangent of an angle is between $-\pi/2$ and $\pi/2$ (-90° and 90°)

so $\theta = \arctan\left(\dfrac{m_2 - m_1}{1 + m_2 m_1} \right)$ always gives the smaller of the angles.

The larger angle is $\pi - \theta$ or $180^o - \theta^o$.

The smaller angle θ formed by two nonperpendicular lines with slopes m_1 and m_2 is

$$\theta = \arctan\left(\frac{m_2 - m_1}{1 + m_2 m_1}\right).$$

Example 9:   Find the point of intersection and the angle between $y = x + 3$ and $y = 2x + 1$.   (Fig. 18)

Solution: Solving the first equation for y and then substituting into the second equation, $(x + 3) = 2x + 1$ so x = 2. Putting this back into either equation, we get y = 5. Each of the lines is in the slope–intercept form so it is easy to see that $m_1 = 1$ and $m_2 = 2$. Then

$$\tan(\theta) = \frac{m_2 - m_1}{1 + m_2 m_1} = \frac{2 - 1}{1 + (2)(1)} = 1/3 \text{ and}$$

$$\theta = \arctan(1/3) = .322 \text{ radians} \approx 18.435^o \ .$$

PROBLEMS

1. Estimate the slope of each line in Figure 19.
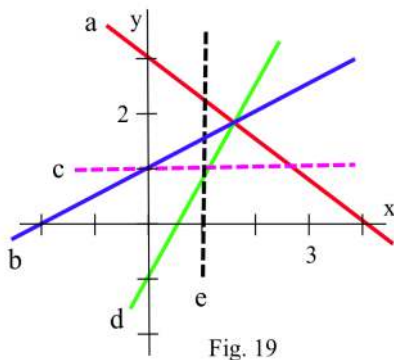
2. Estimate the slope of each line in Figure 20.

Fig. 19
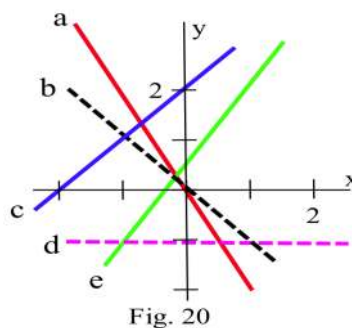
Fig. 20

3. Calculate the slope of the line through each pair of points:
    a) $(2, 4)$, $(5, 8)$
    b) $(-2, 4)$, $(3, -5)$
    c) $(2, 4)$, (x, x^2)
    d) $(2, 5)$, $(2+h, 1+ (2+h)^2)$
    e) $(x, x^2 + 3)$, $(a, a^2 + 3)$

4. Calculate the slope of the line through each pair of points:
    a) $(5, -2)$, $(3, 8)$
    b) $(-2, -4)$, $(5, -3)$
    c) $(x, 3x+5)$, $(a, 3a+5)$
    d) $(4, 5)$, $(4+h, 5-3h)$
    e) $(1, 2)$, $(x, 1+x^2)$
    f) $(2, -3)$, $(2+h, 1- (1+h)^2)$
    g) (x, x^2), $(x+h, x^2+2xh+h^2)$
    h) (x, x^2), $(x-h, x^2-2xh+h^2)$

5.   A small airplane at an altitude of 5000 feet is flying East at 300 feet per second (a bit over 200 miles per

hour), and you are watching it with a small telescope as it passes directly overhead. (Fig. 21)

   a)   What is the **slope** of the telescope $5, 10$ and 20 seconds

         after the plane passes overhead?

   b)   What is the **slope** of the telescope t seconds after the

         plane passes overhead?

   c)   After it passes overhead, is the slope of the telescope

         increasing, decreasing, or staying the same?
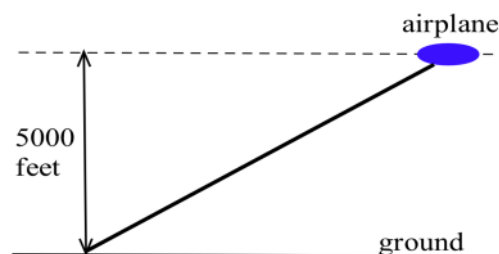
Fig. 21

6.   You are at the origin $(0,0)$ and are watching a small bug at the
        point $(t, 1+t^2)$ at time t seconds.

   a)   What is the **slope** of your line of vision when $t = 5, 10$ and 15 seconds?

   b)   What is the **slope** of your line of vision at any time t?

7.   The blocks in a city are all perfect squares. A friend gives you the following directions to a good restaurant;

     "go north 3 blocks, turn east and go 5 blocks, turn south and go 7 blocks, turn west and go 3 blocks." How

     far away (straight line distance) is the restaurant?

8.   Suppose the directions in problem 7 had been "go north 5 blocks, turn right and go 6 blocks, turn right and

     go 3 blocks, turn left and go 2 blocks." How far away is the restaurant?

9. How far up a wall will a 20 foot long ladder reach if the bottom must be at least 4 feet from the bottom of
    the wall? What will be the slope of the ladder if the bottom is 4 feet from the wall? What angle will the
    ladder make with the ground?

10. Let $P = (1, -2)$ and $Q = (5, 4)$
    (a) Find the point midpoint R on the line segment from P to Q.
    (b) Find the point T which is 1/3 of the way from P to Q: Dist(P,T) = (1/3)·Dist(P,Q) .
    (c) Find the point S which is 2/5 of the way from P to Q: Dist(P,S) = (2/5)·Dist(P,Q) .

11. Let $P = (2, 3)$ and $Q = (8, 11)$. Verify that if $0 \le a \le 1$, then the point $R = (x,y)$ with
    $x = 2a + 8(1-a)$ and $y = 3a + 11(1-a)$ is on the line from P to Q and Dist(P,R) = (1-a)·Dist(P,Q) .

12. What is the longest straight stick which fits into a rectangular box which is 24 inches long, 18 inches wide
    and 12 inches high? What angle, in degrees, does the stick make with the base of the box?

C13.   The lines $y = x$ and $y = 4 - x$ intersect at the point (2, 2).

    a) Use slopes to show that the lines are perpendicular.

    b) Graph them together on your calculator using the "window" $-10 \le x \le 10,\ -10 \le y \le 10.$

        Why do the lines not appear to be perpendicular on the calculator display?

    c) Find a suitable window for the graphs so the lines so that they do appear perpendicular.

C14. a) Find equations for two lines that both go through the point (1, 2), one with slope 3 and one with slope -1/3.

    b) Choose a suitable window so the lines will appear perpendicular, and graph them together on your calculator.

15. Sketch each line which has slope=m and which goes through the point P. Find the equation of each line.
    a)   m = 3, P = (2,5)   b)   m = -2, P = (3,2)   c)   m = -1/2, P=(1,4)

16. Sketch each line which has slope=m and which goes through the point P. Find the equation of each line.
    a) m = 5, P = (2,1)   b) m = - 2/3, P = (1,3)   c) m = π, P = (1,-3)

17. Find the equation of each of the following lines.

    a) L_1 goes through the point (2, 5) and is parallel to $3x - 2y = 9$.

    b) L_2 goes through the point (-1,2) and is perpendicular to $2x = 7-3y$.

    c) L_3 goes through the point (3, -2) and is perpendicular to $y = 1$.

18. a)   Find a value for A so that the line $y = 2x + A$ goes through the point (3,10).

    b)   Find a value for B so that the line $y = Bx + 2$ goes through the point (3,10).

    c)   Find a value for D so that the line $y = Dx + 7$ crosses the y-axis at y = 4.

    d)   Find values for A and B so that the line $Ay = Bx + 1$ goes through the points (1,3) and (5,13).

19. Find the shortest distance between the circles with centers $C_1 = (1, 2)$ and $C_2 = (7, 10)$ and radii

    a) $r_1 = 2$ and $r_2 = 4$ b) $r_1 = 2$ and $r_2 = 7$ c) $r_1 = 5$ and $r_2 = 8$

    d) $r_1 = 3$ and $r_2 = 15$ e) $r_1 = 12$ and $r_2 = 1$

20.   Find the equation of the circle with center C and radius r when
a)   C = (2,7) r = 4   b)   C = (3,–2) r = 1   c)   C = (–5,1) r = 7   d)   C = (–3,–1) r = 4

21. Explain how you can determine, without graphing, whether a given point P = (x,y) is inside, on, or outside the circle with center C = (h,k) and radius r.

22. A box with a height of 2 cm and a width of 8 cm is definitely big enough to hold two semicircular rods with radii of 2 cm (Fig. 22). Will these same two rods fit into a box 2 cm high and 7.6 cm wide? Will they fit in a box 2 cm high and 7.2 cm wide? (Suggestion: turn one of the rods over.)
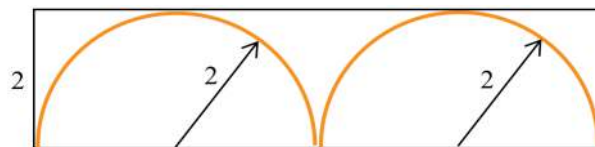
Fig. 22

23. Show that the equation of the circle with center C = (h,k) and radius r is $(x - h)^2 + (y - k)^2 = r^2$.

24. Find the equation of the line which is tangent to the circle $x^2 + y^2 = 25$ at the point P when
a)   P = (3,4)   b)   P = (–4,3)   c)   P = (0,5)   d)   P = (–5,0)

25. Find the slope of the line which is tangent to the circle with center C = (3,1) at the point P when

a)   P = (8,13)   b)   P = (–10,1)   c)   P = (–9,6)   d)   P = (3,14)

26. Find the center C = (h,k) and the radius r of the circle which goes through the three points

a)   (0,1) , (1,0) , and (0,5)   b)   (1,4) , (2,2) , and (8,2)   c)   (1,3) , (4,12) , and (8,4)

27. a)   How close does the line 3x – 2y = 4 come to the point (2,5)?

b)   How close does the line y =5 – 2x come to the point (1,–2)?

c)   How close does the circle with radius 3 and center at (2,3) come to the point (8,3)?

28. a)   How close does the line 2x – 5y = 4 come to the point (1,5)?

b)   How close does the line y = 3 – 2x come to the point (5,–2)?

c)   How close does the circle with radius 4 and center at (4,3) come to the point (10,3)?

29. a)   Show that the line L given by $Ax + By = C$ has slope $m = -A/B$. (Fig. 23)

    b)   Find the equation of the line L* through (0,0) which is perpendicular to
        line L in part (a).

    c)   Show that the lines L and L* intersect at the point

$$(x, y) = \left(\frac{AC}{A^2 + B^2} , \frac{BC}{A^2 + B^2} \right) .$$

    d)   Show that the distance from the origin to the point (x,y) in
        part (c) is

$$\frac{|C|}{\sqrt{A^2 + B^2}} .$$

    Steps (a) – (d) show that the distance from the origin
    to the line $Ax + By = C$ is $\dfrac{|C|}{\sqrt{A^2 + B^2}}$.

Fig. 23

30. Show that the distance from the point (p,q) to the line $Ax + By = C$ is $\dfrac{|Ap + Bq - C|}{\sqrt{A^2 + B^2}}$.

    (The steps will be similar to those in problem 29, but the algebra will be more complicated.)

Section 1.2   **PRACTICE Answers**

Practice 1:   Length = Dist(–7, –2) = | (–7) – (–2) | = | –5 | = **5**.

The midpoint is at $\frac{(-7) + (-2)}{2}$ = $\frac{-9}{2}$ = **– 4.5** .

Practice 2:   Dist(P,Q) = Dist(P,r) so $\sqrt{(x-1)^2 + (y+4)^2}$ = $\sqrt{(x-0)^2 + (y+3)^2}$.

Squaring each side and simplifying, we eventually have **y = x – 4** .

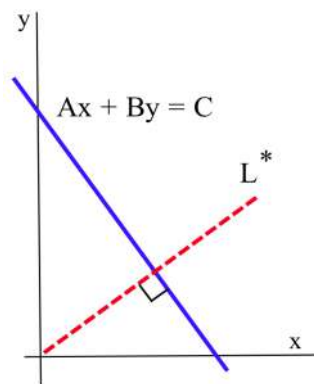**Practice 3**:   The point P = (x , y) is on the circle when it is 5 units from the center C = (–2, 6) so

Dist(P,C) = 5. Then Dist((x,y) , (–2,6)) = 5 so

$\sqrt{(x+2)^2 + (y-6)^2}$ = 5   or   **(x + 2)^2 + (y – 6)^2 = 25** .

Practice 4: $\Delta x = 5 - (-3) = $ **8**, $\Delta y = -14 - 2 = $ **–16** , and slope = $\frac{\Delta y}{\Delta x}$ = $\frac{-16}{8}$ = **– 2** .

Practice 5:   slope = $\frac{\Delta y}{\Delta x}$ = $\frac{(-3 + 5h) - (-3)}{(2 + h) - 2}$ = $\frac{5h}{h}$ = **5** .

The midpoint is at $\left(\frac{(2) + (2 + h)}{2} , \frac{(-3 + 5h) + (-3)}{2} \right) = \left(2 + \frac{h}{2} , -3 + \frac{5h}{2} \right)$.

Practice 6:   slope = $\frac{\Delta y}{\Delta x}$ = $\frac{(3a^2 + 5a) - (3x^2 + 5x)}{a - x}$

= $\frac{3(a^2 - x^2) + 5(a - x)}{a - x}$ = $\frac{3(a + x)(a - x) + 5(a - x)}{a - x}$ = **3(a + x) + 5** .

Practice 7:   Let $y_1 = mx_1 + b$ and $y_2 = mx_2 + b$.   Then

slope = $\frac{\Delta y}{\Delta x}$ = $\frac{(mx_2 + b) - (mx_1 + b)}{x_2 - x_1}$ = $\frac{m(x_2 - x_1)}{x_2 - x_1}$ = **m** .

Practice 8:   The line 3x + 5y = 17 has slope $\frac{-3}{5}$ so the slope of the parallel line is m = $\frac{-3}{5}$.

Using the form y = $\frac{-3}{5}$ x + b and the point (–2, 3) on the line, we have

3 = $\frac{-3}{5}$ (–2) + b so b = $\frac{9}{5}$ and

y = $\frac{-3}{5}$ x + $\frac{9}{5}$ or **5y + 3x = 9** ..

Practice 9:   The line 3y – 7x = 2 has slope $\frac{7}{3}$ so the slope of the perpendicular line is m = $\frac{-3}{7}$.

Using the form y = $\frac{-3}{7}$ x + b and the point (2, –5) on the line, we have

–5 = $\frac{-3}{7}$ (2) + b so b = $\frac{-29}{7}$ and

y = $\frac{-3}{7}$ x + $\frac{-29}{7}$ or **7y + 3x = –29** .

1.3   FUNCTIONS AND THEIR GRAPHS

When you prepared for calculus, you learned to manipulate functions by adding, subtracting, multiplying and dividing them, as well as calculating functions of functions (composition). In calculus, we will still be dealing with functions and their applications. We will create new functions by operating on old ones. We will derive information from the graphs of the functions and from the derived functions. We will find ways to describe the point–by–point behavior of functions as well as their behavior "close to" some points and also over entire intervals. We will find tangent lines to graphs of functions and areas between graphs of functions. And, of course, we will see how these ideas can be used in a variety of fields.

This section and the next one are a review of information and procedures you should already know about functions before we begin calculus.

What is a function?

> **Definition of Function:**
>
> A **function** from a set X to a set Y is a **rule** for assigning to each element of the set X a single element of the set Y. A function assigns a unique (exactly one) output element in the set Y to each input element from the set X.

The **rule** which defines a function is often given by an equation, but it could also be given in words or graphically or by a table of values. In practice, functions are given in all of these ways, and we will use all of them in this book.

In the definition of a function, the set X of all inputs is called the **domain** of the function. The set Y of all outputs produced from these inputs is called the **range** of the function. Two different inputs, elements in the domain, can be assigned to the same output, an element in the range, but one input cannot lead to 2 different outputs.

Most of the time we will work with functions whose domains and ranges are real numbers, but there are other types of functions all around us. Final grades for this course is an example of a function. For each student, the instructor will assign a final grade based on some rule for evaluating that student's performance. The **domain** of this function consists of all students registered for the course, and the **range** consists of the letters A, B, C, D, F, and perhaps W (withdrawn). Two students can receive the same final grade, but only one grade will be assigned to each student.

Function Machines

Functions are abstract structures, but sometimes it is easier to think of them in a more concrete way. One way is to imagine that a function is a special purpose computer, a machine which accepts inputs, does

something to those inputs according to the defining rule, and produces an output. The output is the value of the function for the given input value. If the defining rule for a function f is "multiply the input by itself" , f(input) = (input)(input) , then Fig. 1 shows the results of putting the inputs x, 5, a, c + 3 and x + h into the machine f.
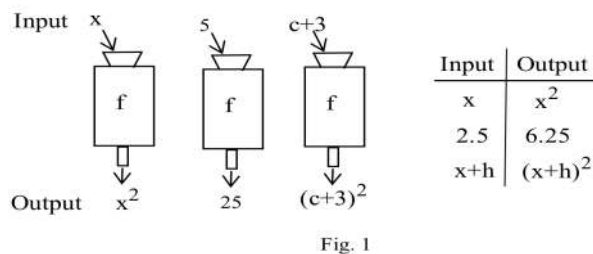

Fig. 1

Practice 1:   If we have a function machine g whose rule is "divide 3 by the input and add 1",

g(x) = 3/x + 1, what outputs do we get from the inputs x, 5, a, c + 3 and x + h ? What happens if we put 0 into the machine g?

You expect your calculator to behave as a function: each time you press the same input sequence of keys you expect to see the same output display. In fact, if your calculator did not produce the same output each time you would need a new calculator. (On many calculators there is a key which does not produce the same output each time you press it. Which key is that?)

Functions Defined by Equations

If the domain consists of a collection of real numbers (perhaps all real numbers) and the range is a collection of real numbers, then the function is called a **numerical function**. The rule for a numerical function can be given in several ways, but it is usually written as a formula. If the rule for a numerical function , f , is "the output is the input number multiplied by itself", then we could write the rule as $f(x) = x \cdot x = x^2$. The use of an "x" to represent the input is simply a matter of convenience and custom. We could also represent the same function by $f(a) = a^2$, $f(\in \#) = \#^2$ or $f(\text{ input }) = (\text{ input })^2$.

For the function f defined by $f(x) = x^2 - x$, we have that $f(3) = 3^2 - 3 = 6$, $f(.5) = (.5)^2 - (.5) = -.25$, and $f(-2) = (-2)^2 - (-2) = 6$. Notice that the two different inputs, 3 and –2, both lead to the output of 6. That is allowable for a function. We can also evaluate f if the input contains variables. If we replace the "x" with something else in the notation "f(x)", then we must replace the "x" with the same thing everywhere in the equation:

$f(\mathbf{c}) = \mathbf{c}^2 - \mathbf{c}$, $f(\mathbf{a+1}) = (\mathbf{a+1})^2 - (\mathbf{a+1}) = (a^2 + 2a + 1) - (a + 1) = a^2 + a$,

$f(\mathbf{x+h}) = (\mathbf{x+h})^2 - (\mathbf{x+h}) = (x^2 + 2xh + h^2) - (x+h)$, and, in general, $f(\mathbf{input}) = (\mathbf{input})^2 - (\mathbf{input})$.

For more complicated expressions, we can just proceed step–by–step:

$$\frac{f(\mathbf{x+h}) - f(x)}{h} \;=\; \frac{\{(\mathbf{x+h})^2 - (\mathbf{x+h})\} - \{x^2 - x\}}{h} \;=\; \frac{\{(x^2 + 2xh + h^2) - (x+h)\} - \{x^2 - x\}}{h}$$

$$=\; \frac{2xh + h^2 - h}{h} \;=\; \frac{h(2x + h - 1)}{h} \;=\; 2x + h - 1 \;.$$

Practice 2:   For the function g   defined by $g(t) = t^2 - 5t$, evaluate $g(1), g(-2), g(w+3), g(x+h)$,

$g(x+h) - g(x)$, and $\dfrac{g(x+h) - g(x)}{h}$.

Functions Defined by Graphs and Tables of Values

The **graph** of a numerical function f consists of a plot of ordered pairs (x,y) where x is in the domain of

f and y = f(x). A **table of values** of a numerical function consists of a list of some of the ordered pairs

(x,y) where y = f(x). Fig. shows a graph of f(x) = sin(x) for $-4 \le x \le 9$.
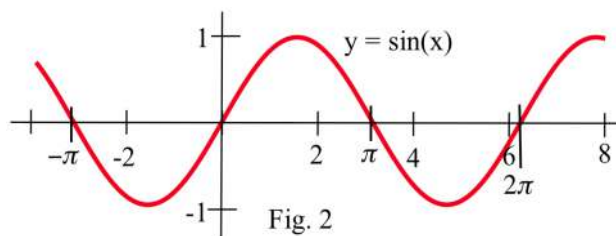
A function can be defined by a graph or by a table of
values, and these types of definitions are common in
applied fields.   The outcome of an experiment will
depend on the input, but the experimenter may not
know the "rule" for predicting the outcome. In that

Fig. 2

case, the experimenter usually represents the experiment function as a table of measured outcome values

verses input values or as a graph of the outcomes verses the inputs. The table and graph in Fig. 3 show

the deflections obtained when weights are loaded at the end of a wooden stick. The graph in Fig. 4 shows

the temperature of a hot cup of tea as a function of the time as it sits in a 68° F room.   In these

experiments, the "rule" for the function is that   f(input) = actual outcome of the experiment.

weight	deflection
1	6.2
2	12.4
3	18.6
4	24.8
5	31.0
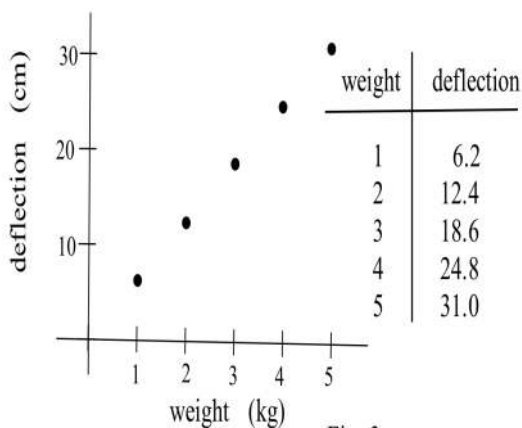
Fig. 3

Tables have the advantage
of presenting the data
explicitly, but it is often
difficult to detect patterns
simply from lists of
numbers. Graphs tend to
obscure some of the
precision of the data, but
patterns are much easier to
detect visually
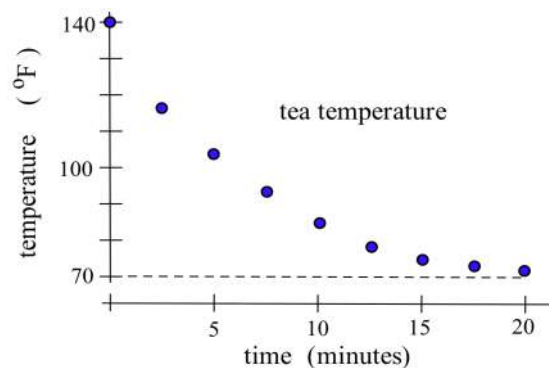
Fig. 4

— we can actually see what is happening with the numbers.

Creating Graphs of Functions

Most people understand and can interpret pictures more quickly than tables of data or equations, so if we have a function defined by a table of values or by an equation, it is often useful and necessary to create a picture of the function, a graph.

A Graph from a Table of Values

If we have a table of values of the function, perhaps consisting of measurements obtained from an experiment, then we can simply plot the ordered pairs in the xy–plane to get a graph which consists of a collection of points.

Fig. 5 shows the lengths and weights of trout caught (and released) during several days of fishing. It also shows a line which comes "close" to the plotted points. From the graph, you could estimate that a 17 inch trout would weigh slightly more than one pound.

| length (in)	weight (pds)
13.5	0.4
14.5	0.9
15.0	0.7
16.0	0.9
18.0	1.2
18.5	1.6
19.5	1.5
20.5	1.7
20.5	2.1
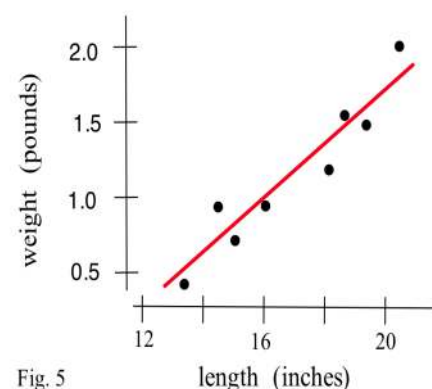
Fig. 5

A Graph from an Equation

Creating the graph of a function given by an equation is similar to creating one from a table of values — we need to plot <u>enough</u> points (x,y) where y = f(x) so we can be confident of the shape and location of the graph of the entire function. We can find a point (x,y) which satisfies y = f(x) by picking a value for x and then calculating the value for y by evaluating f(x). Then we can enter the (x,y) value in a table or simply plot the point (x,y) .

If you recognize the form of the equation and know something about the shape of graphs of that form, you may not have to plot many points. If you do not recognize the form of the equation then you will have to plot more points, maybe 10 or 20 or 234: it depends on how complicated the graph appears and on how important it is to you (or your boss) to have an accurate graph. Evaluating y = f(x) at a lot of different values for x and then plotting the points (x,y) is usually not very difficult, but it can be very time–consuming. Fortunately, there are now calculators and personal computers which will do the evaluations and plotting for you.

Is every graph the graph of a function?

The definition of function requires that each element of the domain, each input value, be sent by the function to <u>exactly one</u> element of the range, to exactly one output value, so for each input x-value there

will be <u>exactly one</u> output y–value, y = f(x). The points (x, y_1) and (x, y_2) cannot both be on the graph of

f unless $y_1 = y_2$. The graphic interpretation of this result is called the Vertical Line Test.

<div style="background-color: yellow">

Vertical Line Test for a Function:

A graph is the graph of a function if and only if   a vertical line drawn

through any point <u>in the domain</u> intersects the graph at <u>exactly one</u> point.

</div>

Fig. 6(a) shows the graph of a function. Figs. 6(b) and
6(c) show graphs which are not the graphs of functions,
and vertical lines are shown which intersect those
graphs at more than one point. Non–functions are not
"bad", and sometimes they are necessary to describe
complicated phenomena.

Fig. 6
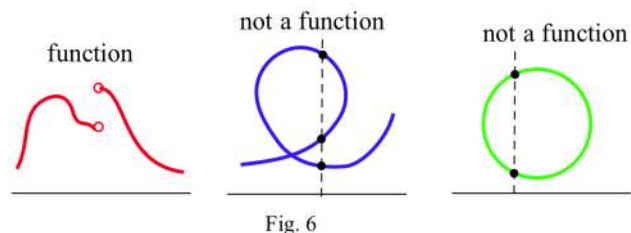
Reading Graphs Carefully

Calculators and computers can help students, reporters, business people and scientific professionals create
graphs quickly and easily, and because of this, graphs are being used more often than ever to present
information and justify arguments. And this text takes a distinctly graphical approach to the ideas and
meaning of calculus. Calculators and computers can help us create graphs, but we need to be able to read
them carefully. The next examples illustrate some types of information which can be obtained by carefully
reading and understanding graphs.

Example 1:   A boat starts from St. Thomas and sails due

west with the velocity shown in Fig. 7

(a) When is the boat traveling the fastest?

(b) What does a negative velocity away from

St. Thomas mean?

(c) When is the boat the farthest from St. Thomas?

Fig. 7
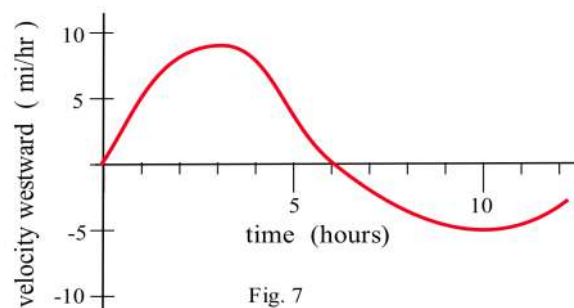
Solution: (a) The greatest speed is 10 mph at t = 3 hours.

(b) It means that the boat is heading back toward St. Thomas.

(c) The boat is farthest from St. Thomas at t = 6 hours. For t < 6 the boat's velocity is positive, and
the distance from the boat to St. Thomas is increasing. For t > 6 the boat's velocity is negative,
and the distance from the boat to St. Thomas is decreasing.

Practice 3:   You and a friend start out together and hike along the same trail but walk at different speeds (Fig. 8).

(a) Who is walking faster at t = 20?

(b) Who is ahead at t = 20?

(c) When are you and your friend farthest apart?

(d) Who is ahead when t = 50?
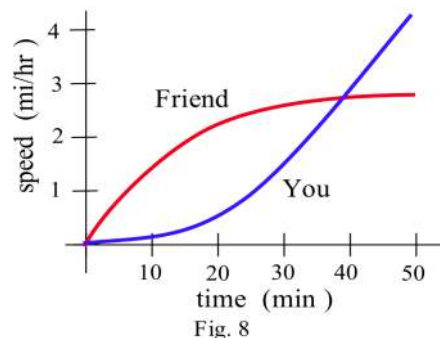
Example 2:   In Fig. 9, which has the largest slope: the line through

the points A and P, the line through B and P, or the line

through C and P?
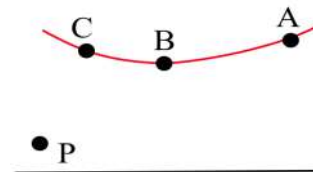
Solution: The line through C and P has the largest slope:

$$m_{PC} > m_{PB} > m_{PA} .$$
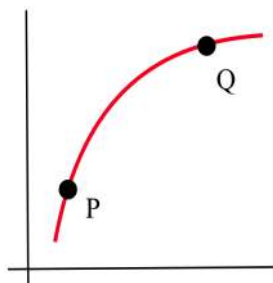
Practice 4:   In Fig. 10, the point Q on the curve is fixed, and the point P is moving

to the right along the curve toward the point Q. As P moves toward Q:

(a) the x–coordinate of P is Increasing, Decreasing, Remaining constant, or None of these.

(b) the x–increment from P to Q is Increasing, Decreasing, Remaining constant, or

None of these

(c) the slope from P to Q is Increasing, Decreasing, Remaining constant, or None of these.

Example 3:   The graph of y = f(x) is shown in Fig. 11. Let

g(x) be the **slope** of the line tangent to the graph of f(x) at the

point (x,f(x)).

(a) Estimate the values g(1), g(2) and g(3).

(b) When does g(x) = 0? (c) At what value(s) of x is g(x)

largest?   (d) Sketch the graph of y = g(x).
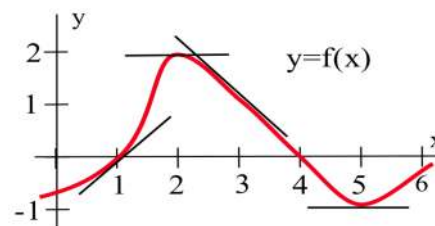
Solution: (a) Fig. 11 shows the graph y = f(x) with several tangent lines to the graph of f. From Fig. 11

we can estimate that g(1) (the <u>slope</u> of the line tangent to the graph of f at (1,0)) is

approximately equal to 1. Similarly, g(2) ≈ 0 and g(3) ≈ –1.

(b) The slope of the tangent line appears to be horizontal (slope = 0) at x = 2 and at x = 5.

(c) The tangent line to the graph appears to have greatest slope (be steepest) near x = 1.5 .

(d) We can build a table of values of g(x) and then sketch the graph of these values.

x	f(x)	g(x) = tangent slope at (x, f(x))
0	−1	.5
1	0	1
2	2	0
3	1	−1
4	0	−1
5	−1	0
6	−.5	.5

The graph y = g(x) is given in Fig. 12.
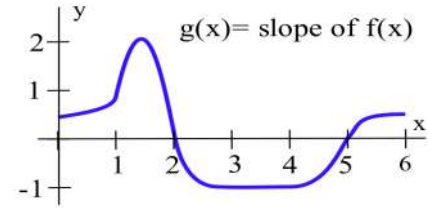
Fig. 12
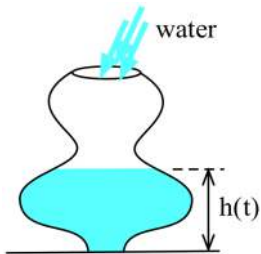
Fig. 13

Practice 5:   Water is flowing into a container (Fig. 13) at a constant rate of 3 gallons per minute. Starting with an empty container, sketch the graph of the height of the water in the container as a function of time.

Problems

In problems 1 – 4, use the shapes and slopes of the data to match the given numerical triples to the graphs in the figures. (For example, A: 3, 3, 6 in Problem 1. is "over and up" so it matches graph (a) in Fig. 14. B is "down and over" so it matches graph (c) in Fig. 14.)

1. Fig. 14. Data: A: 3, 3, 6   B: 12, 6, 6
    C: 7, 7, 3   D: 2, 4, 4

2. Fig. 15. Data: A: 7, 10, 7   B: 17.3, 17.3, 25
    C: 4, 4, 8   D: 12, 8, 16

3. Fig. 16. Data: A: 7, 14, 10   B: 23, 45, 22
    C: 0.8, 1.2, 0.8   D: 6, 9, 3

4. Fig. 17. Data: A: 6, 3, 9   B: 18, 10, 10
    C: 12, 6, 9   D: 3.7, 1.9, 3.6
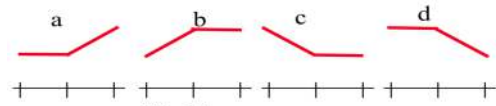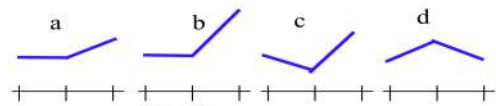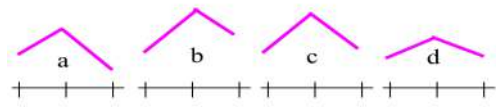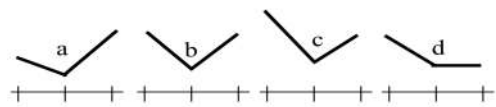
5. Water is flowing into each of the bottles in Fig. 18 at a steady rate. Match each bottle shape with the graph of the height of the water as a function of time.

Fig. 14

Fig. 15

Fig. 16

Fig. 17
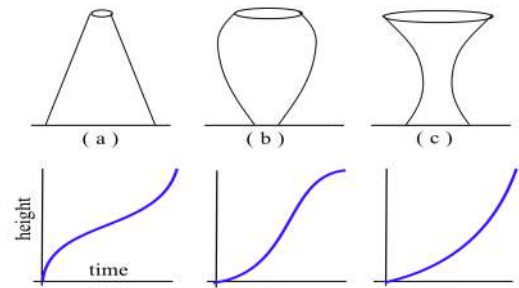
Fig. 18

6. Sketch the shapes of bottles which will have the water height versus time graphs in Fig. 19.
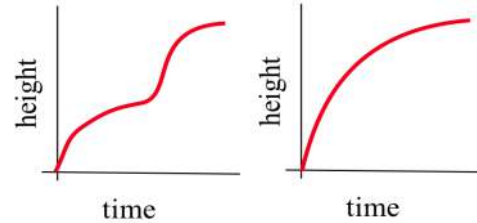
Fig. 19

7. $f(x) = x^2 + 3$, $g(x) = \sqrt{x - 5}$, and $h(x) = \dfrac{x}{x - 2}$

   (a) evaluate $f(1), g(1)$ and $h(1)$ (b)   graph $f(x), g(x)$ and $h(x)$ for $-5 \le x \le 10$

   (c) evaluate $f(3x), g(3x)$ and $h(3x)$ (d)   evaluate $f(x+h), g(x+h)$ and $h(x+h)$

8. Find the slope of the line through the points P and Q when

   (a) $P = (1,3), \ Q = (2,7)$ (b) $P = (x, x^2 + 2), \ Q = (\ x+h \ , (x+h)^2 + 2)$

   (c) $P = (1,3), \ Q = (x, x^2 + 2)$ What are the values of these slopes in (c) if $x = 2, \ x = 1.1, x = 1.002$?

9. Find the slope of the line through the points P and Q when

   (a) $P = (1,5), \ Q = (2,7)$ (b) $P = (x, x^2 + 3x - 1), \ Q = (\ x+h \ , (x+h)^2 + 3(x+h) - 1)$

   (c) $P = (1,3), \ Q = (x, x^2 + 3x - 1)$ What are the values of these slopes in (c) if $x = 1.3, x = 1.1, x = 1.002$?

10. $f(x) = x^2 + x$ and $g(x) = 3/x$. Evaluate and simplify $\dfrac{f(a + h) - f(a)}{h}$ and $\dfrac{g(a + h) - g(a)}{h}$

    when $a = 1, a = 2, a = -1, a = x$.

11. $f(x) = x^2 - 2x$ and $g(x) = \sqrt{x}$. Evaluate and simplify $\dfrac{f(a+h) - f(a)}{h}$ and $\dfrac{g(a+h) - g(a)}{h}$

    when $a = 1, a = 2, a = 3, a = x$.

12. The temperatures in Fig. 20 were recorded during a 12 hour period in Chicago.

   (a) At what time was the temperature the highest? Lowest?

   (b) How fast was the temperature rising at 10 am? At 1 pm?

   (c) What could have caused the drop in temperature between 1 pm and 3 pm?
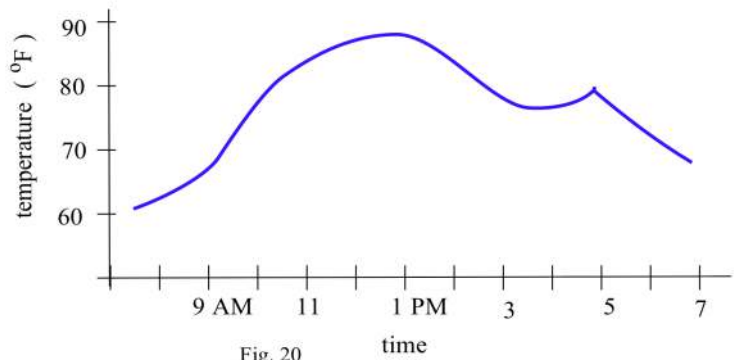
Fig. 20

13. The graph in Fig. 21 shows the distance of an airplane
    from an airport during a several hour flight.
    (a) How far was the airplane from the airport at
         1 pm? At 2 pm?
    (b) How fast was the distance changing at 1 pm?
    (c) How could the distance from the plane to the airport
         remain unchanged from 1:45 pm until 2:30 pm
         without the airplane falling?
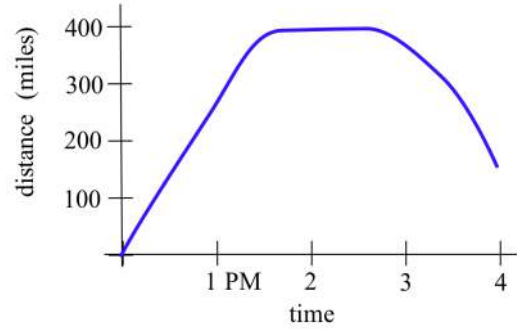
Fig. 21
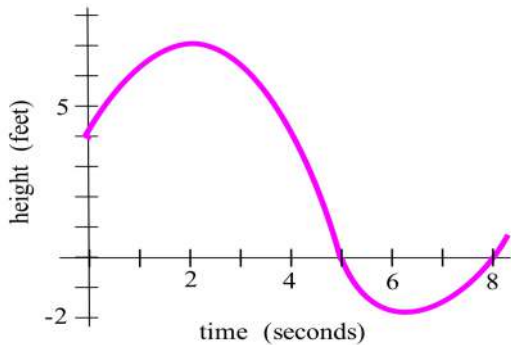
time (seconds)

Fig. 22

14. The graph in Fig. 22 shows the height of a diver above the
    water level at time t seconds.
    (a) What was the height of the diving board?
    (b) When did the diver hit the water?
    (c) How deep did the diver get?
    (d) When did the diver return to the surface?
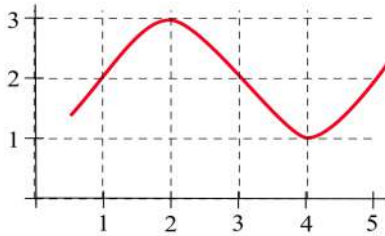
Fig. 23

15. a)   Sketch the lines tangent to the curve in Fig. 23 at $x = 1, 2, 3, 4$, and 5.
    b)   For what value(s) of x is the value of the function largest?   Smallest?
    c)   For what value(s) of x is the slope of the tangent line largest? Smallest?

16. Fig. 24 shows the height of the water
    (above and below mean sea level) at a
    Maine beach.
    a)   At which time(s) was the most
         beach exposed? The least?
    b)   At which time(s) was the current
         the strongest?

Fig. 24

time
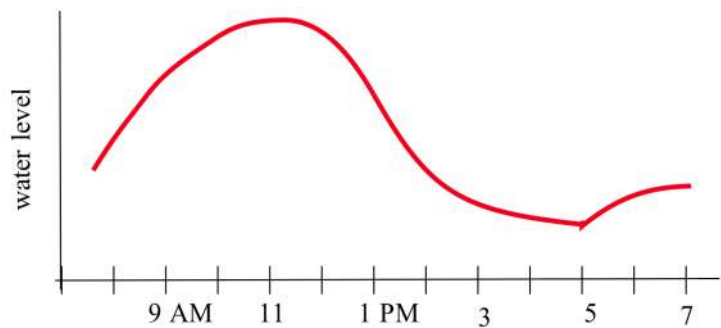
17. Imagine that you are ice skating, from **left to right**, along the path in Fig. 25. Sketch the path you will follow if you fall at points A, B, and C.
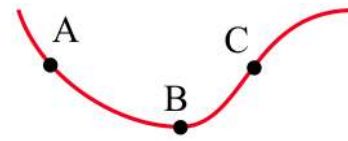
Fig. 25

18. Define s(x) to be the **slope** of the line through the points (0,0) and (x, f(x)) in Fig. 26 . (For example, s(3) = { **slope** of the line through (0,0) and (3, f(3))} = 4/3.)

    a)   Evaluate s(1), s(2), and s(4).

    b)   For which integer value of x between 1 and 7 is s(x) smallest?
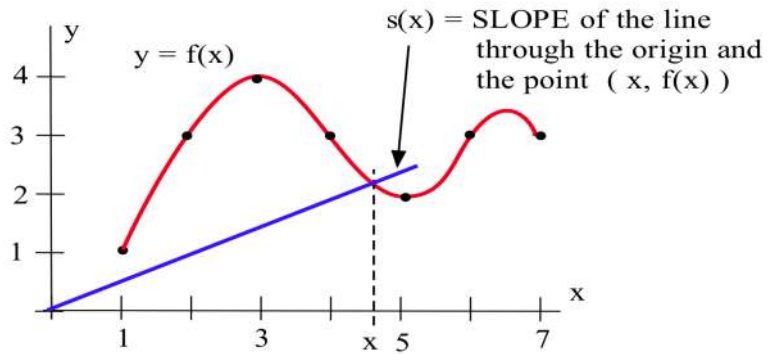
Fig. 26
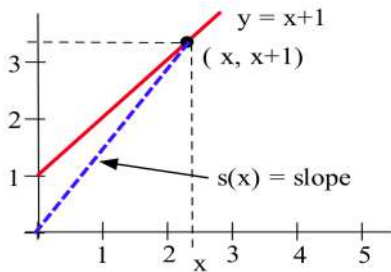
Fig. 27

19. Let f(x) = x + 1 and define s(x) to be the **slope** of the line through the points (0,0) and (x, f(x)) in Fig. 27 . (For example, s(2) = { slope of the line through (0,0) and (2,3)} = 3/2.)

    a)   Evaluate s(1), s(3) and s(4).

    b)   Find an equation for s(x) in terms of x.

20. Define A(x) to be the area of the rectangle bounded by the axes, the line y = 2, and a vertical line at x as in Fig. 28. (For example, A(3) = area of a 2 by 3 rectangle = 6.)

    a)   Evaluate A(1), A(2) and A(5).

    b)   Find an equation for A(x) in terms of x.
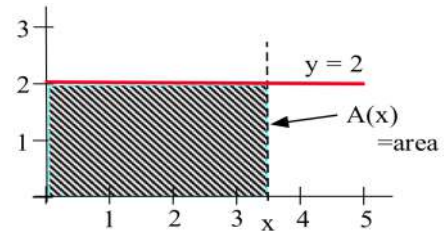
Fig. 28

21. Use the graph of y = f(x) in Fig. 29 to complete the table. (You will have to estimate the values of the slopes.)
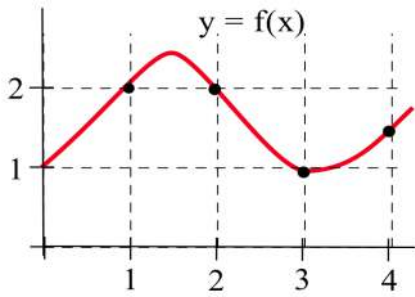
Fig. 29

x	f(x)	slope of the line tangent to the graph of f at (x, f(x))
0	1	1
1		
2		
3		
4		

22. Sketch the graphs of water height versus time for water pouring into a bottle shaped like:

   (a) a milk carton   (b) a spherical glass vase

   (c) an oil drum (cylinder) lying on its side

   (d) a giraffe   (e) you.
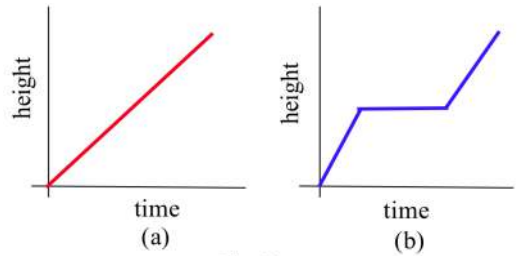
Fig. 30

23. Design bottles whose graphs of (highest) water height versus time will look like those in Fig. 30.

Section 1.3   **PRACTICE Answers**

Practice 1:

Input	Output
x	$\frac{3}{x} + 1$
5	$\frac{3}{5} + 1 = 1.6$
a	$\frac{3}{a} + 1$
0	$g(0) = \frac{3}{0} + 1$ which is not defined because of division by 0.

Input	Output
c + 3	$\frac{3}{c + 3} + 1$
x + h	$\frac{3}{x + h} + 1$

Practice 2: $g(t) = t^2 - 5t$.

$g(1) = 1^2 - 5(1) = \mathbf{-4}$. $g(-2) = (-2)^2 - 5(-2) = \mathbf{14}$.

$g(w + 3) = (w + 3)^2 - 5(w + 3) = w^2 + 6w + 9 - 5w - 15 = \mathbf{w^2 + w - 6}$.

$g(x + h) = (x + h) - 5(x + h) = \mathbf{x^2 + 2xh + h^2 - 5x - 5h}$.

$g(x + h) - g(x) = (x^2 + 2xh + h^2 - 5x - 5h) - (x^2 - 5x) = \mathbf{2xh + h^2 - 5h}$.

$$\frac{g(x + h) - g(x)}{h} = \frac{2xh + h^2 - 5h}{h} = \mathbf{2x + h - 5} .$$

Practice 3:   (a) **Friend**   (b) **Friend**

(c) At **t = 40**. Before that your friend is walking faster and increasing the distance between

you. Then you start to walk faster than your friend and start to catch up.

(d) **Friend**. You are walking faster than your friend at t = 50, but you still have

not caught up.

Practice 4:   (a) The x–coordinate is **increasing**. (b) The x–increment Δx is **decreasing**.

(c) The slope of the line through P and Q is **decreasing**.

Practice 5:   See Fig. 31.

Fig. 13

Fig. 31
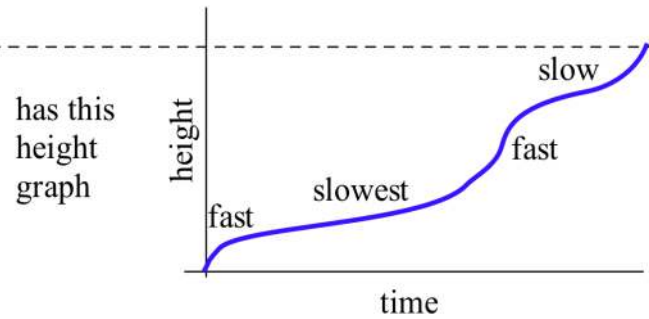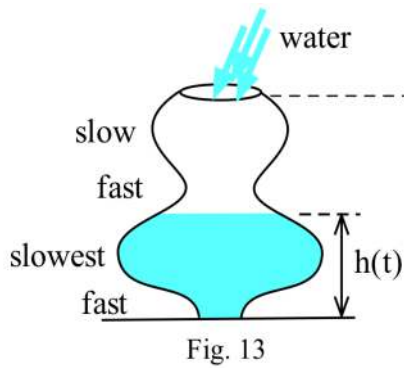
1.4   COMBINATIONS OF FUNCTIONS

Multiline Definitions of Functions -- Putting Pieces Together

Sometimes a physical or economic situation behaves differently depending on circumstances, and a more complicated function may be needed to describe the situation.

Sales Tax: Some states have different rates of sale tax depending on the type of item purchased. A "luxury item" may be taxed at 12%, food may have no tax, and all other items may have a 6% tax. We could describe this situation by using a **multiline** function, a function whose defining rule consists of several pieces. Which piece of the rule we need to use will depend on what we buy. In this example we could define the tax T on an item which costs x to be

$$T(x) = \begin{cases} 0 & \text{if x is the cost of a food} \\ 0.12x & \text{if x is the cost of a luxury item} \\ 0.06x & \text{if x is the cost of any other item.} \end{cases}$$

To find the tax on a \$2 can of stew, we would use the first piece of the rule and find that the tax is 0. To find the tax on a \$30 pair of earrings, we would use the second piece of the rule and find that the tax is \$3.60 . The tax on a \$20 book requires using the third rule, and the tax is \$1.20 .

Wind Chill Index: The rate at which a person's body loses heat depends on the temperature of the surrounding air and on the speed of the air. You lose heat more quickly on a windy day than you do on a day with little or no wind. Scientists have experimentally determined this rate of heat loss as a function of temperature and wind speed, and the resulting function is called the Wind Chill Index, WCI . The WCI is the temperature on a still day (no wind) at which your body would lose heat at the same rate as on the windy day. For example, the WCI value for 30° F air moving at 15 miles per hour is 9° F: your body loses heat as quickly on a 30° F day with a 15 mph wind as it does on a 9° F day with no wind.

If T is the Fahrenheit temperature of the air and v is the speed of the wind in miles per hour, then the WCI is a multiline function of the wind speed v (and of the temperature T):

$$WCI = \begin{cases} T & \text{if } 0 \le v \le 4 \\ 91.4 - \dfrac{10.45 + 6.69\sqrt{v} - 0.447v}{22}(91.5 - T) & \text{if } 4 \le v \le 45 \\ 1.60T - 55 & \text{if } v > 45 \end{cases}$$

The WCI value for a still day $(0 \le v \le 4$ mph) is just the air temperature. The WCI values for wind speeds above 45 mph are the same as the WCI value for a wind speed of 45 mph. The WCI values for wind speeds between 4 mph and 45 mph decrease as the wind speeds increase.

This WCI function depends on two variables, the temperature and the wind speed. However, if the temperature is constant, then the resulting formula for the WCI will only depend on the speed of the wind.   If the air temperature is 30° F (T = 30), then the formula for the Wind Chill Index is

$$WCI_{30} = \begin{cases} 30^{o} & \text{if } 0 \le v \le 4 \;\; mph \\ 62.19 - 18.70\sqrt{v} + 1.25v & \text{if } 4 \le v \le 45 \;\; mph \\ -7^{0} & \text{if } 45 \le v \;\; mph \end{cases}$$

The graphs of the the Wind Chill Indices are shown
on Fig. 1 for temperatures of 40° F, 30° F and
20° F . (From UMAP Module 658, Windchill by
William Bosch and L.G. Cobb, 1984.)
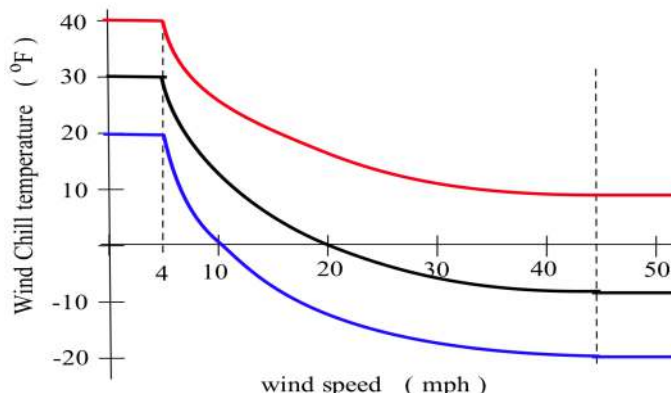
Practice 1:   A motel charges $50 per night for a
room during the tourist season from June 1
through September 15, and $40 per night
otherwise. Define a multiline function which
describes these rates.

Fig. 1

Example 1:   Define $f(x) = \begin{cases} 2 & \text{if } x < 0 \\ 2x & \text{if } 0 \le x < 2 \\ 1 & \text{if } 2 < x \end{cases}$

Evaluate $f(-3), f(0), f(1), f(4)$ and $f(2)$. Graph $y = f(x)$ for $-1 \le x \le 4$.

Solution: To evaluate the function for different values of x, we must first decide which
line of the rule applies. If $x = -3 < 0$, then we need to use the first line of the rule,
and $f(-3) = 2$. When $x = 0$ or $x = 1$, we need the second line of the function
definition, and then $f(0) = 2(0) = 0$ and $f(1) = 2(1) = 2$. At $x = 4$ the
third line is needed, and $f(4) = 1$. Finally, at $x = 2$, none of the lines apply:
the second line requires $x < 2$ and the third line requires $2 < x$, so $f(2)$ is
undefined. The graph of $f(x)$ is given in Fig. 2. Note the "hole" above
$x = 2$ since $f(2)$ is not defined by this rule for f.

y = f(x)
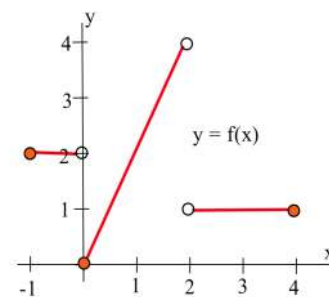
Fig. 2

Practice 2:   Define $g(x) = \begin{cases} x & \text{if } x < -1 \\ 2 & \text{if } -1 \le x < 1 \\ -x & \text{if } 1 < x \le 3 \\ 1 & \text{if } 4 < x. \end{cases}$ Graph $y = g(x)$ for $-3 \le x \le 6$ and

evaluate $g(-3), g(-1), g(0), g(1/2), g(1), g(\pi/3), g(2), g(3), g(4)$ and $g(5)$.

y = f(x)
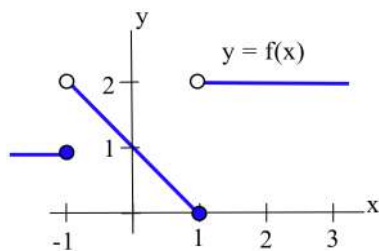
Fig. 3

Practice 3:   Write a multiline function definition for the function
whose graph is given in Fig. 3 .

We can think of a multiline function definition as a machine which first examines the input value to decide which line of the function rule to apply (Fig. 4).
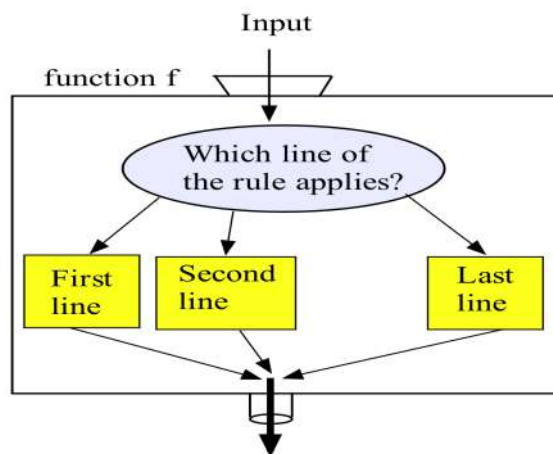
Fig. 4   Function Output = f(Input)

Composition of Functions — Functions of Functions

Basic functions are often combined with each other to describe more complicated situations. Here we will consider the composition of functions, functions of functions.

> **Definition:** The **composite** of two functions f and g , written f∘g , is **f∘g(x) ▪ f(g(x))**.

The domain of the composite function f∘g(x) = f(g(x)) consists of those x–values for which g(x) and

f(g(x)) are both defined — we can evaluate the composition of two functions at a point x only if each step in the composition is defined.

If we think of our functions as machines, then composition is simply a new machine consisting of an arrangement of the original machines. The composition f∘g of the function machines f and g shown in Fig. 5(a) is an arrangement of the machines so that the original input x goes into machine g , the output from machine g becomes the input into machine f , and the output from machine f is our final output. The composition of the function machines f∘g(x) = f(g(x)) is only valid if x is an allowable input into g (x is in the domain of g) and if g(x) is then an allowable input into f . The composition g∘f involves arranging the machines so the original input goes into f , and the output from f then becomes the input for g (Fig. 5(b)).
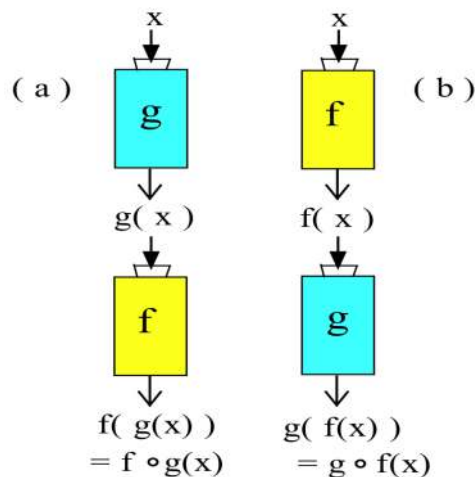
Fig. 5

Example 2:   For $f(x) = \sqrt{x-2}$, $g(x) = x^2$, and $h(x) = \begin{cases} 3x & \text{if } x < 2 \\ x-1 & \text{if } 2 \le x \end{cases}$,

evaluate $f \circ g(3), g \circ f(6), f \circ h(2)$ and $h \circ g(-3)$. Find the equations and domains of $f \circ g(x)$ and $g \circ f(x)$.

Solution: $f \circ g(3) = f(g(3)) = f(3^2) = f(9) = \sqrt{9-2} = \sqrt{7} \approx 2.646$

$g \circ f(6) = g(f(6)) = g(\sqrt{6-2}) = g(\sqrt{4}) = g(2) = 2^2 = 4$

$f \circ h(2) = f(h(2)) = f(2-1) = f(1) = \sqrt{1-2} = \sqrt{-1}$ which is undefined

$h \circ g(-3) = h(g(-3)) = h(9) = 9-1 = 8$.

$f \circ g(x) = f(g(x)) = f(x^2) = \sqrt{x^2 - 2}$, and the domain of $f \circ g$ is those x–values for which

$x^2 - 2 \ge 0$ so the domain of $f \circ g$ is all x such that $x \ge \sqrt{2}$ or $x \le -\sqrt{2}$.

$g \circ f(x) = g(f(x)) = g(\sqrt{x-2}) = \{\sqrt{x-2}\}^2 = x-2$, but we can evaluate the first piece, f, of the

composition only if $f(x) = \sqrt{x-2}$ is defined, so the domain of $g \circ f$ is all $x \ge 2$.

Practice 4:   For $f(x) = \dfrac{x}{x-3}$, $g(x) = \sqrt{1+x}$,   and $h(x) = \begin{cases} 2x & \text{if } x \le 1 \\ 5-x & \text{if } 1 < x. \end{cases}$

Evaluate $f \circ g(3), f \circ g(8), g \circ f(4), f \circ h(1), f \circ h(3), f \circ h(2)$ and $h \circ g(-1)$. Find the equations for $f \circ g(x)$ and $g \circ f(x)$.

Shifting and Stretching Graphs

Some compositions are relatively common and easy, and you should

recognize the effect of the composition on the graphs of the functions.

Example 3:   Fig. 6 shows the graph of $y = f(x)$.

Graph (a) $2 + f(x)$, (b) $3 \cdot f(x)$, and (c) $f(x-1)$.

Fig. 6
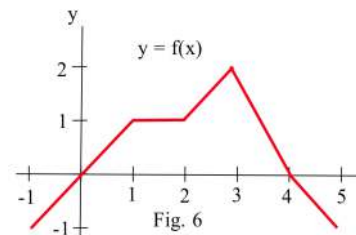
Solution: All of the new graphs are shown below in Fig. 7 .

(a) Adding 2 to all of the values of $f(x)$ **rigidly shifts** the graph of f(x) **2 units upward**.

(b) Multiplying all of the values of f(x) by 3 leaves all of the roots of f fixed: if x is a root of f then $f(x) = 0$ and $3f(x) = 3(0) = 0$ so x is also a root of $3 \cdot f(x)$. If x is not a root of f, then the graph of 3f(x) looks like the graph of f(x) **stretched vertically** by a factor of 3.

(c) The graph of $f(x-1)$ is the graph of f(x) **rigidly shifted 1 units to the right**.

We could also get these results by examining the graph of $y = f(x)$, creating a table of values for f(x)
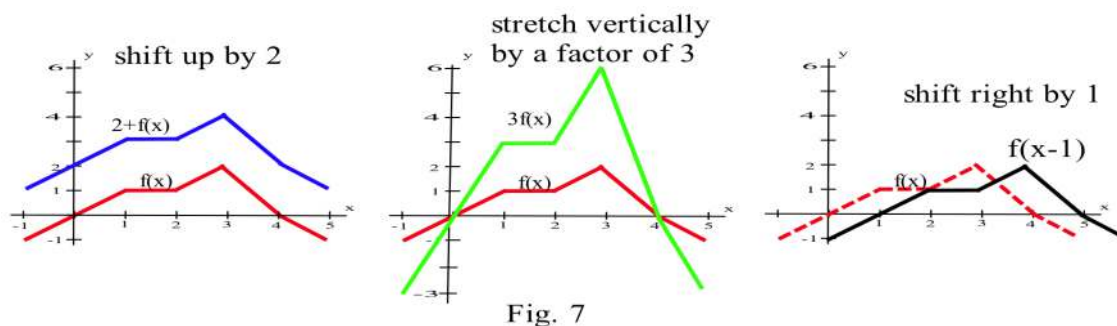
Fig. 7

and the new functions, and then graphing the new functions.

x	f(x)	2 + f(x)	3f(x)	x–1	f(x–1)
–1	–1	1	–3	–2	f(–2) not defined
0	0	2	0	–1	f(0–1) = –1
1	1	3	3	0	f(1–1) = 0
2	1	3	3	1	f(2–1) = 1
3	2	4	6	2	f(3–1) = 1
4	0	2	0	3	f(4–1) = 2
5	–1	1	–3	4	f(5–1) = 0

If k is a positive constant, then

- the graph of $y = k + f(x)$ will be the graph of $y = f(x)$ rigidly **shifted up by k units**,

- the graph of $y = kf(x)$ will have the same roots as the graph of f(x) and will be the graph of $y = f(x)$

    vertically stretched by a factor of k,

- the graph of $y = f(x - k)$ will be the graph of $y = f(x)$ rigidly **shifted right by k units**,

- the grah of $y = f(x + k)$ will be the graph of $y = f(x)$ rigidly **shifted left by k units**.

Practice 5:   Fig. 8 is the graph of g(x).

        Graph (a) 1+g(x), (b) 2g(x), (c) g(x–1) and (d) –3g(x).

Fig. 8
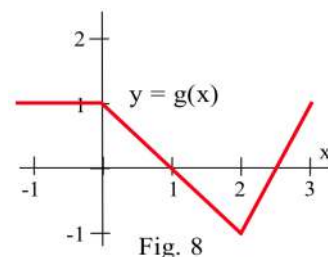
Iteration of Functions

There are applications which feed the output from a function machine back into the same machine as the new input. Each time through the machine is called an **iteration** of the function.

Fig. 9

Example 4:   Suppose $f(x) = \dfrac{5/x + x}{2}$, and we start with the input x = 4 and repeatedly feed the output from f back into f (Fig. 9). What happens?
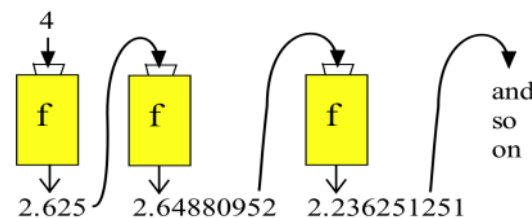
Solution:

Iteration	Input	Output
1	4	$f(4) = \dfrac{5/4 + 4}{2} = 2.625$
2	2.625	$f(f(4)) = \dfrac{5/2.625 + 2.625}{2} = 2.264880952$
3	2.264880952	f(f(f(4))) = 2.236251251
4	2.236251251	2.236067985
5	2.236067985	2.236067977
6	2.236067977	2.236067977

Once we have obtained the output 2.236067977, we will just keep getting the same output. You might

recognize this output value as $\sqrt{5}$. This algorithm always finds $\pm\sqrt{5}$. If we start with any positive input, the

values will eventually get as close to $\sqrt{5}$ as we want. Starting with any negative value for the input will

eventually get us to $-\sqrt{5}$. We cannot start with $x = 0$, since 5/0 is undefined.

Practice 6:   What happens if we start with the input value $x = 1$ and iterate the function

$f(x) = \dfrac{9/x + x}{2}$ several times? Do you recognize the resulting number? What do you think will

happen to the iterates of $g(x) = \dfrac{A/x + x}{2}$? (Try several positive values of A.)

Two Useful Functions: Absolute Value and Greatest Integer

These two functions have useful properties which let us describe situations in which an object abruptly changes

direction or jumps from one value to another value. Their graphs will have corners and breaks.

Absolute Value Function: | x |

The **absolute value** function of a number x, $y = f(x) = | x |$, is the distance between the number x and 0. If x

is greater than or equal to 0, then $| x |$ is simply $x - 0 = x$. If x is negative, then $| x |$ is $0 - x = -x = -1 \cdot x$

which is positive since $-1 \cdot$(negative number) = a positive number. On some calculators and in some computer

programming languages, the absolute value function is represented by **ABS(x)** .

> **Definition of | x | :** $| x | = \begin{cases} x & \textbf{if } x \geq 0 \\ -x & \textbf{if } x < 0 \end{cases}$ or $| x | = \sqrt{x^2}$.

The domain of $y = f(x) = | x |$ consists of all real numbers. The range of $f(x) = | x |$ consists of all numbers

larger than or equal to zero, all non–negative numbers. The graph of $y = f(x) = | x |$

(Fig. 10) has no holes or breaks, but it does have a sharp corner at $x = 0$. The

absolute value will be useful later for describing phenomena such as reflected light

and bouncing balls which change direction abruptly or whose graphs have corners.

The absolute value function has a number of properties which we will use later.

the Absolute Value function
$y = | x |$
Fig. 10

   Properties of | |:   For all real numbers a and b:

   (a) $| a | \geq 0$. $| a | = 0$ if and only if $a = 0$.

   (b) $| ab | = | a | | b |$

   (c) $| a + b | \leq | a | + | b |$
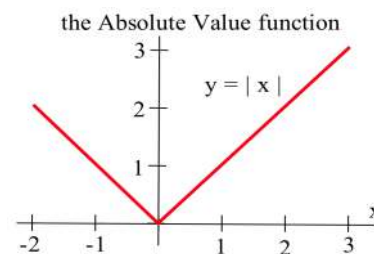
Taking the absolute value of a function has an interesting effect on the graph of the function. Since

$| x | = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, then for any function $f(x)$ we have $| f(x) | = \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ -f(x) & \text{if } f(x) < 0. \end{cases}$

In other words, if $f(x) \geq 0$, then $|f(x)| = f(x)$ so the graph of $|f(x)|$ is the same as the graph of $f(x)$. If $f(x) < 0$, then $|f(x)| = -f(x)$ so the graph of $|f(x)|$ is just the graph of $f(x)$ "flipped" about the x–axis, and it lies above the x–axis. The graph of $|f(x)|$ will always be on or above the x–axis.

Example 5:   Fig. 11 shows the graph of $f(x)$. Graph (a) $|f(x)|$, (b) $|1 + f(x)|$ and (c) $1 + |f(x)|$.

Solution: The graphs are given in Fig. 12. In (b) we shift the graph of f up 1 unit before taking the absolute value. In (c) we take the absolute value before shifting the graph up 1 unit.
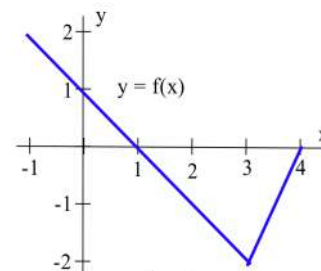
Fig. 11

Practice 7:   Fig. 13 shows the graph of $g(x)$. Graph (a) $|g(x)|$, (b) $|g(x-1)|$, and (c) $g(|x|)$.
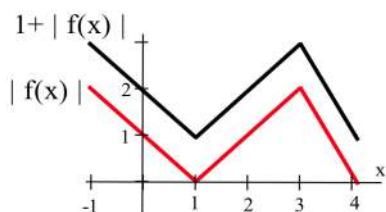
Fig. 12
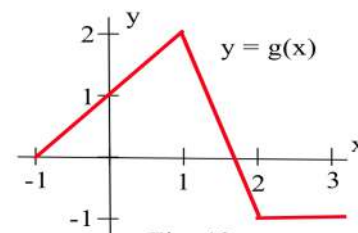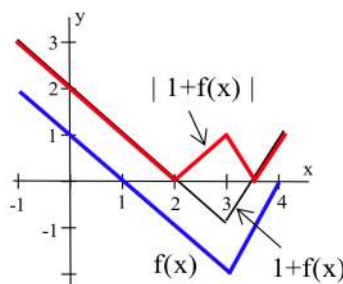
Fig. 13

Greatest Integer Function: [x] or INT(x)

The **greatest integer** function of a number x , $y = f(x) = [\,x\,]$, is the largest **integer** which is less than or equal to x . The value of $[\,x\,]$ is always an integer and $[\,x\,]$ is always less than or equal to x. For example, $[\,3.2\,] = 3$, $[\,3.9\,] = 3$, and $[\,3\,] = 3$. If x is positive, then $[\,x\,]$ truncates x (drops the fractional part of x) to get $[\,x\,]$. If x is negative, the situation is different: $[\,-4.2\,] \neq -4$ since -4 is **not** less than or equal to -4.2 : $[\,-4.2\,] = -5$, $[\,-4.7\,] = -5$ and $[\,-4\,] = -4$. On some calculators and in many programming languages the square brackets $[\,]$ are used for grouping objects or for lists, and the greatest integer function is represented by **INT(x)** .

Definition of [x]:   [x] = the largest integer which is less than or equal to x

$$= \begin{cases} x & \text{if x is an integer} \\ \text{largest integer strictly less than x} & \text{if x is NOT an integer.} \end{cases}$$

the Greatest Integer function

$y = INT(x) = [x]$

Fig. 14

The domain of The $f(x) = [x]$ is all real numbers. The range of $f(x) = [x]$ is only the integers. The graph of $y = f(x) = [x]$ is shown in Fig. 14. It has a jump break, a step, at each integer value of x, and $f(x) = [x]$ is called a **step function**. Between any two consecutive integers, the graph is horizontal with no breaks or holes. The greatest integer function is useful for describing phenomena which change values abruptly such as postage rates as a function of the weight of the letter ("26¢ for the first ounce and 13¢ additional for each additional half ounce"). It can also be used for functions whose graphs are "square waves" such as the on and off of a flashing light.
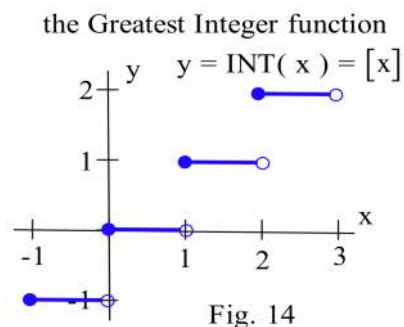
Example 6:   Graph $f(x) = INT(1 + .5 \sin(x))$.

Solution: One way to create this graph is to first graph $y = 1 + 0.5\sin(x)$, the thin curve in Fig. 15, and then apply the greatest integer function to y to get the thicker "square wave" pattern.

$y = 1 + 0.5\sin(x)$ $y = INT(1 + 0.5\sin(x))$
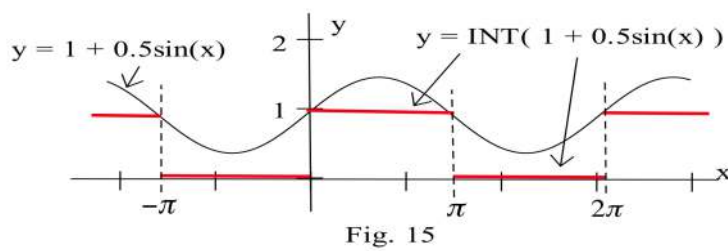
Fig. 15

Practice 8:   Sketch the graph of $y = INT(x^2)$ for $-2 \le x \le 2$.

A Really "Holey" Function

The graph of the greatest integer function has a break or jump at each integer value, but how many breaks can a function have? The next function illustrates just how broken or "holey" the graph of a function can be.

Define $h(x) = \begin{cases} 2 & \text{if } x \text{ is a rational number} \\ 1 & \text{if } x \text{ is an irrational number} \end{cases}$
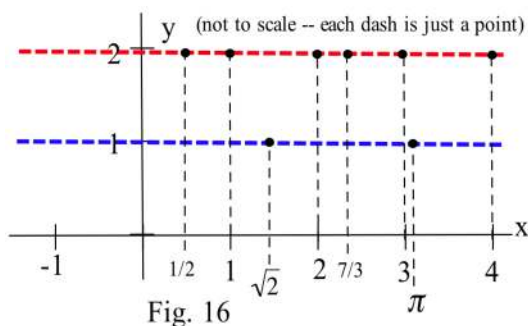
Fig. 16

Then $h(3) = 2, h(5/3) = 2$ and $h(-2/5) = 2$ since 3, 5/3 and $-2/5$ are all rational numbers. $h(\pi) = 1, h(\sqrt{7}) = 1$, and $h(\sqrt{2}) = 1$ since $\pi, \sqrt{7}$ and $\sqrt{2}$ are all irrational numbers. These and some other points are plotted in Fig. 16 .

In order to analyze the behavior of h(x) the following fact about rational and irrational numbers is useful.

Fact: "Every interval contains both rational and irrational numbers" or, equivalently,

"If a and b are real numbers and a < b, then there is

   (i) a rational number R between a and b (a < R < b), and

   (ii) an irrational number I between a and b (a < I < b)."

The Fact tells us that between any two places where the y = h(x) = 1 (because x is rational) there is a place where

y = h(x) is 2 because there is an irrational number between any two distinct rational numbers. Similarly,

between any two places where y = h(x) = 2 (because x is irrational) there

is a place where y = h(x) = 1 because there is a rational number between any two distinct irrational numbers.

The graph of y = h(x) is impossible to actually draw since every two points on the graph are separated by a hole.

This is also an example of a function which your computer or calculator can not graph because in general it can

not determine whether an input value of x is irrational.

Example 7:   Sketch the graph of

$$g(x) = \begin{cases} 2 & \text{if } x \text{ is a rational number} \\ x & \text{if } x \text{ is an irrational number} \end{cases}$$

Solution: A sketch of the graph of y = g(x) is shown in Fig. 17 .

   When x is rational, the graph of y = g(x) looks like the "holey"

   horizontal line y = 2. When x is irrational, the graph of y = g(x)

   looks like the "holey" line y = x.

(not to scale -- each dash is just a point)
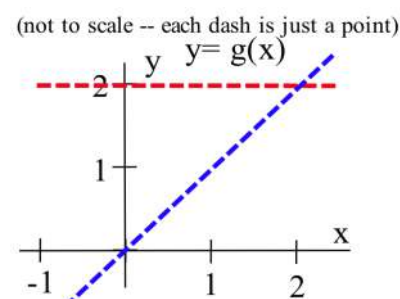
Fig. 17

Practice 9:   Sketch the graph of $r(x) = \begin{cases} 2 & \text{if } x \text{ is a rational number} \\ x & \text{if } x \text{ is an irrational number} \end{cases}$

PROBLEMS

1.   If T is the Celsius temperature of the air and v is the speed of the wind in kilometers per hour, then

$$\text{WCI} = \begin{cases} T & \text{if } 0 \le v \le 6.5 \\ 33 - \dfrac{10.45 + 5.29\sqrt{v} - 0.279v}{22}(33 - T) & \text{if } 6.5 \le v \le 72. \\ 1.6T - 19.8 & \text{if } 72 < v \end{cases}$$

   Determine the Wind Chill Index (a) for a temperature of 0° C and a wind speed of 49 km/hr and

   (b) for a temperature of 11° C and a wind speed of 80 km/hr.

   (c) Write a multiline function definition for the WCI if the temperature is 11° C.

2. Use the graph of y = f(x) in Fig. 18 to evaluate f(0), f(1), f(2), f(3), f(4)

    and f(5). Write a multiline function definition for f.
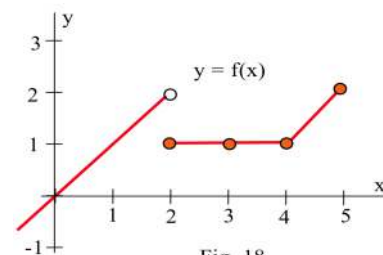
3. Use the graph of y = g(x) in Fig. 19 to evaluate g(0), g(1), g(2), g(3),

    g(4) and g(5). Write a multiline function definition for g.
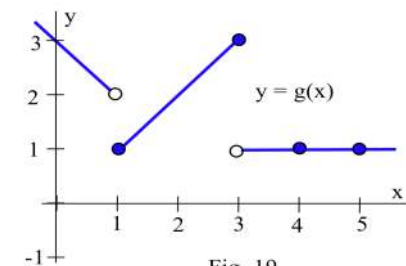
4. Use the values given in the table and   h(x) = 2x + 1 to determine the values

    of f∘g , g∘f and h∘g.

x	f(x)	g(x)	f∘g(x)	g∘f(x)	h∘g(x)
−1	2	0			
0	1	2			
1	−1	1			
2	0	2			
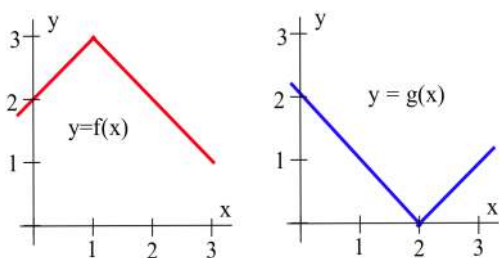
5. Use the graphs in Fig. 20 and the equation h(x) = x − 2 to

    determine the values of

    (a) f(f(1)), f(g(2)), f(g(0)), f(g(1))

    (b) g(f(2)), g(f(3)), g(g(0)), g(f(0))

    (c) f(h(3)), f(h(4)), h(g(0)), h(g(1))

Fig. 20

6. Use the graphs in Fig. 21   and the equation

    h(x) = 5 − 2x to determine the values of

    (a) h(f(0)), f(h(1)), f(g(2)), f(f(3))

    (b) g(f(0)), g(f(1)), g(h(2)), h(f (3))

    (c) f(g(0)), f(g(1)), f(h(2)), h(g(3))
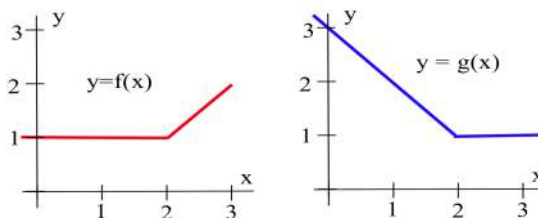
Fig. 21

7. $f(x) = \begin{cases} 3 & \text{if } x < 1 \\ x{-}2 & \text{if } 1 \le x < 3 \\ 1 & \text{if } 3 \le x \end{cases}$ $g(x) = \begin{cases} x^2{-}3 & \text{if } x < 0 \\ \text{INT}(x) & \text{if } 0 \le x \end{cases}$ $h(x) = x - 2.$

    (a) Evaluate f(x), g(x), and h(x) for x = −1, 0, 1, 2, 3, and 4 .

    (b) Evaluate f(g(1)), f(h(1)), h(f(1)), f(f(2)), g(g(3.5)) .

    (c) Graph f(x), g(x) and h(x) for −5 ≤ x ≤ 5 .

8. $f(x) = \begin{cases} x{+}1 & \text{if } x < 1 \\ 1 & \text{if } 1 \le x < 3 \\ 2{-}x & \text{if } 3 \le x \end{cases}$ $g(x) = \begin{cases} |x{+}1| & \text{if } x < 0 \\ 2x & \text{if } 0 \le x \end{cases}$ $h(x) = 3.$

    (a) Evaluate f(x), g(x), and h(x) for x = −1, 0, 1, 2, 3, and 4 .

    (b) Evaluate f(g(1)), f(h(1)), h(f(1)), f(f(2)), g(g(3.5)) .

    (c) Graph f(x), g(x) and h(x) for −5 ≤ x ≤ 5 .

9. You are planning to take a one week vacation in Europe, and the tour brochure says that Monday and Tuesday will be spent in England, Wednesday in France, Thursday and Friday in Germany, and Saturday and Sunday in Italy. Let L(d) be the location of the tour group on day d and write a multiline function definition for L(d).

10. A state has just adopted the following state income tax system: no tax on the first $10,000 earned, 1% of the next $10,000 earned, 2% of the next $20,000 earned, and 3% of all additional earnings. Write a multiline function definition for T(x), the state income tax due on earnings of x dollars.

11. Write a multiline function definition for the curve $y = f(x)$ in Fig. 22.

Fig. 22
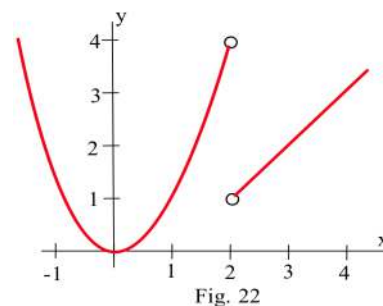
12. Define B(x) to be the **area** of the rectangle whose lower left corner is at the origin and whose upper right corner is at the point $(x, f(x))$ for the function f in Fig. 23. Then, for example, B(3)=6. Evaluate B(1), B(2), B(4) and B(5)

13. Define B(x) to be the **area** of the rectangle whose lower left corner is at the origin and whose upper right corner is at the point $(x, 1/x)$.
    a)   Evaluate B(1), B(2) and B(3).
    b)   Show that B(x) = 1 for all x > 0.

Fig. 23
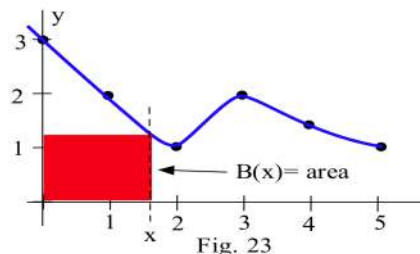
14. For $f(x) = |9 - x|$ and $g(x) = \sqrt{x - 1}$,
    (a) evaluate f∘g(1), f∘g(3), f∘g(5), f∘g(7), f∘g(0)
    (b) evaluate f∘f(2), f∘f(5), f∘f(–2). Does f∘f(x) = |x| for all values of x ?
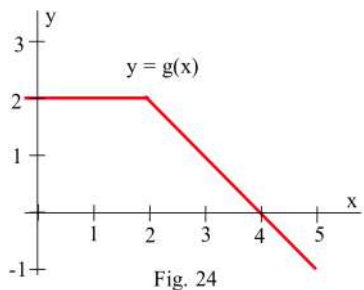
Fig. 24

15. Fig. 24 is the graph of g(x). Graph (a) g(x) – 1,
     (b) g(x–1), (c) | g(x) |, and (d) [g(x)] **.**

16. Fig. 25 is the graph of f(x). Graph (a) f(x) – 2,
     (b) f(x – 2), (c) | f(x) |, and (d) [f(x)] **.**
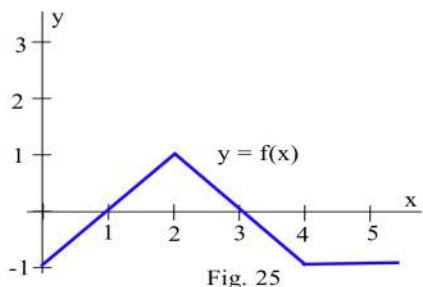
17. (a) Let $f(x) = 3x + 2$ and $g(x) = 2x + A$. Find a value for A so that
         f(g(x)) = g(f(x)).
     (b) Let $f(x) = 3x + 2$ and $g(x) = Bx - 1$. Find a value for B so that
         f(g(x)) = g(f(x)).

18. (a) Let $f(x) = Cx + 3$ and $g(x) = Cx - 1$. Find a value for C so that
         f(g(x)) = g(f(x)).
     (b) Let $f(x) = 2x + D$ and $g(x) = 3x + D$. Find a value for D so that
         f(g(x)) = g(f(x)).

Fig. 25

19. Graph $y = f(x) = x - INT(x)$ for $-1 \le x \le 3$. This function is called the "fractional part of x" and is an example of a "sawtooth" graph.

20. $f(x) = INT(x + 0.5)$ rounds off x to the NEAREST integer. $g(x) = \dfrac{INT(10x + 0.5)}{10}$ rounds off x to the nearest tenth, the first decimal place. What function will round off x to (a) the nearest hundredth (2 decimal places)? (b) the nearest thousandth (3 decimal places)?

21. Modify the function in example 6 to produce a "square wave" graph with a "long on, short off, long on, short off" pattern.

22. Some versions of the computer language BASIC contain a "signum" or "sign" function defined by

$$SGN(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}.$$

   (a) Graph SGN(x)   (b) Graph SGN(x – 2)   (c) Graph SGN(x – 4)

   (d) Graph SGN(x – 2) SGN(x – 4)   (e) Graph 1 – SGN(x – 2) SGN(x – 4)

   (f)   For a < b, describe the graph of 1 – SGN(x – a) SGN(x – b)
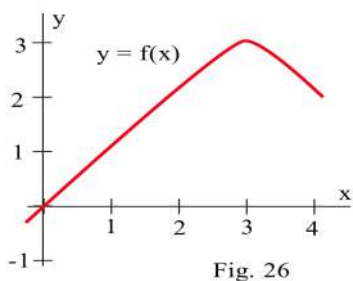
Fig. 26

23. Define g(x) to be the **slope** of the line tangent to the graph of $y = f(x)$ in Fig. 26 at (x,y).

   (a) Estimate g(1), g(2), g(3) and g(4).

   (b) Graph $y = g(x)$ for $0 \le x \le 4$.

24. Define h(x) to be the **slope** of the line tangent to the graph of $y = f(x)$ in Fig. 27 at (x,y).

   (a) Estimate h(1), h(2), h(3) and h(4).

   (b) Graph $y = h(x)$ for $0 \le x \le 4$.

Fig. 27
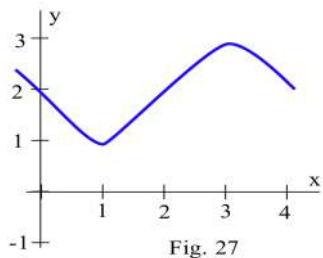
C25.   Pressing the COS (cosine) button on your calculator several times will produce iterates of f(x) = cos(x). What number will the iterates approach if you start with x = 1 and press the COS button 20 or 30 times?   What happens if you start with x = 2 or x = 10? (Be sure your calculator is in radian mode.)

C26.   Let $f(x) = 1 + \sin(x)$. What happens if you start with x = 1 and repeatedly feed the output from f back into f ? What happens if we start with x = 2 and examine the iterates of f ? (Be sure your calculator is in radian mode.)

C27.   Starting with x = 1, do the iterates of $f(x) = \dfrac{x^2 + 1}{2x}$ approach a number? What happens if you start with x = .5 or x = 4?

C28.   Let $f(x) = \frac{x}{2} + 3$. (a) What are the iterates of f if you start with x = 2? 4? 6?

    (b) Find a number c so that $f(c) = c$. This value of c is called a **fixed point** of f.

    (c) Find a fixed point of $g(x) = \frac{x}{2} + A$.

C29.   Let $f(x) = \frac{x}{3} + 4$. (a) What are the iterates of f if you start with x = 2? 4? 6?

    (b) Find a number c so that $f(c) = c$. (c) Find a fixed point of $g(x) = \frac{x}{3} + A$.

Some iterative procedures are geometric rather than numerical.

30.   Start with an equilateral triangle with sides of length 1 (Fig, 28a).

    (i)   Remove the middle third of each line segment.

    (ii)   Replace the removed portion with 2 segments with the same length as the removed segment.

    The first two iterations of this procedure are shown in Fig. 28b and Fig. 28c. Repeat steps (i) and (ii)

    several more times, each time removing the middle third of each line segment and replacing it with two new

    segments. What happens to the length of the shape with each iteration?   (The result of iterating over and

    over with this procedure is called Koch's Snowflake, named for Helga von Koch)

Koch;s Snowflake

(a)   (b)   (c)
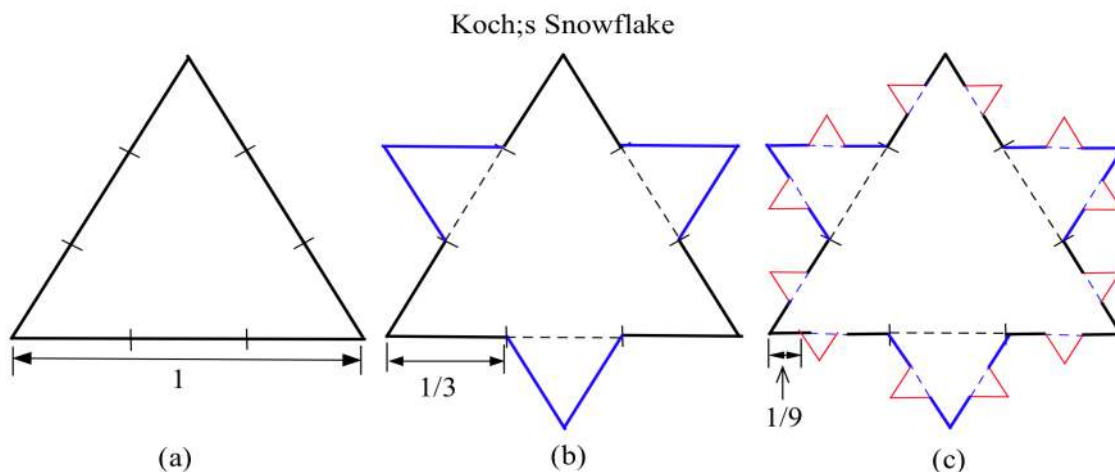
Fig. 28

31. (Optional) Sketch the graph of $p(x) = \begin{cases} 3 - x & \text{if } x \text{ is a rational number} \\ 1 & \text{if } x \text{ is an irrational number} \end{cases}$

32. (Optional) Sketch the graph of $q(x) = \begin{cases} x^2 & \text{if } x \text{ is a rational number} \\ x + 1 & \text{if } x \text{ is an irrational number} \end{cases}$

Section 1.4   **PRACTICE Answers**

Practice 1:   C(x) is the cost for one night on date x .

$$C(x) = \begin{cases} \$50 & \text{if } x \text{ is between June 1 and September 15} \\ \$40 & \text{if } x \text{ is any other date} \end{cases}$$

Practice 2:

x	g(x)
–3	–3
–1	2
0	2
1/2	2
1	undefined

x	g(x)
$\pi/3$	$-\pi/3$
2	–2
3	–3
4	undefined
5	1

Fig. 29
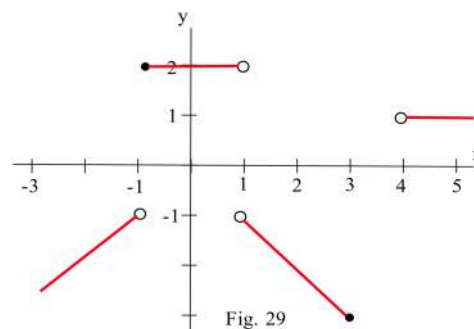
Practice 3: $f(x) = \begin{cases} 1 & \text{if } x \le -1 \\ 1-x & \text{if } -1 < x \le 1 \\ 2 & \text{if } 1 < x \end{cases}$

Practice 4:   f∘g(3) = f(2) = 2/–1 = **–2**   f∘g(8) = f(3) is **undefined**   g∘f(4) = g(4) = $\sqrt{5}$

f∘h(1) = f(2) = 2/–1 = **–2**   f∘h(3) = f(2) = **–2**   f∘h(2) = f(3) is **undefined**

h∘g(–1) = h(0) = **0**   f∘g(x) = f($\sqrt{1+x}$) = ($\sqrt{1+x}$)/($\sqrt{1+x}$ – 3) , g∘f(x) = g($\frac{x}{x-3}$) = $\sqrt{1 + \frac{x}{x-3}}$

Practice 5:   See Fig. 30.

(a)   (b)   (c)   (d)

Fig. 30
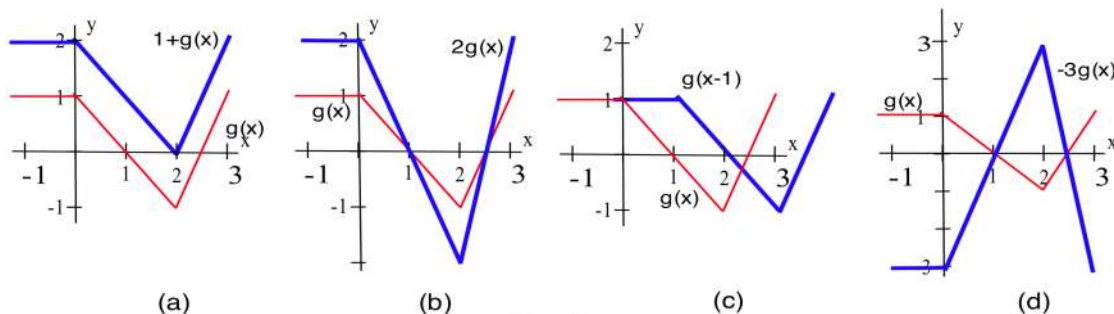
Practice 6: $f(x) = \frac{9/x + x}{2}$.

$f(1) = \frac{9/1 + 1}{2} = 5, \;\; f(5) = \frac{9/5 + 5}{2} = 3.4, \;\; f(3.4) \approx 3.023529412,$

$f(3.023529412) \approx 3.000091554,$ and $f(3.000091554) \approx 3.000000001$.

These values are approaching 3, the square root of 9.

Putting A = 6, then $f(x) = \frac{6/x + x}{2}$.

$f(1) = \frac{6/1 + 1}{2} = 3.5, \;\; f(3.5) = \frac{6/3.5 + 3.5}{2} = 2.607142857,$

$f(2.607142857) \approx 2.45425636, f(2.45425636) \approx 2.449494372,$

$f(2.449494372) \approx 2.449489743$.

$f(2.449489743) \approx 2.449489743$ (the output is the same as the input for 9 decimal places)

These values are approaching 2.449489743, the square root of 6.

For any positive value A, the iterates of $f(x) = \frac{A/x + x}{2}$ (starting with any positive x) will approach $\sqrt{A}$.

Practice 7:   Fig. 31 shows some of the intermediate steps and final graphs.

(a)   (b)   (c)

g(x)   ———

g(x-1)   -----

| g(x-1) | ━━━
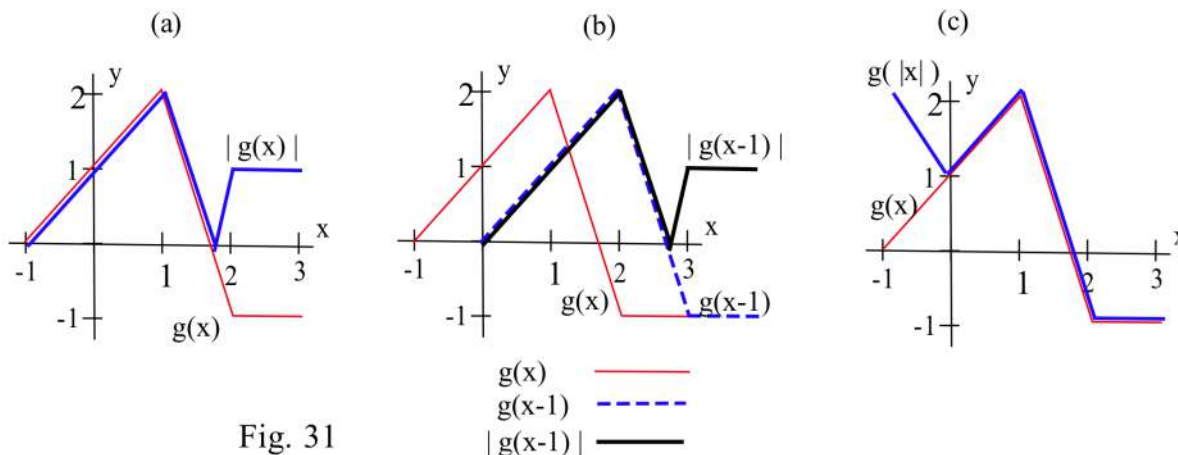
Fig. 31

Practice 8:   Fig. 32 shows the graph of $y = x^2$
and the graph (thicker) of $y = INT(x^2)$.

Fig. 32

$y = x^2$ ━━━
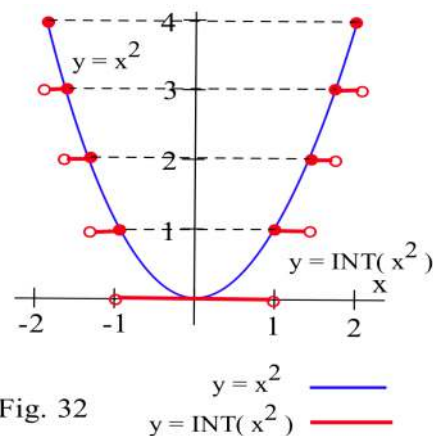
$y = INT(x^2)$ ━━━

Practice 9:   Fig. 33 shows the "holey" graph of $y = x$ with a hole
at each rational value of x and the "holey" graph of
$y = sin(x)$ with a hole at each irrational value of x.
Together they form the graph of r(x) .
(This is a very crude image since we can't really see
the individual holes which have zero width.)
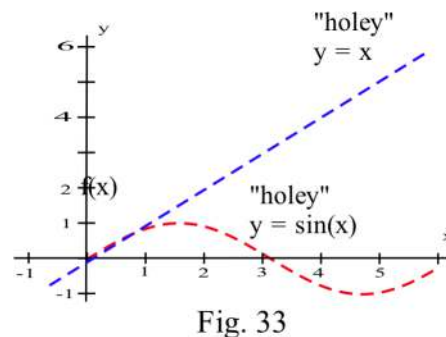
Fig. 33

1.5   MATHEMATICAL LANGUAGE

The calculus concepts we will explore in this book are simple and powerful, but sometimes subtle. To succeed in calculus you will have to master some techniques, but, more importantly, you will have to understand ideas and be able to work with the ideas in words and pictures -- very clear words and pictures. You also need to understand some of the common linguistic constructions used in mathematics. In this section we will discuss a few of the most common mathematical phrases, the meaning of these phrases and some of their equivalent forms.

Your calculus teacher is going to use these types of statements, and it is very important that you understand exactly what the teacher means. You have reached the level in mathematics where the precise use of language is important.

EQUIVALENT STATEMENTS

Two statements are **equivalent** if they always have the same logical value (a logicl value is either "true" or "fale", that is, if they are both true or are both false. The statements "x = 3" and "x + 2 = 5" are equivalent statements because if one of them is true then so is the other, and if one of them is false then so is the other. The statements "x = 3" and "$x^2 - 4x + 3 = 0$" are not equivalent since x = 1 makes the second statement true but the first one false.

AND and OR

> The compound statement "A **and** B are true" is equivalent to "**both of** A and B are true."

If A or if B or if both are false, then the statement "A and B are true" is false. The statement "$x^2 = 4$ and x > 0" is true when x = 2 and is false for every other value of x.

> The compound statement "A **or** B is true" is equivalent to "**at least one** of A or B is true."

If both A and B are false, then the statement "A or B is true" is false. The statement "$x^2 = 4$ or x > 0" is true if x = –2 or x is any positive number. The statement is false when x = –3 and for lots of other values of x.

Practice 1: Which values of x make each statement true?

    (a) "x < 5"   (b) "x + 2 = 6"   (c) "$x^2 - 10x + 24 = 0$"   (d) "(a) and (b)"   (e) "(a) or (c)"

NEGATION OF A STATEMENT

For some simple statements we can construct the negation just by adding the word "not."

Statement	Negation of the Statement
x is equal to 3 ($x = 3$)	x is **not** equal to 3 ($x \neq 3$)
x is less than 5 ($x < 5$)	x is **not** less than 5 ($x \nless 5$) x is greater than or equal to 5 ($x \geq 5$)

When the statement contains words such as "all", "no", or "some," then its negation is more complicated.

Statement	Negation of the Statement
All x satisfy A. Every x satisfies A. }	At least one x does not satisfy A. There is an x which does not satisfy A. Some x does not satisfy A.
No x satisfies A. Every x does not satisfy A. }	At least one x satisfies A. Some x satisfies A.
There is an x which satisfies A. At least one x satisfies A. Some x satisfies A.	No x satisfies A. Every x does not satisfy A.

We can also negate compound statements containing "and" and "or".

Statement	Negation of the Statement
A and B are both true.	At least one of A or B is not true.
A and B and C are all true.	At least one of A or B or C is not true.
A or B is true.	Both A and B are not true.

Practice 2:   Write the negation of each of these statements.

(a) $x + 5 \geq 3$ (b) All prime numbers are odd.   (c) $x^2 < 4$

(d) x divides 2 and x divides 3.   (e) No mathematician can sing well.

IF ... THEN ... : A Very Common Structure in Mathematics

The most common and basic structure used in mathematical language is the

> "**If** (some hypothesis) **then** (some conclusion)"

sentence. Almost every result in mathematics can be stated using one or more "If … then … " sentences.

> "If A then B"   means that when the hypothesis A is true, then the conclusion B must also be true.

If the hypothesis is false, then the "If … then … " sentence makes no claim about the truth or falsity of the

conclusion — the conclusion may be either true or false.

Even in everyday life you have probably encountered "If ... then ..." statements for a long time. A parent might try to encourage a child with a statement such as " If you clean your room then I will buy you an ice cream cone."

To show that an "If . . . then . . . " statement is not valid (not true), all we need to do is find a **single** example where the hypothesis is true and the conclusion is false. Such an example with a true hypothesis and false conclusion is called a **counterexample** for the "If . . . then . . . " statement. A valid "If . . . then . . . " statement has no counterexample.

> A **counterexample** to the statement "If A then B" is an example in which A is true and B is false.

The only way for the statement " If you clean your room then I will buy you an ice cream cone" to be false is if the child cleaned the room and the parent did not buy the ice cream cone. If the child did not clean the room but the parent still bought the ice cream cone we would say that the statement was true.

The statement "If n is a positive integer, then $n^2 + 5n + 5$ is a prime number" has hypotheses "n is a positive integer" and conclusion "$n^2 + 5n + 5$ is a prime number." This "If ... then" statement is false since replacing n with the number 5 will make the hypothesis true and the conclusion false. The number 5 is a counterexample for the statement. Every invalid "If . . . then . . . " statement has at least one counterexample, and the most convincing way to show that a statement is not valid is to find a counterexample to the statement.

A number of other language structures can be translated into the "If ... then ..." form. The statements below all mean the same as "If (A) then (B)" :

"All (A) are (B)."	"Every (A) is (B)."	"Each (A) is (B)."
"Whenever (A), then (B)."	"(B) whenever (A)."	"(A) only if (B)."
"(A) implies (B)."	"(A) $\Rightarrow$ (B)" (the symbol " $\Rightarrow$ " means "implies")	

Practice 3:   Restate "**If** (a shape is a square) **then** (the shape is a rectangle)" as many ways as you can.

"If … then … " statements occur hundreds of times in every mathematics book, including this one. It is important that you are able to recognize the various forms of "If … then … " statements and that you are able to distinguish the hypotheses from the conclusions.

Contrapositive Form of an "If … then …" Statement

The statement "If (A) then (B)" means that if the hypothesis A is true, then the conclusion B is guaranteed to be true.

Suppose we know that in a certain town the statement

"If (a building is a church) then (the building is green)"

is a true statement,. What can we validly conclude about a red building? Based on the information we have, we can validly conclude that the red building is "not a church" since every church is green. We can also conclude that a blue building is not a church. In fact, we can conclude that every "not green" building is "not a church." That is, if the conclusion of a valid "If … then … " statement is **false**, then the hypothesis must also be **false**.

The **contrapositive** form of "If (A) then (B)" is

"If (negation of B) then (negation of A)"   or "If (B is false) then (A is false)."

The statement "If (A) then (B)" and its contrapositive "If (not B) then (not A)" are **equivalent**.

What about a green building in this town? The green building may or may not be a church – perhaps every post office is also painted green. Or perhaps every building in town is green, in which case the statement "If (a building is a church) then (the building is green)" is certainly true.

Practice 4:   Write the contrapositive form of each of the following statements.

  (a) If a function is differentiable then it is continuous.   (b) All men are mortal.

  (c) If (x equals 3) then $(x^2 – 5x + 6$ equals 0)   (d) If (2 divides x and 3 divides x) then (6 divides x).

Converse of an "If … then …" Statement

If we switch the hypotheses and the conclusion of an "If A then B" statement we get the converse "If B then A."

The converse of an "If … then … " statement is a new statement with the hypothesis and conclusion switched: the converse of "If (A) then (B)" is "If (B) then (A)." For example, the converse of "If (a building is a church) then (the building is green)" is "If (a building is green) then (the building is a church)."   The converse of an "If … then … " statement **is not equivalent** to the original "If … then … " statement. The statement "If $x = 2$, then $x^2 = 4$" is true, but the converse statement "If $x^2 = 4$, then $x = 2$" is not true because $x = –2$ makes the hypothesis of the converse true and the conclusion false.

The **converse** of "If (A) then (B)" is "If (B) then (A)."

The statement "If (A) then (B)" and its converse "If (B) then (A)" are **not equivalent**.

Wrap–up

The precise use of language by mathematicians (and mathematics books) is an attempt to clearly communicate ideas from one person to another, but that requires that both people understand the use and rules of the language. If you don't understand this usage, the communication of the ideas will almost certainly fail.

PROBLEMS

In problems 1 and 2, let $A = \{1,2,3,4,5\}$, $B = \{0,2,4,6\}$, and $C = \{-2,-1,0,1,2,3\}$. Which values of x satisfy each statement.

1. a) x is in A **and** x is in B. b) x is in A **or** x is in C. c) x is not in B **and** x is in C.

2. a) x is not in B **or** C. b) x is in B and C but not in A. c) x is not in A but is in B or C.

In problems 3 – 5, list or describe all the values of x which make each statement true.

3. a) $x^2 + 3 > 1$ b) $x^3 + 3 > 1$ c) $[\, x \,] \le |\, x \,|$

4. a) $\dfrac{x^2 + 3x}{x} = x + 3$ b) $x > 4$ and $x < 9$ c) $|\, x \,| = 3$ and $x < 0$

5. a) $x + 5 = 3$ or $x^2 = 9$ b) $x + 5 = 3$ and $x^2 = 9$ c) $|\, x + 3 \,| = |\, x \,| + 3$

In problems 6 – 8, write the **contrapositive** of each statement. If the statement is false, give a **counterexample**.

6. a) If $x > 3$ then $x^2 > 9$. b) Every solution of $x^2 - 6x + 8 = 0$ is even.

7. a) If $x^2 + x - 6 = 0$ then $x = 2$ or $x = -3$. b) All triangles have 3 sides.

8. a) Every polynomial has at least one zero. b) If I exercise and eat right then I will be healthy.

In problems 9 – 11, write the **contrapositive** of each statement. If necessary, first write the original statement in the "If . . . then . . . " form.

9. a) If your car is properly tuned, it will get at least 24 miles per gallon.

b) You can have dessert if you eat your vegetables.

10. a) A well–prepared student will miss less than 15 points.

b) I feel good when I jog.

11. a) If you love your country, you will vote for me.

b) If guns are outlawed then only outlaws will have guns.

In problems 12 – 15, write the **negation** of each statement.

12. a) It is raining.   b) Some equations have solutions.   c) f(x) and g(x) are polynomials.

13. a) f(x) or g(x) is positive.   b) x is positive.   c) 8 is a prime number.

14. a) Some months have 6 Mondays.   b) All quadratic equations have solutions.

    c) The absolute value of a number is positive.

15. a) For all numbers a and b, | a + b | = | a | + | b |.   b) All snakes are poisonous.

    c) No dog can climb trees.

16. Write an "If . . . then . . . " statement which is true but whose converse is false.

17. Write an "If . . . then . . . " statement which is true and whose converse is true.

18. Write an "If . . . then . . . " statement which is false and whose converse is false.

In problems 19 – 22, state whether each statement is true or false. If the statement is false, give a counterexample.

19. a)   If a and b are real numbers then $(a + b)^2 = a^2 + b^2$.

    b)   If $a > b$ then $a^2 > b^2$.   c)   If $a > b$ then $a^3 > b^3$.

20. a)   For all real numbers a and b, | a + b | = | a | + | b |

    b)   For all real numbers a and b, [a] + [b] ≤ [a + b]   ([ ] represents the greatest integer function.)

    c)   If f(x) and g(x) are linear functions then f(g(x)) is a linear function.

21. a)   If f(x) and g(x) are linear functions then f(x) + g(x) is a linear function.

    b)   If f(x) and g(x) are linear functions then f(x)g(x) is a linear function.

    c)   If x divides 6   then x divides 30.

22. a)   If x divides 50 then x divides 10.   b) If x divides yz then x divides y or z.

    c)   If x divides a^2 then x divides a.

In problems 23 – 26, rewrite each statement as an "If … then … " statement and state whether it is true or

false. If the statement is false, give a counterexample.

23. a)   The sum of two prime numbers is a prime.   b)   The sum of two prime numbers is never a prime.

    c)   Every prime number is odd.   d)   Every prime number is even.

24. a)   Every square has 4 sides.   b)   All 4–sided polygons are squares.

    c)   Every triangle has 2 equal sides.   d)   Every 4–sided polygon with equal sides is a square.

25. a)   Every solution of x+5=9 is odd.   b)   Every 3–sided polygon with equal sides is a triangle.

    c)   Every calculus student studies hard.   d)   All (real number) solutions of $x^2 – 5x + 6 = 0$ are even.

26. a)   Every straight line intersects the x–axis.   b)   Every (real number) solution of $x^2 + 3 = 0$ is even.

    c)   All birds can fly.   d)   No mammal can fly.

Section 1.5   **PRACTICE Answers**

Practice 1:   (a) All values of x less than 5.   (b) x = 4

(c) Both x = 4 and x = 6.   (d) x = 4

(e) x = 6 and all x less than 5.

Practice 2:   (a) $x + 5 < 3$.

(b) At least one prime number is even.

There is an even prime number.

(c) $x^2 \geq 4$.

(d) x does not divide 2 or x does not divide 3.

(e) At least one mathematician can sing well.

There is a mathematician who can sing well.

Practice 3:   Here are several ways to restate "**If** (a shape is a square) **then** (the shape is a rectangle)."

All squares are rectangles.

Every square is a rectangle.

Each square is a rectangle.

Whenever a shape is a square, then it is a rectangle.

A shape is a rectangle whenever it is a square.

A shape is a square only if it is a rectangle.

A shape is a square implies that it is a rectangle.

Being a square implies being a rectangle.

Practice 4:   (a) statement   "If a function is differentiable then it is continuous."

contrapositive "If a function is not continuous then it is not differentiable."

(b) statement   "All men are mortal."

contrapositves "All immortals are not men."

"If a thing is not mortal then it is not human."

(c) statement   "If (x equals 3) then $(x^2 – 5x + 6$ equals 0)."

contrapositive "If $(x^2 – 5x + 6$ does not equal 0) then (x does not equal 3)."

(d) statement   "If (2 divides x and 3 divides x) then (6 divides x)."

contrapositive "If (6 does not divide x) then (2 does not divide x **or** 3 does not divide x)."

1.6   Solutions to Odd Numbered Problems

Important Note about Precision of Answers:

In many of the problems in this book you are required to read information from a graph and to calculate with that information. You should take reasonable care to read the graphs as accurately as you can (a small straightedge is helpful), but even skilled and careful people make slightly different readings of the same graph. That is simply one of the drawbacks of graphical information. When answers are given to graphical problems, the answers should be viewed as the best approximations we could make, and they usually include the word "approximately" or the symbol "≈" meaning "approximately equal to." Your answers should be close to the given answers, but you should not be concerned if they differ a little. (Yes those are vague terms, but it is all we can say when dealing with graphical information.)

Section 1.1
1.   approx. 1, 0, –1

3.   (a) Approx. $\dfrac{70 - 150 \text{ deg.}}{10 - 0 \text{ min}} = -8$ deg/min. Avg. rate of cooling ≈ 8 deg/min. (b) Approx. 6 deg/min cooling, and 5 deg/min cooling. (c) Approx. 5.5 deg/min cooling, and 10 deg/min cooling. (d) When t = 6 min.

5.   We estimate that the area is approximately (very approximate) 9 cm^2.

7.   Method 1: Measure the diameter of the coffee can, then fill it about half full of wine and measure the height of the wine and calculate the volume. Submerge the bulb, measure the height of the wine again, and calculate the new volume. The volume of the bulb is the difference of the two calculated volumes.
   Method 2: Fill the can completely full of wine and weigh the full can. Submerge the bulb (displacing a volume of wine equal to the volume of the bulb), remove the bulb, and weigh the can again. By subtracting, find the weight of the displaced wine and then use the fact that the density of wine is approximately 1 gram per 1 cubic centimeter to determine the volume of the bulb.

Section 1.2
1.   (a) –3/4 (b) 1/2 (c) 0 (d) 2 (e) undefined

3.   (a) $\dfrac{4}{3}$ (b) $\dfrac{-9}{5}$ (c) $x + 2$ (if $x \neq 2$) (d) $4 + h$ (if $h \neq 0$) (e) $a + x$ (if $a \neq x$)

5.   (a) t = 5: $\dfrac{5000}{1500} = \dfrac{10}{3}$, t = 10: $\dfrac{5000}{3000} = \dfrac{5}{3}$, t = 20: $\dfrac{5000}{6000} = \dfrac{5}{6}$ (b) any t > 0: $\dfrac{5000}{300t} = \dfrac{50}{3t}$
   (c) decreasing, since the numerator remains constant at 5000 while the denominator increases.

7.   The restaurant is 4 blocks south and 2 blocks east. The distance is $\sqrt{4^2 + 2^2} = \sqrt{20} \approx 4.47$ blocks.

9. $y = \sqrt{20^2 - 4^2} = \sqrt{384} \approx 19.6$ feet, $m = \dfrac{\sqrt{384}}{4} \approx 4.9$. $\tan(q) = \dfrac{\sqrt{384}}{4} \approx 4.9$ so $q \approx 1.37$ $(\approx 78.5^{\text{o}})$.

11.   The equation of the line through $P = (2,3)$ and $Q = (8,11)$ is $y - 3 = \dfrac{8}{6}(x - 2)$ or $6y - 8x = 2$. Substituting
   $x = 2a + 8(1{-}a) = 8 - 6a$ and $y = 3a + 11(1{-}a) = 11 - 8a$ into the equation for the line, we
   get $6(11 - 8a) - 8(8 - 6a) = 66 - 48a - 64 + 48a$ which equals 2 for every value of a, so the point
   with $x = 2a + 8(1{-}a)$ and $y = 3a + 11(1{-}a)$ is on the line through P and Q for every value of a.
   The $\text{Dist(P,Q)} = \sqrt{6^2 + 8^2} = 10$. $\text{Dist(P,R)} = \sqrt{(8{-}6a - 2)^2 + (11{-}8a - 3)^2}$
   $= \sqrt{(6 - 6a)^2 + (8 - 8a)^2} = \sqrt{6^2(1{-}a)^2 + 8^2(1{-}a)^2} = \sqrt{100(1{-}a)^2} = 10{\cdot}|1{-}a| = |1{-}a| {\cdot} \text{Dist(P,Q)}$.

13. (a) $m_1 \cdot m_2 = (1)(-1) = -1$ so the lines are perpendicular. (b) Because 20 units of x-values are physically wider on the screen than 20 units of y-values. (c) Set the window so (xmax − xmin) ≈ 1.7 (ymax − ymin).

15. (a) y − 5 = 3(x − 2) or y = 3x − 1. (b) y − 2 = −2(x − 3) or y = 8 − 2x (c) y − 4 = $-\frac{1}{2}$(x − 1) or y = $-\frac{1}{2}$x + $\frac{9}{2}$

17. (a) y − 5 = $\frac{3}{2}$(x−2) or y = $\frac{3}{2}$x + 2 (b) y − 2 = $\frac{3}{2}$(x+1) or y = $\frac{3}{2}$x + $\frac{7}{2}$ (c) x = 3.

19. The distance between the centers is $\sqrt{6^2 + 8^2}$ = 10. (a) 10–2–4 = 4 (b) 10–2–7 = 1 (c) 0 (they intersect) (d) 15–10–3 = 2 (e) 12–10–1 = 1.

21. Find Dist(P,C) = $\sqrt{(x–h)^2 + (y–k)^2}$, and compare the value to r:

$$P \text{ is } \begin{cases} \text{inside the circle} & \text{if Dist(P,C) < r} \\ \text{on the circle} & \text{if Dist(P,C) = r} \\ \text{outside the circle} & \text{if Dist(P,C) > r} \end{cases}$$

23. A point P =(x,y) lies on the circle if and only if its distance from C = (h,k) is r : Dist(P,C) = r. So P is on the circle if and only if $\sqrt{(x–h)^2 + (y–k)^2}$ = r or $(x–h)^2 + (y–k)^2 = r^2$.

25. (a) slope is $-\frac{5}{12}$ (b) undefined (vertical line) (c) $\frac{12}{5}$ (d) 0 (horizontal line)

27. (a) distance ≈ 2.22 . (b) Distance ≈ 2.24 .
(c) (by inspection) 3 units which occurs at the point (5, 3).

29. (a) If B $\neq$ 0, we may solve for y: y = $-\frac{A}{B}$ x + $\frac{C}{B}$. The slope is the coefficient of x: m = $-\frac{A}{B}$.

(b) The required slope is B/A (the negative reciprocal of −A/B) so the equation is y = $\frac{B}{A}$ x or Bx−Ay = 0.

(c) Solve { Ax + By = C, Bx − Ay = 0 } to get x = $\frac{AC}{A^2 + B^2}$ and y = $\frac{BC}{A^2 + B^2}$.

(d) Distance = $\sqrt{(\frac{AC}{(A^2 + B^2)})^2 + (\frac{BC}{A^2 + B^2})^2}$ = $\sqrt{\frac{A^2C^2}{(A^2 + B^2)^2} + \frac{B^2C^2}{(A^2 + B^2)^2}}$

= $\sqrt{\frac{(A^2 + B^2)C^2}{(A^2 + B^2)^2}}$ = $\sqrt{\frac{C^2}{A^2 + B^2}}$ = $\frac{|C|}{\sqrt{A^2 + B^2}}$

Section 1.3

1. A–a, B–c, C–d, D–b   3. A–b, B–c, C–d, D–a
5. (a)–C, (b)–A, (c)–B

6. The bottles are sketched in Fig. 0.3P6.

7. f(x) = x^2 + 3, g(x) = $\sqrt{x - 5}$, h(x) = $\frac{x}{x - 2}$

Fig. 0.3P6

(a) f(1) = 4, g(1) is undefined , h(1) = −1.
(b) Graphs of f, g and s are shown in Fig. 0.3P7.
(c) f(3x) = (3x $)^2$ + 3 = $9x^2$ + 3, g(3x) = $\sqrt{3x - 5}$ (for x ≥ 5/3) h(3x) = $\frac{3x}{3x - 2}$

(d) f(x+h) = (x+h$)^2$ + 3 = x^2 + 2xh + h^2 + 3 , g(x+h) = $\sqrt{x + h - 5}$, h(x + h) = $\frac{x+h}{x+h - 2}$
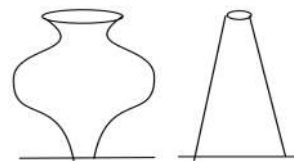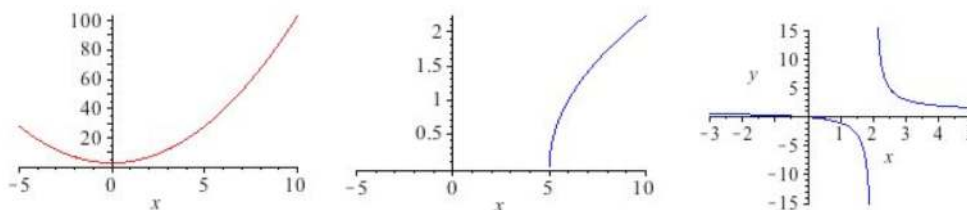
Fig. 0.3P7

9. (a) m = 2   (b) m = 2x + 3 + h .   (c) m = x + 4 (if x≠1)
    If x = 1.3, then m = 5.3 . If x = 1.1, then m = 5.1 . If x = 1.002, then m = 5.002 .

11. $f(x) = x^2 - 2x, \ g(x) = \sqrt{x}$.

$m = \dfrac{f(\,a+h\,) - f(\,a\,)}{h} = 2a + h - 2 \ (h\neq 0)$. If a = 1, then m = h. If a = 2, then m = 2+h.

If a = 3, then m = 4 + h. If a = x, then m = 2x + h − 2.

$m = \dfrac{g(\,a+h\,) - g(\,a\,)}{h} = \dfrac{1}{\sqrt{a+h} + \sqrt{a}} = \dfrac{\sqrt{a+h} - \sqrt{a}}{h}$. If a = 1, then m = $\dfrac{\sqrt{1+h} - 1}{h}$.

If a = 2, then m = $\dfrac{\sqrt{2+h} - \sqrt{2}}{h}$. If a = 3, then m = $\dfrac{\sqrt{3+h} - \sqrt{3}}{h}$.

If a = x, then m = $\dfrac{\sqrt{x+h} - \sqrt{x}}{h}$.

13. (a) Approx. 250 miles, 375 miles. (b) Approx. 200 miles/hour.
     (c) By flying along a circular arc about 375 miles from the airport.
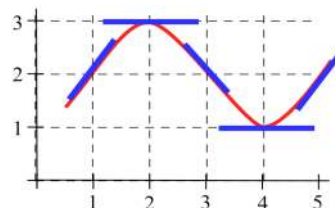
15. (a) See Fig. 0.3P15.   (b) Max at x = 2. Min at x = 4.
     (c) Largest at x = 5. Smallest at x = 3.

Fig. 0.3P15

17. The path of the slide is a straight line tangent to the graph of the
     path at the point of fall. See Fig. .

19. (a) s(1) = 2, s(3) = 4/3. s(4) = 5/4. (b) s(x) = $\dfrac{x+1}{x}$.
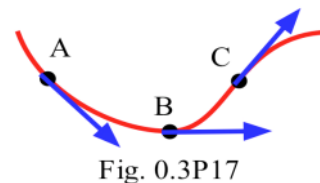
Fig. 0.3P17

21.

x	f(x)	slope of the line tangent to the graph of f at (x, f(x))
0	1	1
1	2	1
2	2	−1
3	1	0
4	1.5	0.5

23. On your own.

Section 1.4 Answers

1. (a) ≈ -18 . (b) -2.2

(c) If $T = 11^{\circ}C$, WCI $= \begin{cases} 11 & \text{if } 0 \le v \le 6.5 \\ 33 - \dfrac{10.45 + 5.29\sqrt{v} - 0.279\,v}{22}(22) & \text{if } 6.5 < v \le 72 \\ -2.2 & \text{if } 72 < v \end{cases}$.

3. $g(0){=}3$, $g(1){=}1$, $g(2){=}2$, $g(3){=}3$, $g(4){=}1$, $g(5){=}1$. $g(x) = \begin{cases} 3 - x & \text{if } x < 1 \\ x & \text{if } 1 \le x \le 3 \\ 1 & \text{if } x > 3 \end{cases}$

5. (a) f(f(1)) = 1, f(g(2)) = 2, f(g(0)) = 2, f(g(1)) = 3
    (b) g(f(2)) = 0, g(f(3)) = 1, g(g(0)) = 0, g(f(0)) = 0
    (c) f (h(3)) = 3, f(h(4)) = 2, h(g(0)) = 0, h(g(1)) = −1

7. (a)

x	−1	0	1	2	3	4
f(x)	3	3	−1	0	1	1
g(x)	−2	0	1	2	3	4
h(x)	−3	−2	−1	0	1	2

(b) f(g(1)) = −1, f(h(1)) = 3, h(f(1)) = −3, f(f(2)) = 3, g(g(3.5)) = 3

(c) See Fig. 0.4P7 for the graphs of f, g, and h.

9. $L(d) = \begin{cases} \text{England} & \text{if } d = \text{Mon. or Tue.} \\ \text{France} & \text{if } d = \text{Wed.} \\ \text{Germany} & \text{if } d = \text{Thur. or Fri.} \\ \text{Italy} & \text{if } d = \text{Sat. or Sun.} \end{cases}$
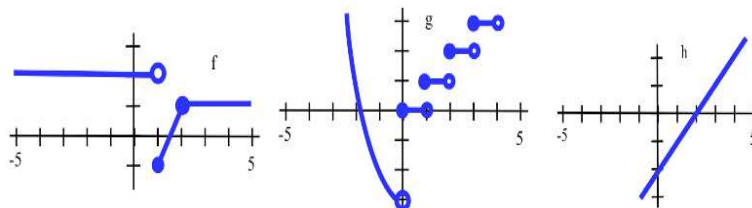
Fig. 0.4P7

11. $f(x) = \begin{cases} x^2 & \text{if } x < 2 \\ x - 1 & \text{if } x > 2 \end{cases}$

13. (a) $B(1) = 1{\cdot}f(1) = 1{\cdot}\dfrac{1}{1} = 1$,

$B(2) = 2{\cdot}f(2) = 2{\cdot}\dfrac{1}{2} = 1$,

$B(3) = 3{\cdot}f(3) = 3{\cdot}\dfrac{1}{3} = 1$.

(b) For $x > 0$, $B(x) = x{\cdot}f(x) = x{\cdot}\dfrac{1}{x} = 1$.
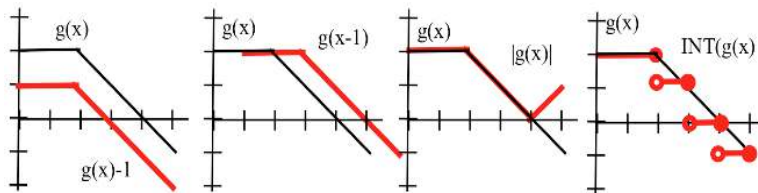
Fig. 0.4P15

15. See Fig. 0.4P15.

17. (a) f(g(x)) = 6x + 2 + 3A, g(f(x)) = g(3x+2) = 6x + 4 + A. If f(g(x)) = g(f(x)), then   A = 1.
    (b) f(g(x)) = 3Bx − 1, g(f(x)) = 3Bx + 2B − 1. If f(g(x)) = g(f(x)), then B = 0.

19. See Fig. 0.4P19 for the graph of f(x) = x − [x] = x − INT(x).

21. f(x) = [1.3 + 0.5·sin(x)] works. The value of 0.5 < A < 1.5 in
    f(x) = [A + 0.5·sin(x)] determines the relative lengths of the long
    and short parts of the pattern..
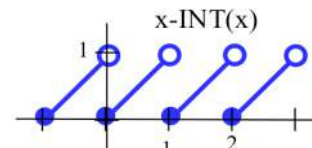
23. (a) g(1) = 1, g(2) = 1, g(3) = 0, g(4) = −1. Now graph g

25. ≈ 0.739 starting with x = 1, 2, 10, or any value.

Fig. 0.4P19

27. $f(x) = (x^2 + 1)/(2x)$. (note that this is the corrected version of the function f)
    $f(1) = 2/2 = 1$.
    $f(0.5) = 1.25$, $f(1.25) = 1.025$, $f(1.025) \approx 1.0003049$, $f(1.0003049) \approx 1.000000046$, ...
    $f(4) = 2.125$, $f(2.125) \approx 1.297794$, $f(1.297794) \approx 1.034166$, $f(1.034166) \approx 1.000564$, ...

29. (a) $f(2) = 14/3 \approx 4.7$, $f(14/3) = 50/9 \approx 5.6$, $f(50/9) = 158/27 \approx 5.85$, $f(158/27) = 482/81 \approx 5.95$
       $f(4) = 16/3 \approx 5.3$, $f(16/3) = 52/9 \approx 5.8$, $f(52/9) = 160/27 \approx 5.93$, $f(160/27) = 484/81 \approx 5.975$
       $f(6) = 6$.
    (b) $c = 6$.
    (c) Solve $c = g(c) = c/3 + A$ to get $3c = c + 3A$ and $2c = 3A$ so $c = \dfrac{3A}{2}$ is a fixed point of g.

31. On your own.

Section 1.5 Answers
1. (a) $x = 2, 4$ (b) $x = -2, -1, 0, 1, 2, 3, 4, 5$ (c) $x = -2, -1, 1, 3$

3. (a) all x (all real numbers)   (b) $x > \sqrt[3]{-2}$ (c) all x

5. (a) $x = -2, -3, 3$ (b) no values of x   (c) $x \geq 0$

7. (a) If $x \neq 2$ and $x \neq -3$, then $x^2 + x - 6 \neq 0$. True.
   (b) If an object does not have 3 sides, then it is not a triangle. True.

9. (a) If your car does not get at least 24 miles per gallon, then it is not tuned properly.
   (b) If you can not have dessert, then you did not eat your vegetables.

11. (a) If you will not vote for me, then you do not love your country.
    (b) If not only outlaws have guns, then guns are not outlawed. (poor English)
        If someone legally has a gun, then guns are not illegal.

13. (a) Both f(x) and g(x) are not positive.   (b) x is not positive. ($x \leq 0$)
    (c) 8 is not a prime number.

15. (a) For some numbers a and b, $|a+b| \neq |a| + |b|$.   (b) Some snake is not poisonous.
    (c) Some dog can climb trees.

17. If x is an integer, then 2x is an even integer. True.
    Converse: If 2x is an even integer, then x is an integer. True.
    (It is not likely that these were the statements you thought of. There are lots of other examples.)

19. (a) False. Put a = 3 and b = 4. Then $(a + b)^2 = (7)^2 = 49$, but $a^2 + b^2 = 3^2 + 4^2 = 9 + 16 = 25$.
    (b) False. Put a = -2 and b = -3. Then a > b, but $a^2 = 4 < 9 = b^2$.
    (c) True.

21. (a) True.   (b) False. Put f(x) = x + 1 and g(x) = x + 2. Then $f(x)g(x) = x^2 + 3x + 2$ is not a linear function.
    (c) True.

23. (a) If a and b are prime numbers, then a + b is prime. False: take a = 3 and b = 5.
    (b) If a and b are prime numbers, then a + b is not prime. False: take a = 2 and b = 3.
    (c) If x is a prime number, then x is odd. False: take x = 2. (this is the only counterexample)
    (d) If x is a prime number, then x is even. False: take x = 3 (or 5 or 7 or ...)

25. (a) If x is a solution of x + 5 = 9, then x is odd. False: take x = 4.
    (b) If a 3–sided polygon has equal sides, then it is a triangle. True. (We also have nonequilateral triangles .)
    (c) If a person is a calculus student, then that person studies hard. False (unfortunately), but we won't
        mention names.
    (d) If x is a (real number) solution of $x^2 - 5x + 6 = 0$, then x is even. False: take x = 3.

2.1   TANGENT LINES, VELOCITIES, GROWTH

In section 1.2, we estimated the slope of a line tangent to the graph of a function <u>at a point</u>. At the end of section
1.3, we constructed a new function which was the slope of the line tangent to the graph of a function
<u>at each point</u>. In both cases, before we could calculate a slope, we had to **estimate** the tangent line from the graph of
the function, a method which required an accurate graph and good estimating. In this section we will start to look at a
more precise method of finding the slope of a tangent line which does not require a graph or any estimation by us.
We will start with a nonapplied problem and then look at two applications of the same idea.

The Slope of a Line Tangent to a Function at a Point

Our goal is to find a way of exactly determining the slope of the line which is tangent to a function (to the graph of the
function) at a point in a way which does not require us to have the graph of the function.
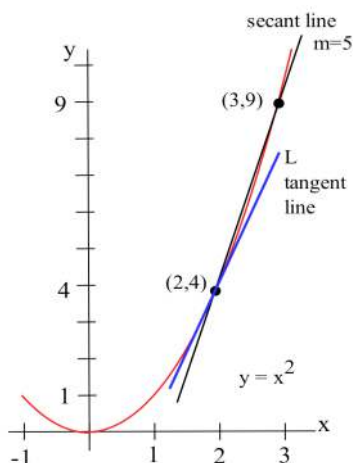
Let's start with the problem of finding the slope of the line L (Fig. 1) which is tangent
to $f(x) = x^2$ at the point (2,4). We could estimate the slope of L from the graph,
but we won't. Instead, we can see that the line through (2,4) and (3,9) on the graph
of f is an approximation of the slope of the tangent line, and we can calculate that
slope exactly: $m = \Delta y/\Delta x = (9-4)/(3-2) = 5$. But m = 5 is only an estimate of the
slope of the tangent line and not a very good
estimate. It's too big. We can get a better estimate
by picking a second point on the graph of f which
is closer to (2,4) — the point (2,4) is fixed and it
must be one of the points we use. From Fig. 2, we
can see that the slope of the line through the points
(2,4) and (2.5,6.25) is a better approximation of the slope of the tangent line at
(2,4): $m = \Delta y/\Delta x = (6.25 - 4)/(2.5 - 2) = 2.25/.5 = 4.5$, a better estimate, but still
an approximation. We can continue picking points closer and closer to (2,4) on
the graph of f, and then calculating the slopes of the lines through each of these
points and the point (2,4):

Fig. 1
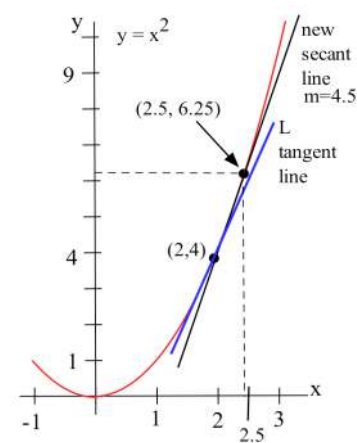
Fig. 2

Points to the left of (2,4)			Points to the right of (2,4)		
x	$y = x^2$	slope of line through (x,y) and (2,4)	x	$y = x^2$	slope of line through (x,y) and (2,4)
1.5	2.25	3.5	3	9	5
1.9	3.61	3.9	2.5	6.25	4.5
1.99	3.9601	3.99	2.01	4.0401	4.01

The only thing special about the x–values we picked is that they are numbers which are close, and very close, to
x = 2. Someone else might have picked other nearby values for x. As the points we pick get closer and closer to

the point (2,4) on the graph of $y = x^2$, the slopes of the lines through the points and (2,4) are better

approximations of the slope of the tangent line, and these slopes are getting closer and closer to 4.

Practice 1:   What is the slope of the line through (2,4) and (x, y) for $y = x^2$ and $x = 1.994$? $x = 2.0003$?

We can bypass much of the calculating by not picking the points one at a time: let's look at a general point near
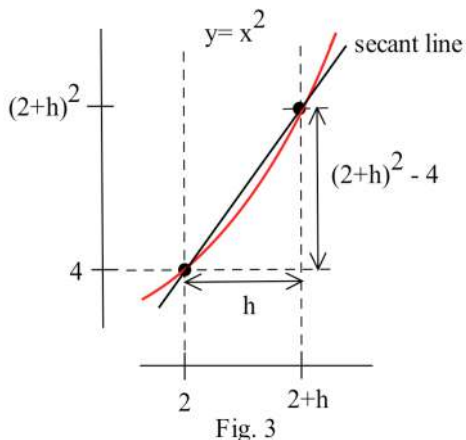
(2,4). Define $x = 2 + h$ so h is the increment from 2 to x

(Fig. 3). If h is small, then $x = 2 + h$ is close to 2 and the point

$(2+h, f(2+h)) = (2+h, (2+h)^2)$ is close to (2,4). The slope m

of the line through the points (2,4) and $(2+h, (2+h)^2)$ is a good

approximation of the slope of the tangent line at the point (2,4):

$$m = \frac{\Delta y}{\Delta x} = \frac{(2+h)^2 - 4}{(2+h) - 2}$$

$$= \frac{\{4 + 4h + h^2\} - 4}{h} = \frac{4h + h^2}{h} = \frac{h(4 + h)}{h} = 4 + h .$$

If h is very small, then $m = 4 + h$ is a very good approximation to the slope of the tangent line, and $m = 4 + h$

is very close to the value 4. The value $m = 4 + h$ is called the slope of the **secant line** through the two points

(2,4) and $(2+h, (2+h)^2)$. The limiting value 4 of $m = 4 + h$ as h gets smaller and smaller is called the

slope of the tangent line to the graph of f at (2,4).

Example 1:   Find the slope of the line tangent to $f(x) = x^2$ at the point (1,1) by evaluating the slope of the

secant line through (1,1) and $(1+h, f(1+h))$ and then determining what

happens as h gets very small (Fig. 4).
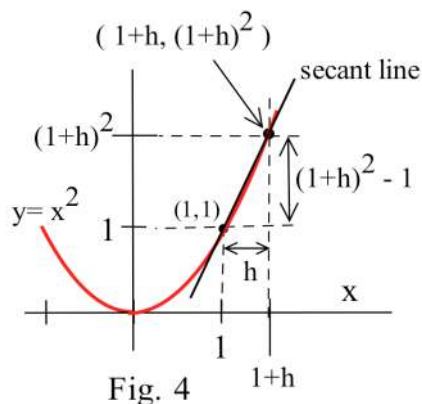
Solution: The slope of the secant line through the points (1,1) and

$(1+h, f(1+h))$ is

$$m = \frac{f(1+h) - 1}{(1+h) - 1} = \frac{(1+h)^2 - 1}{h} = \frac{\{1 + 2h + h^2\} - 1}{h}$$

$$= \frac{2h + h^2}{h} = 2 + h.$$ As h gets very small, the value of m approaches

the value 2, the slope of tangent line at the point (1,1).

Practice 2:   Find the slope of the line tangent to the graph of $y = f(x) = x^2$ at the point (−1,1) by finding the

slope of the secant line, m_{sec} , through the points (−1,1) and $(−1+h, f(−1+h))$ and then determining

what happens to m_{sec} as h gets very small.

FALLING TOMATO

Suppose we drop a tomato from
the top of a 100 foot building
(Fig. 5) and time its fall .

Fig. 5

time (s)	height (ft)
0	100
0.5	96
1.0	84
1.5	64
2.0	36
2.5	0
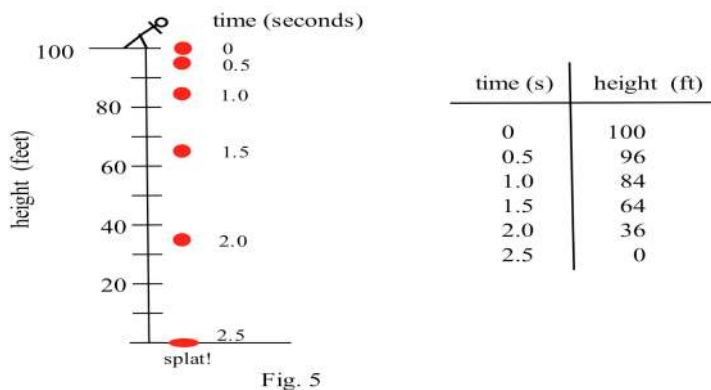
Some questions are easy to answer directly from the table:

   (a) How long did it take for the tomato n to drop 100 feet?   (2.5 seconds)

   (b) How far did the tomato fall during the first second?   (100 – 84 = 16 feet)

   (c) How far did the tomato fall during the last second?   (64 – 0 = 64 feet)

   (d) How far did the tomato fall between t =.5 and t = 1?   (96 – 84 = 12 feet)

Some other questions require a little calculation:

   (e) What was the average velocity of the tomato during its fall?

$$\textbf{Average velocity} \;=\; \frac{\textbf{distance fallen}}{\textbf{total time}} \;=\; \frac{\Delta \, \text{position}}{\Delta \, \text{time}} \;=\; \frac{-100 \text{ ft}}{2.5 \text{ s}} \;=\; -40 \text{ ft/s} \;.$$

   (f) What was the average velocity between t=1 and t=2 seconds?

$$\text{Average velocity} \;=\; \frac{\Delta \, \text{position}}{\Delta \, \text{time}} \;=\; \frac{36 \text{ ft} - 84 \text{ ft}}{2 \text{ s} - 1 \text{ s}} \;=\; \frac{-48 \text{ ft}}{1 \text{ s}} \;=\; -48 \text{ ft/s} \;.$$

Some questions are more difficult.

   (g) How fast was the tomato falling 1 second after it was dropped?

This question is significantly different from the previous two questions about average velocity. Here we want
the **instantaneous velocity,** the velocity at an instant in time. Unfortunately the tomato is not equipped with a
speedometer so we will have to give an approximate answer.

One crude approximation of the instantaneous velocity after 1 second is simply the average velocity during
the entire fall, –40 ft/s . But the tomato fell slowly at the beginning and rapidly near the end so the
"–40 ft/s" estimate may or may not be a good answer.

We can get a better approximation of the instantaneous velocity at t=1 by calculating the average velocities
over a short time interval near t = 1 . The average velocity between t = 0.5 and t = 1 is $\frac{-12 \text{ feet}}{0.5 \text{ s}} = -24$ ft/s,
and the average velocity between t = 1 and t = 1.5 is $\frac{-20 \text{ feet}}{.5 \text{ s}} = -40$ ft/s so we can be reasonably sure that
the instantaneous velocity is between –24 ft/s and –40 ft/s.

In general, the shorter the time interval over which we calculate the average velocity, the better the average velocity will approximate the instantaneous velocity. The average velocity over a time interval is $\dfrac{\Delta \text{ position}}{\Delta \text{ time}}$, which is the slope of the **secant line** through two points on the graph of height versus time (Fig. 6). The instantaneous velocity at a particular time and height is the slope of the **tangent line** to the graph at the point given by that time and height.
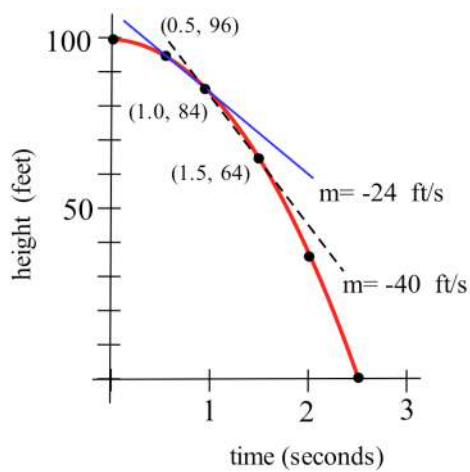
Fig. 6

Average velocity $= \dfrac{\Delta \text{ position}}{\Delta \text{ time}}$

$=$ **slope of the secant line through 2 points.**

Instantaneous velocity $=$ **slope of the line tangent to the graph.**

Practice 3: Estimate the velocity of the tomato 2 seconds after it was dropped.

GROWING BACTERIA

Suppose we set up a machine to count the number of bacteria growing on a petri plate (Fig. 7). At first there are few bacteria so the population grows slowly. Then there are more bacteria to divide so the population grows more quickly. Later, there are more bacteria and less room and nutrients available for the expanding population, so the population grows slowly again. Finally, the bacteria have used up most of the nutrients, and the population declines as bacteria die.
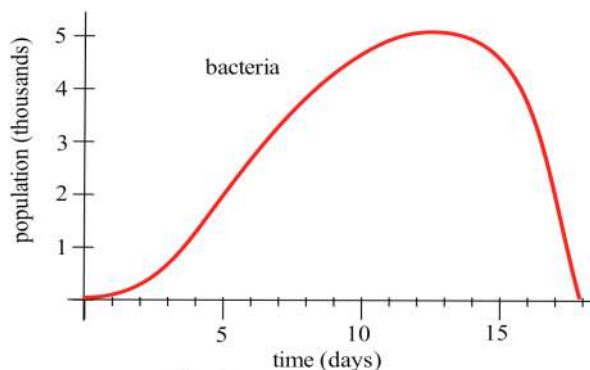
Fig. 7

The population graph can be used to answer a number of questions.

(a) What is the bacteria population at time $t = 3$ days?

(Answer: about 500 bacteria)

(b) What is the population increment from $t = 3$ to $t = 10$ days? (about 4000 bacteria)

(c) What is the **rate** of population growth from $t = 3$ to $t = 10$ days? (Fig. 7)

Solution: The rate of growth from $t = 3$ to $t = 10$ is the average change in population during that time:

average change in population $= \dfrac{\text{change in population}}{\text{change in time}} = \dfrac{\Delta \text{ population}}{\Delta \text{ time}}$

$= \dfrac{4000 \text{ bacteria}}{7 \text{ days}} \approx 570 \text{ bacteria/day}$.

This is the slope of the secant line through the two points $(3, 500)$ and $(10, 4500)$.

(d) What is the **rate** of population growth on the third day, at t = 3 ?
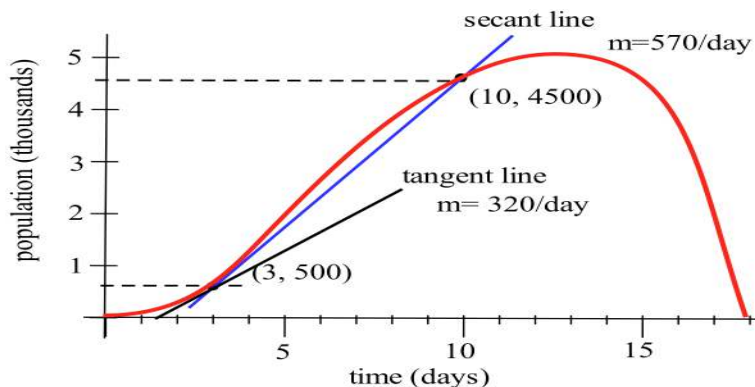
Fig. 8

Solution: This question is asking for the **instantaneous** rate of population change, the slope of the line

which is **tangent** to the population curve at (3, 500). If we sketch a line approximately tangent to the

curve at (3, 500) and pick two points near the ends of the tangent line segment (Fig. 8), we can estimate

that instantaneous rate of population growth is approximately   320 bacteria/day .

Average population growth rate   = $\dfrac{\Delta \text{ population}}{\Delta \text{ time}}$

                                                                      = **slope of the secant line through 2 points.**

Instantaneous population growth rate   =   **slope of the line tangent to the graph.**

Practice 4:   Approximately what was the average change in population between t = 9 and t = 13?

                          Approximately what was rate of population growth at t = 9 days?

The tangent line problem, the instantaneous velocity problem and the instantaneous growth rate problem are all

similar. In each problem we wanted to know how rapidly something was **changing at an instant in time**, and

each problem turned out to be finding the **slope of a tangent line**. The approach in each problem was also the

same: find an approximate solution and then examine what happened to the approximate solution over shorter

and shorter intervals. We will often use this approach of finding a limiting value, but before we can use it

effectively we need to describe the concept of a limit with more precision.

PROBLEMS

1.   What is the slope of the line through   (3,9) and (x, y) for y = x^2 and x = 2.97?   x = 3.001?

x = 3+h?   What happens to this last slope when h is very small (close to 0)?   Sketch the graph

of y = x^2 for x near 3.

2.   What is the slope of the line through   (–2,4) and (x, y) for y = x^2 and x = –1.98?   x = –2.03?

x = –2+h?   What happens to this last slope when h is very small (close to 0)?   Sketch the graph of

= x^2 for x near –2.

3. What is the slope of the line through   (2,4) and (x, y) for $y = x^2 + x - 2$ and x = 1.99?
   x = 2.004?   x = 2+h? What happens to this last slope when h is very small?   Sketch the graph of
   $y = x^2 + x - 2$ for x near 2.

4. What is the slope of the line through   (–1,–2) and (x, y) for $y = x^2 + x - 2$ and x = –.98?
   x = –1.03?   x = –1+h? What happens to this last slope when h is very small?   Sketch the graph of
   $y = x^2 + x - 2$ for x near –1.
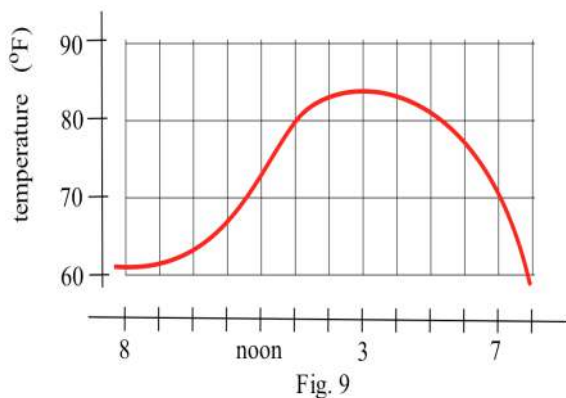
Fig. 9

5. Fig. 9 shows the temperature during a day in Ames.

   (a)   What was the average change in temperature from 9 am
         to 1 pm?

   (b)   Estimate how fast the temperature was rising **at** 10 am
         and **at** 7 pm?

time (seconds)
Fig. 10
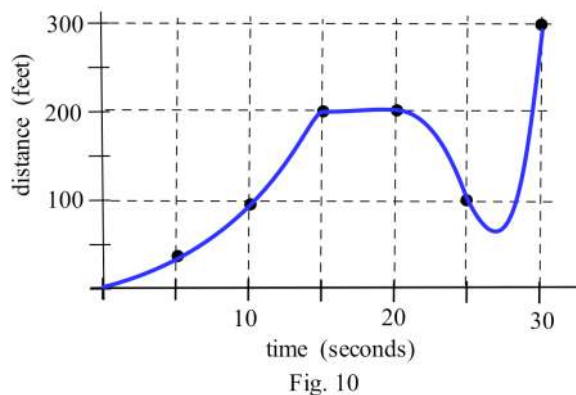
6. Fig. 10 shows the distance of a car from a measuring position
   located on the edge of a straight road.

   (a)   What was the average velocity of the car from t = 0
         to t = 30 seconds?

   (b)   What was the average velocity of the car from
         t = 10 to t = 30 seconds?

   (c)   About how fast was the car traveling **at** t = 10
         seconds? **at** t = 20 s ? **at** t = 30 s ?

   (d)   What does the horizontal part of the graph between
         t = 15 and t = 20 seconds mean?

   (e)   What does the negative velocity at t = 25 represent?

time (seconds)
Fig. 11
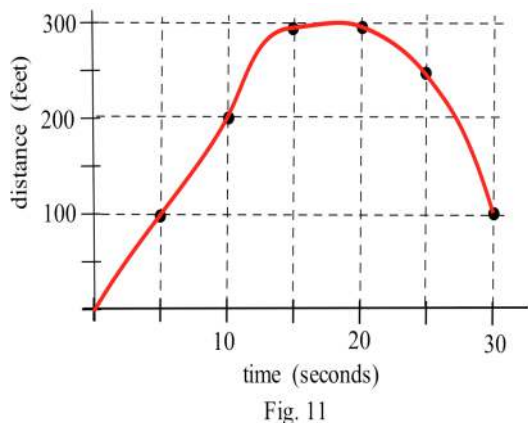
7. Fig. 11 shows the distance of a car from a measuring
   position located on the edge of a straight road.

   (a)   What was the average velocity of the car from
         t = 0 to t = 20 seconds?

   (b)   What was the average velocity from t = 10 to
         t = 30 seconds?

   (c)   About how fast was the car traveling **at** t = 10
         seconds? **at** t = 20 s ? **at** t = 30 s ?

8.   Fig. 12 shows the composite developmental skill level of chessmasters at different ages as determined by their performance against other chessmasters. (From "Rating Systems for Human Abilities", by W.H. Batchelder and R.S. Simpson, 1988. UMAP Module 698.)
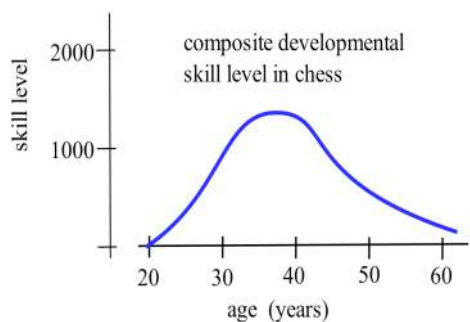
Fig. 12

(a)   At what age is the "typical" chessmaster playing the best chess?

(b)   At approximately what age is the chessmaster's skill level increasing most rapidly?

(c)   Describe the development of the "typical" chessmaster's skill in words.

(d)   Sketch graphs which you **think** would reasonably describe the performance levels versus age for an athlete, a classical pianist, a rock singer, a mathematician, and a professional in your major field.

Problems 9 and 10 define new functions A(x) in terms of AREAS bounded by the functions y = 3 and y = x + 1. This may seem a strange way to define a functions A(x), but this idea will become important later in calculus. We are just getting an early start here.
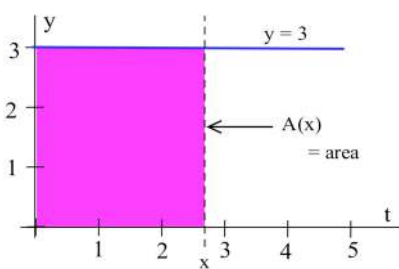
Fig. 13

9.   Define A(x) to be the **area** bounded by the x and y axes, the horizontal line y = 3, and the vertical line at x (Fig. 13). For example, A(4) = 12 is the area of the 4 by 3 rectangle.

a)   Evaluate A(0), A(1), A(2), A(2.5) and A(3).

b)   What area would A(4) – A(1) represent in the figure?
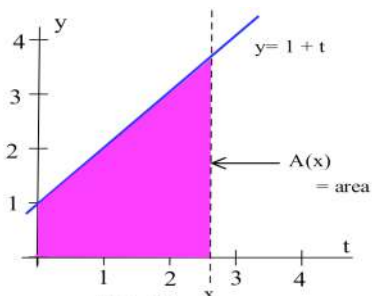
c)   Graph y = A(x) for 0 ≤ x ≤ 4.

Fig. 14

10. Define A(x) to be the **area** bounded by the x and y axes, the line y = x + 1, and the vertical line at x (Fig. 14). For example, A(4) = 12.

a)   Evaluate A(0), A(1), A(2), A(2.5) and A(3).

b)   What area would A(3) – A(1) represent in the figure?

c)   Graph y = A(x) for 0 ≤ x ≤ 4.

Section 2.1   **PRACTICE Answers**

Practice 1: $y = x^2$

If x = 1.994, then y = 3.976036   so the slope between (2, 4) and (x ,y) is

$$\frac{4 - y}{2 - x} \; = \; \frac{4 - 3.976036}{2 - 1.994} \; = \; \frac{0.023964}{0.006} \; \approx \; 3.994 \; .$$

If x = 2.0003, then y ≈ 4.0012   so the slope between (2, 4) and (x ,y) is

$$\frac{4 - y}{2 - x} \; = \; \frac{4 - 4.0012}{2 - 2.0003} \; = \; \frac{-0.0012}{0.0003} \; \approx \; 4.0003 \; .$$

Practice 2: $m_{sec} = \frac{f(-1 + h) - (1)}{(-1 + h) - (-1)} \; = \; \frac{(-1 + h)^2 - 1}{h} \; = \; \frac{1 - 2h + h^2 - 1}{h} \; = \; \frac{h(-2 + h)}{h} \; = \; \mathbf{-2 + h}$

As h → 0, m_{sec} = –2 + h → **–2** .

Practice 3:   The average velocity between t = 1.5 and t = 2.0 is $\frac{36 - 64 \text{ feet}}{2.0 - 1.5 \text{ sec}}$ = –56 feet per second.

The average velocity between t = 2.0 and t = 2.5 is $\frac{0 - 36 \text{ feet}}{2.5 - 2.0 \text{ sec}}$ = –72 feet per second.

The velocity **at** t = 2.0 is somewhere between –56 ft/sec and –72 ft/sec, probably about the

middle of this interval: $\frac{(-56) + (-72)}{2}$ = **–64 ft/sec**.

Practice 4:   (a)   When t = 9 days, the population is approximately P = 4,200 bacteria. When t = 13,

P ≈ 5,000. The average change in population is approximately

$$\frac{5000 - 4200 \text{ bacteria}}{13 - 9 \text{ days}} \; = \; \frac{800 \text{ bacteria}}{4 \text{ days}} \; = \; \textbf{200 bacteria per day}.$$
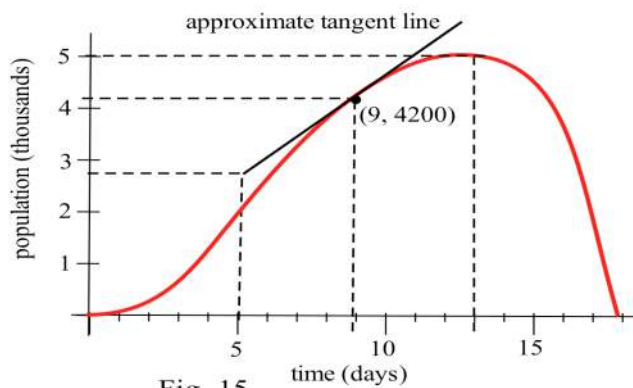
Fig. 15

(b) To find the rate of population growth at t = 9 days,
     sketch the line tangent to the population curve
     at the point (9, 4200) and then use (9, 4200) and
     another point on the tangent line to calculate the
     slope of the line.   Using the approximate values
     (5, 2800) and (9, 4200), the slope of the tangent
     line at the point (9, 4200) is approximately

$$\frac{4200 - 2800 \text{ bacteria}}{9 - 5 \text{ days}} \; = \; \frac{1400 \text{ bacteria}}{4 \text{ days}} \; \approx \; \textbf{350 bacteria/day} \; .$$

2.2   THE LIMIT OF A FUNCTION

THE IDEA, Informally

Calculus has been called the study of continuous change, and the **limit** is the basic concept which allows us to describe and analyze such change. An understanding of limits is necessary to understand derivatives, integrals and other fundamental topics of calculus.
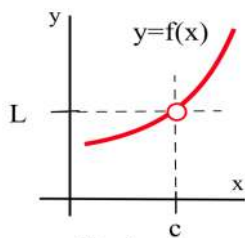
The limit of a function describes the behavior of the function when the variable is near, **but does not equal** , a specified number (Fig. 1). If the values of f(x) get closer and closer , as close as we want, to one number L as we take values of x very close to (but not equal to) a number c, then we

Fig. 1

say **"the limit of f(x), as x approaches c, is L "**   and we

write " $\lim\limits_{x \to c}$ **f(x) = L."**   (The symbol " → " means "approaches" or "gets very close to.")

f(c)   is a single number that describes the behavior (value) of f **AT** the point x = c.

$\lim\limits_{x \to c}$ **f(x)**   is a single number that describes the behavior of f **NEAR, <u>BUT NOT AT,</u>** the point x = c.

If we have a graph of the function near x = c , then it is usually easy to determine $\lim\limits_{x \to c}$ f(x) .

y=f(x)
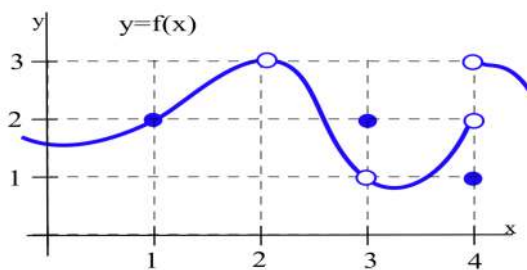
Fig. 2

Example 1:   Use the graph of y = f(x) in Fig. 2 to determine the following limits:

(a) $\lim\limits_{x \to 1}$ f(x)   (b) $\lim\limits_{x \to 2}$ f(x)

(c) $\lim\limits_{x \to 3}$ f(x)   (d) $\lim\limits_{x \to 4}$ f(x)

Solution: (a) $\lim\limits_{x \to 1}$ f(x) = 2 . When x is very close to 1, the values of f(x) are very close to

y = 2. In this example, it happens that f(1) = 2, but that is <u>irrelevant</u> for the limit. The only thing that matters is what happens for x close to 1 but x ≠ 1.

(b) f(2) is undefined, but we only care about the behavior of f(x) for x <u>close to</u> 2 and <u>not equal to</u> 2. When x is close to 2, the values of f(x) are close to 3. If we restrict x close enough to 2, the

values of y will be as close to 3 as we want, so $\lim\limits_{x \to 2}$ f(x) = 3.

(c) When x is close to 3 (or as x approaches the value 3), the values of f(x) are close to 1 (or

approach the value 1), so $\lim\limits_{x \to 3} f(x) = 1$. For this limit it is completely irrelevant that f(3) = 2, We only
care about what happens to f(x) for x close to and not equal to 3.

(d) This one is harder and we need to be careful. When x is close to 4 and slightly **less than** 4 (x is just to

the left of 4 on the x–axis), then the values of f(x) are close to 2. But if x is close to 4 and slightly

larger than 4 then the values of f(x) are close to 3.   If we only know that x is very close to 4, then we

cannot say whether y = f(x) will be close to 2 or close to 3 — it depends on whether x is on the right or

the left side of 4. In this situation, the f(x) values are not close to a <u>single</u> number so we say

$\lim\limits_{x \to 4} f(x)$ **does not exist**. It is irrelevant that f(4) = 1. The limit, as x approaches 4, would still
be undefined if f(4) was 3 or 2 or anything else.
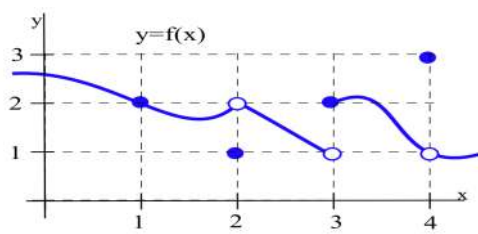
Fig. 3

Practice 1:   Use the graph of y = f(x) in Fig. 3 to determine

the following limits:

(a) $\lim\limits_{x \to 1} f(x)$ (b) $\lim\limits_{t \to 2} f(t)$

(c) $\lim\limits_{x \to 3} f(x)$ (d) $\lim\limits_{w \to 4} f(w)$

Example 2: Determine the value of $\lim\limits_{x \to 3} \dfrac{2x^2 - x - 1}{x - 1}$.

Solution: We need to investigate the values of f(x) when x is close to 3. If the f(x) values get

arbitrarily close to or even equal some number L, then L will be the limit. One way to keep track of both

the x and the f(x) values is to set up a table and to pick several x values which are closer and closer (but

not equal) to 3. We can pick some values of x which approach 3 from the left, say   x = 2.91, 2.9997,

2.999993, and 2.9999999 , and some values of x which approach 3 from the right, say x = 3.1, 3.004,

3.0001, and 3.000002 .   The only thing important about these particular values for x is that they get closer

and closer to 3 without equaling 3. You should try some other values "close to 3" to see what happens.

Our table of values is

x	f(x)	x	f(x)
2.9	6.82	3.1	7.2
2.9997	6.9994	3.004	7.008
2.999993	6.999986	3.0001	7.0002
2.9999999	6.9999998	3.000002	7.000004
↓	↓	↓	↓
3	**7**	**3**	**7**

As the x values get closer and closer to 3, the f(x) values are getting closer and closer to 7. In

fact, we can get f(x) as close to 7 as we want ("arbitrarily close") by taking the values of x very

close ("sufficiently close") to 3. $\lim\limits_{x \to 3} \dfrac{2x^2 - x - 1}{x - 1} = 7$.

Instead of using a table of values , we could have graphed y = f(x) for x close to 3 , Fig. 4, and used

the graph to answer the limit question. This graphic approach is easier, particularly if you have a

calculator or computer do the graphing work for you, but it is really very similar to the "table of values"

method: in each case you need to evaluate y = f(x) at many values of x near 3.

You might have noticed that if we just evaluate f(3), then we get the correct answer 7. That works for this

particular problem, but it often fails.   The next example illustrates the difficulty.
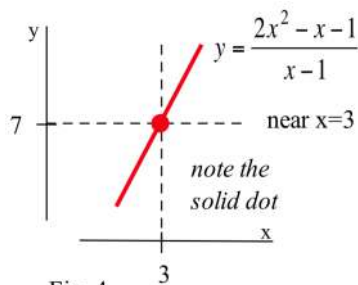
Example 3:   Find $\lim\limits_{x \to 1} \dfrac{2x^2 - x - 1}{x - 1}$. (Same as Example 2 but with x→1.)

Solution: You might try to evaluate $f(x) = \dfrac{2x^2 - x - 1}{x - 1}$ at x = 1,

but f is not defined at x = 1. It is tempting, **but wrong**, to

conclude that this function does not have a limit as x approaches 1.

Fig. 4

$y = \dfrac{2x^2 - x - 1}{x - 1}$

near x=3

note the solid dot

Table Method: Trying some "test" values for x which get closer and closer to 1 from both the left

and the right, we get

x	f(x)		x	f(x)
0.9	2.82		1.1	3.2
0.9998	2.9996		1.003	3.006
0.999994	2.999988		1.0001	3.0002
0.9999999	2.9999998		1.000007	3.000014
↓	↓		↓	↓
1	**3**		1	**3**

The function f is not defined at x = 1, but when x is **close to** 1, the values of f(x) are getting very close to 3.

We can get f(x) as close to 3 as we want by taking x very close to 1 so $\lim\limits_{x \to 1} \dfrac{2x^2 - x - 1}{x - 1} = 3$.

Graph Method: We can graph $y = f(x) = \dfrac{2x^2 - x - 1}{x - 1}$ for x close to 1, Fig. 5, and notice that

whenever x is close to 1, the values of y = f(x) are close to 3. f is not defined at x = 1, so the graph has

a hole above x = 1, but we only care about what f(x) is doing for x close to but **not equal to** 1.

Fig. 5

$y = \dfrac{2x^2 - x - 1}{x - 1}$

near x=1

note the open dot
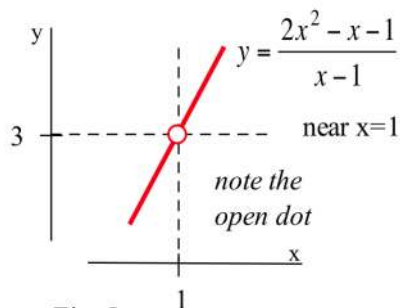
Algebra Method: We could have found the same result by noting

that $f(x) = \dfrac{2x^2 - x - 1}{x - 1} = \dfrac{(2x+1)(x-1)}{x-1} = 2x+1$ as long as

x ≠ 1. (If x≠1, then x–1 ≠ 0 so it is valid to divide the

numerator and denominator by the factor x–1.) The "x→1"

part of the limit means that x is close to 1 but **not equal to 1**,

so our division step is valid and

$$\lim\limits_{x \to 1} \dfrac{2x^2 - x - 1}{x - 1} = \lim\limits_{x \to 1} 2x + 1 = 3 \text{ , the correct answer.}$$

THREE METHODS FOR EVALUATING LIMITS

The Algebra Method

The algebra method involves algebraically simplifying the function before trying to evaluate its limit. Often, this simplification just means factoring and dividing, but sometimes more complicated algebraic or even trigonometric steps are needed.

The Table and Graph Methods

To evaluate a limit of a function f(x) as x approaches c, the table method involves calculating the values of f(x) for "enough" values of x very close to c so that we can "confidently" determine which value f(x) is approaching. If f(x) is well–behaved, we may not need to use very many values for x. However, this method is usually used with complicated functions, and then we need to evaluate f(x) for lots of values of x.

A computer or calculator can often make the function evaluations easier, but their calculations are subject to "round off" errors. The result of any computer calculation which involves both large and small numbers should be viewed with some suspicion.   For example, the function

$$f(x) = \frac{\{ (0.1)^x + 1 \} - 1}{(0.1)^x} = \frac{(0.1)^x}{(0.1)^x} = 1 \text{ for every value of x , and my calculator gives the correct}$$

answer for some values of x : $f(3) = \dfrac{\{ (0.1)^3 + 1 \} - 1}{(0.1)^3} = 1$, and f(8) and f(9) both equal 1.

But my calculator says $\{ (0.1)^{10} + 1 \} - 1 = 0$ so it evaluates f(10) to be 0, definitely an **incorrect** value. Your calculator may evaluate f(10) correctly, but try f(35) or f(107).

Calculators are too handy to be ignored, but they are too prone

to these types of errors to be believed underlined uncritically. Be careful.

The graph method is closely related to the table method, but we create a graph of the function instead of a table of values, and then we use the graph to determine which value f(x) is approaching.

Which Method Should You Use?

In general, the algebraic method is preferred because it is precise and does not depend on which values of x we chose or the accuracy of our graph or precision of our calculator. **If you can evaluate a limit algebraically, you should do so.** Sometimes, however, it will be very difficult to evaluate a limit algebraically, and the table or graph methods offer worthwhile alternatives. Even when you can algebraically evaluate the limit of a function, it is still a good idea to graph the function or evaluate it at a few points just to verify your algebraic answer.

The table and graph methods have the same advantages and disadvantages. Both can be used on very complicated functions which are difficult to handle algebraically or whose algebraic properties you don't know.

Often both methods can be easily programmed on a calculator or computer. However, these two methods are very time–consuming by hand and are prone to round off errors on computers. You need to know how to use these methods when you can't figure out how to use the algebraic method, but you need to use these two methods warily.

Example 4:   Evaluate   (a) $\displaystyle\lim_{x\to 0}\frac{x^2+5x+6}{x^2+3x+2}$ and   (b) $\displaystyle\lim_{x\to -2}\frac{x^2+5x+6}{x^2+3x+2}$.

Solution: The function in each limit is the same but x is approaching a different number in each of them.

(a) Since x→0, we know that x is getting closer and closer to 0 so the values of the x^2 , 5x and 3x terms get as close to 0 as we want. The numerator approaches 6 and the denominator approaches 2, so the values of the whole function get arbitrarily close to 6/2 = 3, the limit.

(b) As x approaches –2, the numerator and denominator approach 0, and a small number divided by a small number can be almost anything — the ratio depends on the size of the top compared to the bottom. <u>More investigation is needed</u>.

<u>Table Method</u>: If we pick some values of x close to (but not equal to) –2, we get the table

x	x^2+5x+6	x^2+3x+2	$\dfrac{x^2+5x+6}{x^2+3x+2}$
−1.97	0.0309	−0.0291	−1.061856
−2.005	−0.004975	0.005025	−0.990050
−1.9998	0.00020004	−0.00019996	−1.00040008
−2.00003	−0.00002999	0.0000300009	−0.9996666
↓	↓	↓	↓
−2	0	0	**−1**

Even though the numerator and denominator are each getting closer and closer to 0, their ratio is getting arbitrarily close to −1 which is the limit.

$y = \dfrac{x^2+5x+6}{x^2+3x+2}$
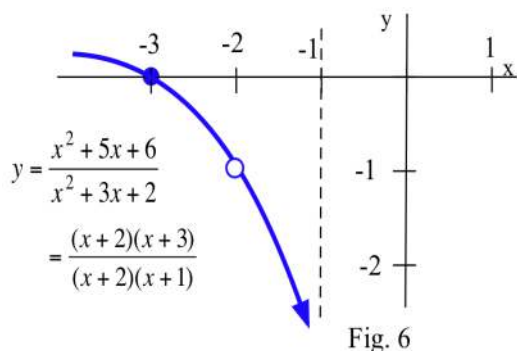
$= \dfrac{(x+2)(x+3)}{(x+2)(x+1)}$

Fig. 6

<u>Graph Method</u>: The graph of y = f(x) = $\dfrac{x^2+5x+6}{x^2+3x+2}$ in Fig. 6 shows that the values of f(x) are very close to −1 when the x–values are close to −2.

<u>Algebra Method</u>: f(x) = $\dfrac{x^2+5x+6}{x^2+3x+2}$ = $\dfrac{(x+2)(x+3)}{(x+2)(x+1)}$.

We know x → −2 so x ≠ −2, and we can divide the top and bottom by (x+2).   Then f(x) = (x+3)/(x+1) so f(x) → 1/−1 = −1 as x → −2.

> If $\displaystyle\lim_{x\to c}\left\{\dfrac{\textbf{polynomial}}{\textbf{polynomial}}\right\}$ **approaches** $\dfrac{\textbf{0}}{\textbf{0}}$ **, try dividing the top and bottom by x − c.**

Practice 2:   Evaluate   (a) $\lim\limits_{x\to 2} \dfrac{x^2 - x - 2}{x - 2}$ (b) $\lim\limits_{t\to 0} \dfrac{t \cdot \sin(t)}{t^2 + 3t}$ (c) $\lim\limits_{w\to 2} \dfrac{w - 2}{\ln(w/2)}$.

ONE–SIDED LIMITS

Fig. 7

Sometimes, what happens to us at a place depends on the direction we use to approach that place. If we approach Niagara Falls from the upstream side, then we will be 182 feet higher and have different worries than if we approach from the downstream side. Similarly, the values of a function near a point may depend on the direction we use to approach that point. If we let x approach 3 from the left (x is close to 3 and x < 3), then the values of [x] = INT(x) equal 2 (Fig. 7). If we let x approach 3 from the right (x is close to 3 and x > 3), then the values of [x] = INT(x) equal 3.
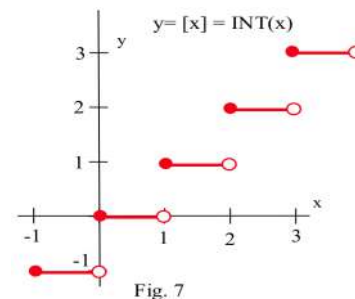
On the number line we can approach a point from the left or right, and that leads to **one–sided limits**.

Definition of Left and Right Limits:

The **left limit** as x approaches c of f(x) is L if the values of f(x) get as close to L as we

want when x is very close to and **left of c**, x < c: $\lim\limits_{x\to c^-}$ f(x) = L .

The **right limit**, written with $x \to c^+$, requires that x lie to the **right of c**, x > c.

Example 5:   Evaluate $\lim\limits_{x\to 2^-} (x - \lfloor x \rfloor)$ and $\lim\limits_{x\to 2^+} (x - \lfloor x \rfloor)$.
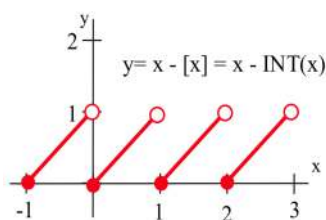
Fig. 8

Solution: The left–limit notation x → 2⁻ requires that x be close to 2 and that

x be to the left of 2, so x < 2.

If 1 < x < 2, then [x] = 1 so $\lim\limits_{x\to 2^-} (x - \lfloor x \rfloor) = 2 - 1 = 1.$

If x is close to 2 and is to the right of 2, then 2 < x < 3 so [x] = 2

and $\lim\limits_{x\to 2^+} (x - \lfloor x \rfloor) = 2 - 2 = 0.$

The graph of f(x) = x – [x] is shown in Fig. 8 .

If the left and right limits have the same value, $\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x) = L$, then the value of f(x) is close to

L whenever x is close to c, and it does not matter if x is left or right of c so $\lim\limits_{x\to c} f(x) = L$. Similarly, if

$\lim\limits_{x\to c} f(x) = L$, then f(x) is close to L whenever x is close to c and less than c and whenever x is close to c

and greater than c, so $\lim\limits_{x\to c^-} f(x) = \lim\limits_{x\to c^+} f(x) = L$. We can combine these two statements into a single

theorem.

Corollary: If $\lim_{x\to c^-} f(x) \;\neq\; \lim_{x\to c^+} f(x)$, then $\lim_{x\to c} f(x)$ does not exist.

One–sided limits are particularly useful for describing the behavior of functions which have steps or jumps.

To determine the limit of a function involving the greatest integer or absolute value or a multiline definition, definitely consider both the left and right limits.
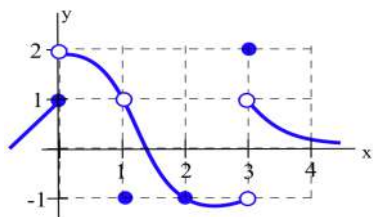
Fig. 9

Practice 3:   Use the graph in Fig. 9 to evaluate the one and two–sided limits of f at $x = 0, 1, 2,$ and 3.

Practice 4:   Let f(x) = $\begin{cases} 1 & \text{if } x < 1 \\ x & \text{if } 1 < x < 3 \\ 2 & \text{if } 3 < x \end{cases}$.

Find the one and two–sided limits of f at 1 and 3.

PROBLEMS

1.   Use the graph in Fig. 10 to determine the following limits.

   (a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 2} f(x)$
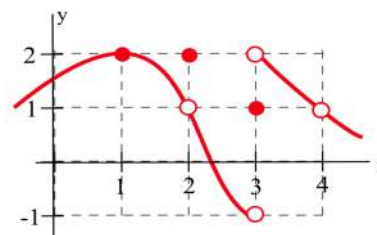
   (c) $\lim_{x\to 3} f(x)$ (d) $\lim_{x\to 4} f(x)$

Fig. 10
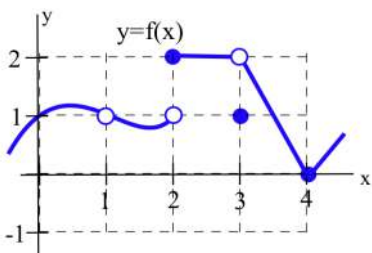
Fig. 11

2.   Use the graph in Fig. 11 to determine the following limits.

   (a) $\lim_{x\to 1} f(x)$ (b) $\lim_{x\to 2} f(x)$

   (c) $\lim_{x\to 3} f(x)$ (d) $\lim_{x\to 4} f(x)$

3. Use the graph in Fig. 12 to determine the following limits.

   (a) $\lim\limits_{x\to1} f(2x)$ (b) $\lim\limits_{x\to2} f(x-1)$

   (c) $\lim\limits_{x\to3} f(2x-5)$ (d) $\lim\limits_{x\to0} f(4+x)$
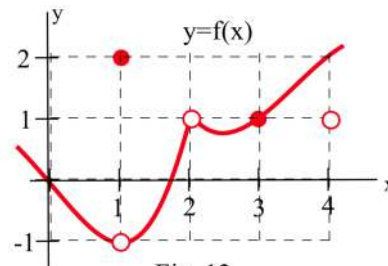
Fig. 12
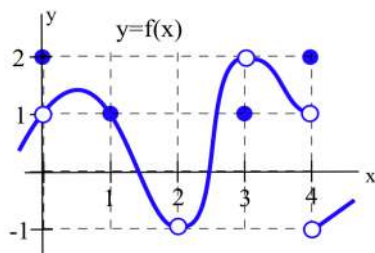
4. Use the graph in Fig. 13 to determine the following limits.

   (a) $\lim\limits_{x\to1} f(3x)$ (b) $\lim\limits_{x\to2} f(x+1)$

   (c) $\lim\limits_{x\to3} f(2x-4)$ (d) $\lim\limits_{x\to0} |f(4+x)|$

Fig. 13

5. Evaluate   (a) $\lim\limits_{x\to1} \dfrac{x^2+3x+3}{x-2}$ (b) $\lim\limits_{x\to2} \dfrac{x^2+3x+3}{x-2}$

6. Evaluate   (a) $\lim\limits_{x\to0} \dfrac{x+7}{x^2+9x+14}$ (b) $\lim\limits_{x\to3} \dfrac{x+7}{x^2+9x+14}$

    (c) $\lim\limits_{x\to-4} \dfrac{x+7}{x^2+9x+14}$ (d) $\lim\limits_{x\to-7} \dfrac{x+7}{x^2+9x+14}$

7. Evaluate   (a) $\lim\limits_{x\to1} \dfrac{\cos(x)}{x}$ (b) $\lim\limits_{x\to\pi} \dfrac{\cos(x)}{x}$ (c) $\lim\limits_{x\to-1} \dfrac{\cos(x)}{x}$

8. Evaluate   (a) $\lim\limits_{x\to7} \sqrt{x-3}$ (b) $\lim\limits_{x\to9} \sqrt{x}-3$ (c) $\lim\limits_{x\to9} \dfrac{\sqrt{x}-3}{x-9}$

9. Evaluate   (a) $\lim\limits_{x\to0^-} |x|$ (b) $\lim\limits_{x\to0^+} |x|$ (c) $\lim\limits_{x\to0} |x|$

10. Evaluate   (a) $\lim\limits_{x\to0^-} \dfrac{|x|}{x}$ (b) $\lim\limits_{x\to0^+} \dfrac{|x|}{x}$ (c) $\lim\limits_{x\to0} \dfrac{|x|}{x}$

11. Evaluate   (a) $\lim\limits_{x\to5} |x-5|$ (b) $\lim\limits_{x\to3} \dfrac{|x-5|}{x-5}$ (c) $\lim\limits_{x\to5} \dfrac{|x-5|}{x-5}$

12. $f(x) = \begin{cases} x & \text{if } x<0 \\ \sin(x) & \text{if } 0<x\le2 \\ 1 & \text{if } 2<x \end{cases}$. Find the one and two–sided limits of f as $x\to0,1,$ and 2.

13. $g(x) = \begin{cases} 1 & \text{if } x\le2 \\ 8/x & \text{if } 2<x<4 \\ 6-x & \text{if } 4<x \end{cases}$. Find the one and two–sided limits of g as $x\to1,2,4,$ and 5.

In problems 14 – 17 use a calculator or computer to get approximate answers accurate to 2 decimal places.

14. (a) $\displaystyle\lim_{x\to 0}\ \frac{2^x-1}{x}$ (b) $\displaystyle\lim_{x\to 1}\ \frac{\log_{10}(x)}{x-1}$ 15. (a) $\displaystyle\lim_{x\to 0}\ \frac{3^x-1}{x}$ (b) $\displaystyle\lim_{x\to 1}\ \frac{\ln(x)}{x-1}$

16. (a) $\displaystyle\lim_{x\to 5}\ \frac{\sqrt{x-1}-2}{x-5}$ (b) $\displaystyle\lim_{x\to 0}\ \frac{\sin(3x)}{5x}$ 17. (a) $\displaystyle\lim_{x\to 16}\ \frac{\sqrt{x}-4}{x-16}$ (b) $\displaystyle\lim_{x\to 0}\ \frac{\sin(7x)}{2x}$

18. Define A(x) to be the **area** bounded by the x and y axes, the
bent line in Fig. 14, and the vertical line at x. For example,
A(4) = 10.

   a)   Evaluate A(0), A(1), A(2), and A(3).

   b)   Graph y = A(x) for 0 ≤ x ≤ 4.

   c)   What area does A(3) – A(1) represent?
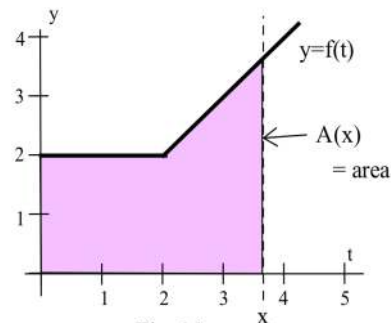
Fig. 14

19. Define A(x) to be the **area** bounded by the x and y axes, the
line y = $\frac{1}{2}$ x + 2 , and the vertical line at x. (Fig. 15).

   For example, A(4) = 12.

   a)   Evaluate A(0), A(1), A(2), and A(3).

   b)   Graph y = A(x) for 0 ≤ x ≤ 4.

   c)   What area does A(3) – A(1) represent?
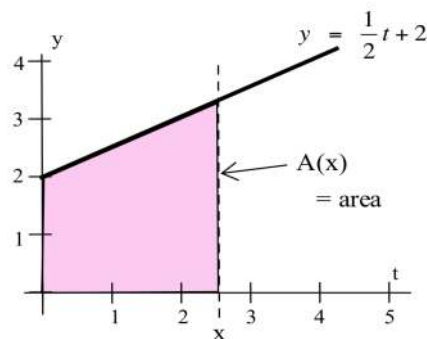
Fig. 15

20. Sketch $f(x)=\sqrt{4x-x^2}$ for $0\le x\le 4$ (this is a semicircle) .

   Define A(x) to be the **area** bounded by the x and y axes, the graph y = f(x) , and the vertical line at x.

   a)   Evaluate A(0), A(2), and A(4).

   b)   Graph y = A(x) for 0 ≤ x ≤ 4.

   c)   What area does A(3) – A(1) represent?

Section 2.2   PRACTICE Answers

Practice 1:   (a) **2**   (b) **2**   (c) **does not exist** (no limit)   (d) **1**

Practice 2:   (a) $\lim\limits_{x \to 2} \dfrac{(x+1)(x-2)}{x-2} \;=\; \lim\limits_{x \to 2} (x+1) \;=\; 3$

(b) $\lim\limits_{t \to 0} \dfrac{t \cdot \sin(t)}{t(t+3)} \;=\; \lim\limits_{t \to 0} \dfrac{\sin(t)}{t+3} \;=\; \dfrac{0}{3} \;=\; 0$

(c) $\lim\limits_{w \to 2} \dfrac{w-2}{\ln(w/2)} \;=\; 2.$ Try this one numerically or using a graph.

w	$\dfrac{w-2}{\ln(w/2)}$	w	$\dfrac{w-2}{\ln(w/2)}$
2.2	2.098411737	1.9	1.949572575
2.01	2.004995844	1.99	1.994995823
2.003	2.001499625	1.9992	1.999599973
2.0001	2.00005	1.9999	1.99995

Practice 3: $\lim\limits_{x \to 0^-} f(x) = \mathbf{1}$ $\lim\limits_{x \to 0^+} f(x) = \mathbf{2}$ $\lim\limits_{x \to 0} f(x)$ **does not exist**

$\lim\limits_{x \to 1^-} f(x) = \mathbf{1}$ $\lim\limits_{x \to 1^+} f(x) = \mathbf{1}$ $\lim\limits_{x \to 1} f(x)$ **1**

$\lim\limits_{x \to 2^-} f(x) = \mathbf{-1}$ $\lim\limits_{x \to 2^+} f(x) = \mathbf{-1}$ $\lim\limits_{x \to 2} f(x) = \mathbf{-1}$

$\lim\limits_{x \to 3^-} f(x) = \mathbf{-1}$ $\lim\limits_{x \to 3^+} f(x) = \mathbf{1}$ $\lim\limits_{x \to 3} f(x)$ **does not exist**

Practice 4: $\lim\limits_{x \to 1^-} f(x) = \mathbf{1}$ $\lim\limits_{x \to 1^+} f(x) = \mathbf{1}$ $\lim\limits_{x \to 1} f(x) = \mathbf{1}$

$\lim\limits_{x \to 3^-} f(x) = \mathbf{3}$ $\lim\limits_{x \to 3^+} f(x) = \mathbf{2}$ $\lim\limits_{x \to 3} f(x)$ **does not exist**

2.3 PROPERTIES OF LIMITS

This section presents results which make it easier to calculate limits of combinations of functions or to show that a limit does not exist. The main result says we can determine the limit of "elementary combinations" of functions by calculating the limit of each function separately and recombining these results for our final answer.

Main Limit Theorem:

If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$,

then   (a) $\lim\limits_{x \to a} \{f(x) + g(x)\}$ = $\lim\limits_{x \to a} f(x) + \lim\limits_{x \to a} g(x)$ = $L + M$

(b) $\lim\limits_{x \to a} \{f(x) - g(x)\}$ = $\lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x)$ = $L - M$

(c) $\lim\limits_{x \to a} k\, f(x)$ = $k \lim\limits_{x \to a} f(x)$ = kL

(d) $\lim\limits_{x \to a} f(x) \cdot g(x)$ = $\{ \lim\limits_{x \to a} f(x)\} \cdot \{ \lim\limits_{x \to a} g(x)\} = L \cdot M$

(e) $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ = $\dfrac{\lim\limits_{x \to a} f(x)}{\lim\limits_{x \to a} g(x)}$ = $\dfrac{L}{M}$ (if $M \neq 0$) .

(f) $\lim\limits_{x \to a} \{ f(x) \}^{n}$ = $\left\{ \lim\limits_{x \to a} f(x) \right\}^{n}$ = L^{n}

(g) $\lim\limits_{x \to a} \sqrt[n]{f(x)}$ = $\sqrt[n]{\lim\limits_{x \to a} f(x)}$ = $\sqrt[n]{L}$ (if $L > 0$ when n is even)

The Main Limit Theorem says we get the same result if we first perform the algebra and then take the limit or if we take the limits first and then perform the algebra: e.g., (a) the limit of the sum equals the sum of the limits. A proof of the Main Limit Theorem is not inherently difficult, but it requires a more precise definition of the limit concept than we have given, and it then involves a number of technical difficulties.

Practice 1:   For $f(x) = x^{2} - x - 6$ and $g(x) = x^{2} - 2x - 3$, evaluate the following limits:

(a) $\lim\limits_{x \to 1} \{f(x) + g(x)\}$ (b) $\lim\limits_{x \to 1} f(x)g(x)$ (c) $\lim\limits_{x \to 1} f(x)/g(x)$ (d) $\lim\limits_{x \to 3} \{f(x) + g(x)\}$

(e) $\lim\limits_{x \to 3} f(x)g(x)$ (f) $\lim\limits_{x \to 3} f(x)/g(x)$ (g) $\lim\limits_{x \to 2} \{ f(x) \}^{3}$ (h) $\lim\limits_{x \to 2} \sqrt{1 - g(x)}$

Limits of Some Very Nice Functions: Substitution

As you may have noticed in the previous example, for some functions f(x) it is possible to calculate the limit as x approaches a simply by substituting x = a into the function and then evaluating f(a), but sometimes this method does not work. The Substitution Theorem uses the following Two Easy Limits and the Main Limit Theorem to partially answer when such a substitution is valid.

> **Two Easy Limits:** $\lim\limits_{x\to a} k = k$ and $\lim\limits_{x\to a} x = a$.

> **Substitution Theorem For Polynomial and Rational Functions:**
>
> If   P(x) and Q(x) are **polynomials** and a is any number,
>
> then $\lim\limits_{x\to a} P(x) = P(a)$ and $\lim\limits_{x\to a} \dfrac{P(x)}{Q(x)} = \dfrac{P(a)}{Q(a)}$ if Q(a) ≠ 0 .

The Substitution Theorem says that we can calculate the limits of polynomials and rational functions by substituting as long as the substitution does not result in a division by zero.

Practice 2:   Evaluate   (a) $\lim\limits_{x\to 2} 5x^3 - x^2 + 3$ (b) $\lim\limits_{x\to 2} \dfrac{x^3 - 7x}{x^2 + 3x}$ (c) $\lim\limits_{x\to 2} \dfrac{x^2 - 2x}{x^2 - x - 2}$

Limits of Other Combinations of Functions

So far we have concentrated on limits of single functions and elementary combinations of functions. If we are working with limits of other combinations or compositions of functions, the situation is slightly more difficult, but sometimes these more complicated limits have useful geometric interpretations.

Example 1:   Use the function defined by the graph in Fig. 1 to evaluate

(a) $\lim\limits_{x\to 1} \{ 3 + f(x) \}$ (b) $\lim\limits_{x\to 1} f(2+x)$ (c) $\lim\limits_{x\to 0} f(3-x)$ (d) $\lim\limits_{x\to 2} f(x+1) - f(x)$

Solution: (a) $\lim\limits_{x\to 1} \{ 3 + f(x) \}$ is a straightforward application of part (a) of the Main Limit Theorem:

$$\lim\limits_{x\to 1} \{ 3 + f(x) \} = \lim\limits_{x\to 1} 3 + \lim\limits_{x\to 1} f(x) = 3 + 2 = 5 .$$
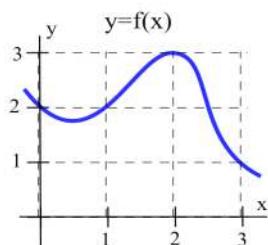
Fig. 1

(b)   We first need to examine what happens to the quantity 2+x , as x→1 , before we can consider the limit of f(2+x). When x is very close to 1, the value of 2+x is very close to 3, so the limit of f(2+x) as x→1 is equivalent to the limit of f(w) as w→3   (w=2+x), and it is clear from the graph that $\lim\limits_{x\to 3} f(w) = 1$: $\lim\limits_{x\to 1} f(2+x) = \lim\limits_{x\to 3} f(w) = 1$ (w represents 2+x).

In most cases it is not necessary to formally substitute a new variable w for the quantity 2+x, but it is still necessary to think about what happens to the quantity 2+x as x→1.

(c) As x→0, the quantity 3–x will approach 3 so we want to know what happens to the values of f when the variable is approaching 3: $\lim\limits_{x\to 0} f(3-x) = 1$.

(d) $\lim\limits_{x\to 2} \{ f(x+1) - f(x) \} = \lim\limits_{x\to 2} f(x+1) - \lim\limits_{x\to 2} f(x)$ replace x+1 with w

$$= \lim\limits_{x\to 3} f(w) - \lim\limits_{x\to 2} f(x) = 1 - 3 = -2 .$$
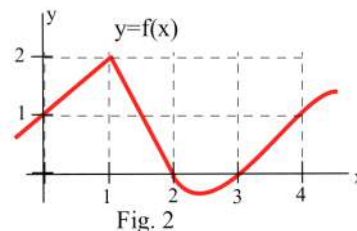
Practice 3:   Use the function defined by the graph in Fig. 2 to evaluate

(a) $\lim\limits_{x\to 1} f(2x)$ (b) $\lim\limits_{x\to 2} f(x-1)$

(c) $\lim\limits_{x\to 0} 3{\cdot}f(4+x)$ (d) $\lim\limits_{x\to 2} f(3x-2)$.

Fig. 2
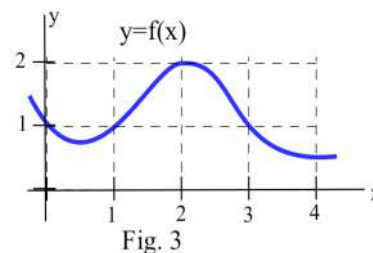
Example 2:   Use the function defined by the graph in Fig. 3 to evaluate

(a) $\lim\limits_{h\to 0} f(3+h)$ (b) $\lim\limits_{h\to 0} f(3)$

(c) $\lim\limits_{h\to 0} \{ f(3+h) - f(3) \}$ (d) $\lim\limits_{h\to 0} \dfrac{f(3+h) - f(3)}{h}$

Fig. 3

Solution: Part (d) is a common form of limit, and parts (a) – (c) are the steps we need to evaluate (d).

(a) As h→0, the quantity w = 3+h will approach 3 so $\lim\limits_{h\to 0} f(3+h) = \lim\limits_{x\to 3} f(w) = 1$.

(b) f(3) is the constant 1 and f(3) does not depend on h in any way so $\lim\limits_{h\to 0} f(3) = 1$.

(c) The limit in part (c) is just an algebraic combination of the limits in (a) and (b):

$$\lim\limits_{h\to 0} \{ f(3+h) - f(3) \} = \lim\limits_{h\to 0} f(3+h) - \lim\limits_{h\to 0} f(3) = 1 - 1 = 0 .$$

The quantity f(3+h) – f(3) also has a geometric interpretation — it is the change in the y–coordinates, the Δy, between the points (3,f(3)) and (3+h,f(3+h)). (Fig. 4)

Fig. 4
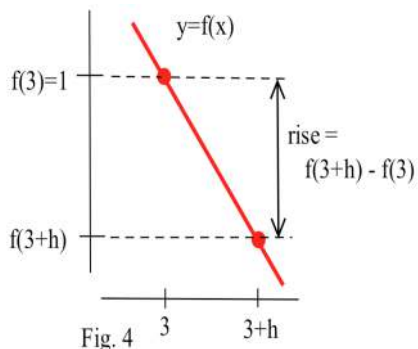
(d)   As h→0, the numerator and denominator of $\dfrac{f(3+h) - f(3)}{h}$ both approach 0 so we cannot immediately determine the value of the limit. But if we recognize that f(3+h) – f(3) = Δy for the two points (3, f(3)) and (3+h, f(3+h)) and that h = Δx for the same two points, then we can interpret $\dfrac{f(3+h) - f(3)}{h}$ as $\dfrac{\Delta y}{\Delta x}$ which is the slope of the secant line through the two points. So

$$\lim_{h \to 0} \frac{f(3+h)-f(3)}{h} \;=\; \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \;=\; \lim_{\Delta x \to 0} \{\text{ slope of the secant line }\}$$

$$= \text{ slope of the tangent line at } (\,3, f(3)\,) \approx -2 \,.$$

This limit, representing the slope of line tangent to the graph of f at the point $(\,3, f(3)\,)$, is a

pattern we will see often in the future.

Tangent Lines as Limits

If we have two points on the graph of a function, $(\,x, f(x)\,)$ and $(\,x{+}h, f(x{+}h)\,)$, then $\Delta y = f(x{+}h) - f(x)$

and $\Delta x = (x{+}h) - (x) = h$ so the slope of the secant line through those points is $m_{\text{secant}} \;=\; \dfrac{\Delta y}{\Delta x}$

and the slope of the line tangent to the graph of f at the point $(\,x, f(x)\,)$ is, by definition,

$$m_{\text{tangent}} \;=\; \lim_{\Delta x \to 0} \{\text{ slope of the secant line }\} \;=\; \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} \;.$$

Example 3:   Give a geometric interpretation for the following limits and **estimate** their values for the

function in Fig. 5:   (a) $\lim\limits_{h \to 0} \dfrac{f(1+h)-f(1)}{h}$ (b) $\lim\limits_{h \to 0} \dfrac{f(2+h)-f(2)}{h}$

Fig. 5
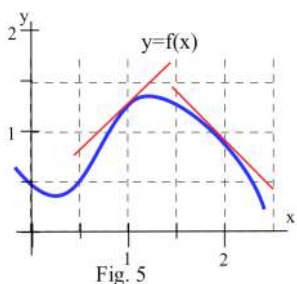
Solution: Part (a) represents the slope of the line tangent to the graph of f(x) at the

point $(\,1, f(1)\,)$ so

$\lim\limits_{h \to 0} \dfrac{f(1+h)-f(1)}{h} \approx 1$. Part (b) represents the slope of the line tangent to the

graph of f(x) at the point $(\,2, f(2)\,)$ so $\lim\limits_{h \to 0} \dfrac{f(2+h)-f(2)}{h} \approx -1$.

Practice 4:   Give a geometric interpretation for the following limits and

estimate their values for the function in Fig. 6:

$$\lim_{h \to 0} \frac{g(1+h)-g(1)}{h} \qquad \lim_{h \to 0} \frac{g(3+h)-g(3)}{h} \qquad \lim_{h \to 0} \frac{g(h)-g(0)}{h} \;.$$

Fig. 6
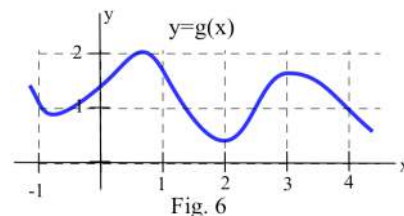
Comparing the Limits of Functions

Sometimes it is difficult to work directly with a function. However, if we can compare our difficult function

with easier ones, then we can use information about the easier functions to draw conclusions about the

difficult one. If the complicated function is always between two functions whose limits are equal, then we

know the limit of the complicated function.

Squeezing Theorem (Fig. 7):

   If $g(x) \le f(x) \le h(x)$ for all x near c (for all x close to but not equal to c)

   and $\lim\limits_{x \to c} g(x) = \lim\limits_{x \to c} h(x) = L$

   then   for x near c, f(x) will be squeezed between g(x) and h(x), and $\lim\limits_{x \to c} f(x) = L$.
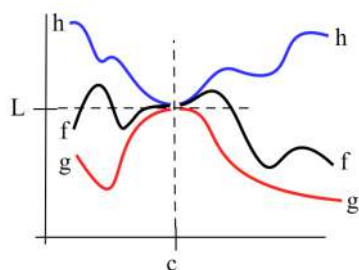
Fig. 7

Example 4:   Use the inequality $-|x| \le \sin(x) \le |x|$ to determine $\lim\limits_{x \to 0} \sin(x)$

                and $\lim\limits_{x \to 0} \cos(x)$.

Solution: $\lim\limits_{x \to 0} |x| = 0$ and $\lim\limits_{x \to 0} -|x| = 0$ so, by the Squeezing Theorem,

         $\lim\limits_{x \to 0} \sin(x) = 0$. If $-\pi/2 < x < \pi/2$ then $\cos(x) = +\sqrt{1 - \sin^2(x)}$ so

         $\lim\limits_{x \to 0} \cos(x) = \lim\limits_{x \to 0} +\sqrt{1 - \sin^2(x)} = +\sqrt{1 - 0} = 1$.

Example 5:   Evaluate $\lim\limits_{x \to 0} x \cdot \sin(\frac{1}{x})$.

Solution: The graph of $y = \sin(\frac{1}{x})$ for values of x near 0 is shown in Fig. 8. The y–values of this

graph change very rapidly for values of x near 0, but they all lie between −1 and +1:

$-1 \le \sin(\frac{1}{x}) \le +1$. The fact that

$\sin(\frac{1}{x})$ is bounded between −1 and +1

implies that $x \sin(\frac{1}{x})$ is stuck between

−x and +x , so the function we are

interested in , $x \cdot \sin(\frac{1}{x})$, is squeezed

between two "easy" functions, −x and

x (Fig. 9). Both "easy" functions

approach 0 as x→0 , so $x \cdot \sin(\frac{1}{x})$

must also approach 0 as x→0 :

$\lim\limits_{x \to 0} x \cdot \sin(\frac{1}{x}) = 0$.
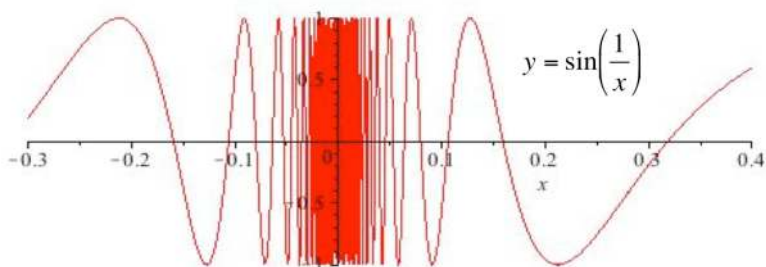
Fig. 8
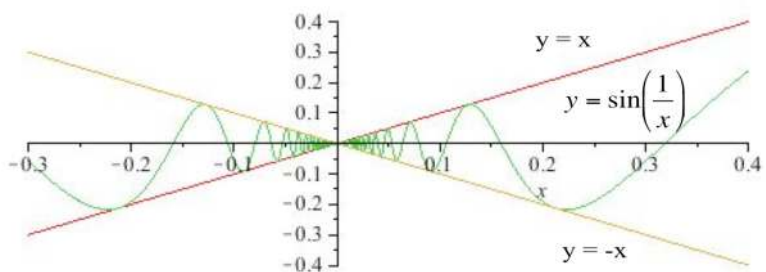
Fig. 9

Practice 5:   If   f(x) is always between $x^2 + 2$ and $2x + 1$, then $\lim\limits_{x \to 1} f(x) = ?$

Practice 6:   Use the relation $\cos(x) \le \dfrac{\sin(x)}{x} \le 1$ to show that $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x} = 1$. (The steps for

          deriving the inequalities are shown in problem 19.)

List Method for Showing that a Limit Does Not Exist

If the limit, as x approaches c, exists and equals L, then we can guarantee that the values of f(x) are
as close to L as we want by restricting the values of x to be very, very close to c. To show that a limit,
as x approaches c, does not exist, we need to show that no matter how closely we restrict the values of x
to c, the values of f(x) are not all close to a single, finite value L. One way to

demonstrate that $\lim\limits_{x \to c}$ f(x) does not exist is to show that the left and right limits exist but are not equal.

Another method of showing that $\lim\limits_{x \to c}$ f(x)   does not exist is to find two infinite lists of numbers,
$\{a_1, a_2, a_3, a_4, \dots\}$ and $\{b_1, b_2, b_3, b_4, \dots\}$, which approach arbitrarily close to the value c as the
subscripts get larger, but so that the lists of function values, $\{f(a_1), f(a_2), f(a_3), f(a_4), \dots\}$ and
$\{f(b_1), f(b_2), f(b_3), f(b_4), \dots\}$, approach two different numbers as the subscripts get larger.

Example 6:   For f(x) = $\begin{cases} 1 & \text{if } x < 1 \\ x & \text{if } 1 < x < 3 \\ 2 & \text{if } 3 < x \end{cases}$, show that $\lim\limits_{x \to 3}$ f(x) does not exist.

Solution: We can use one–sided limits to show that this limit does not exist, or we can use the list
    method by selecting values for one list to approach 3 from the right and values for the other list to
    approach 3 from the left.

One way to define values of $\{a_1, a_2, a_3, a_4, \dots\}$ which approach 3 from the right is to define
$a_1 = 3 + 1$, $a_2 = 3 + \frac{1}{2}$, $a_3 = 3 + \frac{1}{3}$, $a_4 = 3 + \frac{1}{4}$ and, in general, $a_n = 3 + \frac{1}{n}$. Then $a_n > 3$
so $f(a_n) = 2$ for all subscripts n , and the values in the list $\{f(a_1), f(a_2), f(a_3), f(a_4), \dots\}$ are
approaching 2 . In fact, all of the $f(a_n) = 2$.

We can define values of $\{b_1, b_2, b_3, b_4, \dots\}$ which approach 3 from the left by $b_1 = 3 - 1$,
$b_2 = 3 - \frac{1}{2}$, $b_3 = 3 - \frac{1}{3}$, $b_4 = 3 - \frac{1}{4}$ and, in general, $b_n = 3 - \frac{1}{n}$. Then $b_n < 3$ so
$f(b_n) = b_n = 3 - \frac{1}{n}$ for each subscript n , and the values in the list

$\{ f(b_1), f(b_2), f(b_3), f(b_4), \dots \} = \{ 2, 2.5, 2.67, 2.75, 2.8, \dots, 3 - \frac{1}{n} , \dots \}$ approach 3.

Since the values in the lists $\{ f(a_1), f(a_2), f(a_3), f(a_4), \dots \}$ and $\{ f(b_1), f(b_2), f(b_3), f(b_4), \dots \}$
approach two different numbers, we can conclude that $\lim\limits_{x \to 3}$ f(x) does not exist.

Example 7:   Let $h(x) = \begin{cases} 2 & \text{if } x \text{ is a } \textbf{rational} \text{ number} \\ 1 & \text{if } x \text{ is an } \textbf{irrational} \text{ number} \end{cases}$ be the "holey" function

introduced in Section 0.4 . Use the list method to show that $\lim\limits_{x \to 3} h(x)$ does not exist.

Solution: Let $\{ a_1, a_2, a_3, a_4, \ldots \}$ be a list of rational numbers which approach 3, for example, $a_1 = 3 +$

1, $a_2 = 3 + 1/2, \ldots, a_n = 3 + 1/n$. Then f(a_n) always equals 2 so $\{ f(a_1), f(a_2), f(a_3), f(a_4), \ldots \} =$

$\{ 2, 2, 2, \ldots \}$ and the f(a_n) values "approach" 2. If $\{ b_1, b_2, b_3, b_4, \ldots \}$ is a list of irrational

numbers which approach 3, for example, $b_1 = 3 + \pi, b_2 = 3 + \pi/2, \ldots, b_n = 3 + \pi/n$. then $\{ f(b_1), f(b_2),$

$f(b_3), f(b_4), \ldots \} = \{ 1, 1, 1, \ldots \}$ and the f(b_n) "approach" 1. Since the f(a_n) and f(b_n) values

approach different numbers, the limit as x→3 does not exist.

A similar argument will work as x approaches any number c, so for every c we have that $\lim\limits_{x \to c} h(x)$

does not exist. The "holey" function does not have a limit as x approaches any value c.

PROBLEMS

1. Use the functions f and g defined by the graphs
    in Fig. 10 to determine the following limits.

    (a) $\lim\limits_{x \to 1} \{ f(x) + g(x) \}$ (b) $\lim\limits_{x \to 1} f(x){\cdot}g(x)$

    (c) $\lim\limits_{x \to 1} f(x)/g(x)$ (d) $\lim\limits_{x \to 1} f(g(x))$
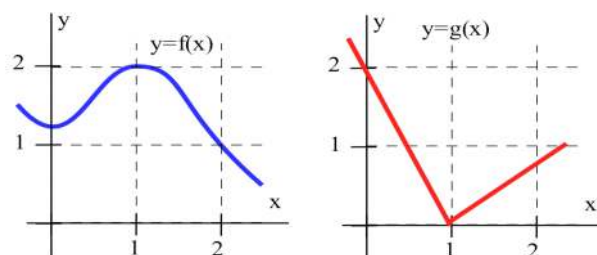
Fig. 10

2. Use the functions f and g defined by the graphs
    in Fig. 10 to determine the following limits.

    (a) $\lim\limits_{x \to 2} \{ f(x) + g(x) \}$ (b) $\lim\limits_{x \to 2} f(x){\cdot}g(x)$

    (c) $\lim\limits_{x \to 2} f(x)/g(x)$ (d) $\lim\limits_{x \to 2} f(g(x))$

3. Use the function h defined by the graph in Fig. 11 to determine
    the following limits.

    (a) $\lim\limits_{x \to 2} h(2x - 2)$ (b) $\lim\limits_{x \to 2} \{ x + h(x) \}$

    (c) $\lim\limits_{x \to 2} h(1 + x)$ (d) $\lim\limits_{x \to 3} h(x/2)$

4. Use the function h defined by the graph in Fig. 11 to determine
    the following limits.

    (a) $\lim\limits_{x \to 2} h(5 - x)$ (b) $\lim\limits_{x \to 2} x{\cdot}h(x - 1)$

    (c) $\lim\limits_{x \to 0} \{ h(3 + x) - h(3) \}$ (d) $\lim\limits_{x \to 0} \dfrac{h(3 + x) - h(3)}{x}$
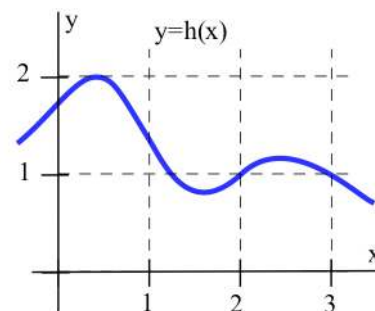
Fig. 11
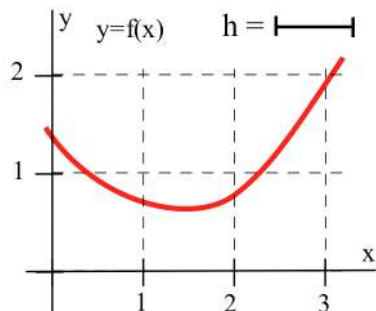
Fig. 12

5. Label the parts of the graph of f (Fig. 12) which are described by

   (a) 2 + h   (b) f(2)   (c) f(2 + h)

   (d) f(2 + h) – f(2)   (e) $\dfrac{f(2 + h) – f(2)}{(2 + h) – (2)}$ (f) $\dfrac{f(2 – h) – f(2)}{(2 – h) – (2)}$

6. Label the parts of the graph of f (Fig. 13) which are described by

   (a) a + h   (b) g(a)   (c) g(a + h)

   (d) g(a + h) – g(a)   (e) $\dfrac{g(a + h) – g(a)}{(a + h) – (a)}$ (f) $\dfrac{g(a – h) – g(a)}{(a – h) – (a)}$
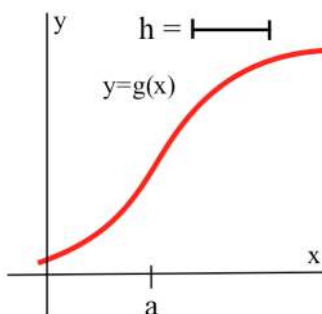
Fig. 13

7. Use the function f defined by the graph in Fig. 14 to determine the following limits.

   (a) $\lim\limits_{x \to 1^+}$ f(x)   (b) $\lim\limits_{x \to 1^-}$ f(x)   (c) $\lim\limits_{x \to 1}$ f(x)

   (d) $\lim\limits_{x \to 3^+}$ f(x)   (e) $\lim\limits_{x \to 3^-}$ f(x)   (f) $\lim\limits_{x \to 3}$ f(x)

   (g) $\lim\limits_{x \to -1^+}$ f(x)   (h) $\lim\limits_{x \to -1^-}$ f(x)   (i) $\lim\limits_{x \to -1}$ f(x)
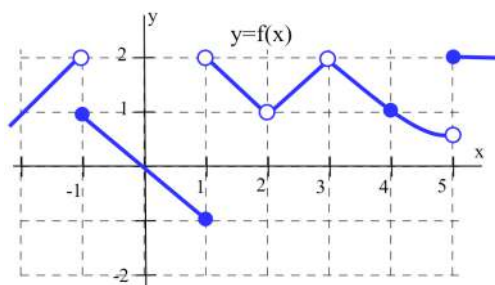
Fig. 14

8. Use the function f defined by the graph in Fig. 14 to determine the following limits.

   (a) $\lim\limits_{x \to 2^+}$ f(x)   (b) $\lim\limits_{x \to 2^-}$ f(x)   (c) $\lim\limits_{x \to 2}$ f(x)

   (d) $\lim\limits_{x \to 4^+}$ f(x)   (e) $\lim\limits_{x \to 4^-}$ f(x)   (f) $\lim\limits_{x \to 4}$ f(x)

   (g) $\lim\limits_{x \to -2^+}$ f(x)   (h) $\lim\limits_{x \to -2^-}$ f(x)   (i) $\lim\limits_{x \to -2}$ f(x)

9. The Lorentz Contraction Formula in relativity theory says the length L of an object moving at v miles per second with respect to an observer is $L = A \cdot \sqrt{1 - \dfrac{v^2}{c^2}}$ where c is the speed of light (a constant).

   a)   Determine the "rest length" of the object (v = 0).   b)   Determine $\lim\limits_{v \to c^-}$ L .

10. (a) $\lim\limits_{x \to 2^+}$ INT(x)   (b) $\lim\limits_{x \to 2^-}$ INT(x)   (c) $\lim\limits_{x \to -2^+}$ INT(x)   (d) $\lim\limits_{x \to -2^-}$ INT(x)

   (e) $\lim\limits_{x \to -2.3}$ INT(x)   (f) $\lim\limits_{x \to 3}$ INT(x/2)   (g) $\lim\limits_{x \to 3}$ INT(x)/2   (h) $\lim\limits_{x \to 0^+} \dfrac{INT(2 + x) – INT(2)}{x}$

11. $f(x) = \begin{cases} 1 & \text{if } x < 1 \\ x & \text{if } 1 < x \end{cases}$ and $g(x) = \begin{cases} x & \text{if } x \neq 2 \\ 3 & \text{if } x = 2 \end{cases}$.

(a) $\lim\limits_{x \to 2} \{ f(x) + g(x) \}$ (b) $\lim\limits_{x \to 2} f(x)/g(x)$ (c) $\lim\limits_{x \to 2} f(g(x))$

(d) $\lim\limits_{x \to 0} g(x)/f(x)$ (e) $\lim\limits_{x \to 1} f(x)/g(x)$ (f) $\lim\limits_{x \to 1} g(f(x))$

Problems 12 – 15 require a calculator.

12. Give geometric interpretations for the following

limits and use a calculator to estimate their values.

(a) $\lim\limits_{h \to 0} \dfrac{\arctan(0 + h) - \arctan(0)}{h}$

(b) $\lim\limits_{h \to 0} \dfrac{\arctan(1 + h) - \arctan(1)}{h}$

(c) $\lim\limits_{h \to 0} \dfrac{\arctan(2 + h) - \arctan(2)}{h}$

Fig. 15
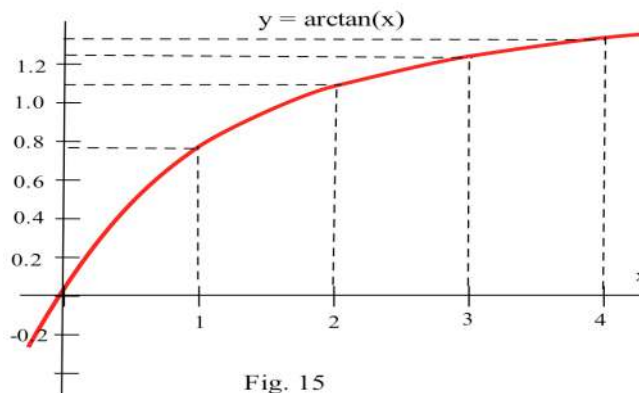
13. (a) What does $\lim\limits_{h \to 0} \dfrac{\cos(h) - 1}{h}$ represent on the graph of $y = \cos(x)$?

(It may help to recognize that $\dfrac{\cos(h) - 1}{h} = \dfrac{\cos(0 + h) - \cos(0)}{h}$)

(b) Graphically and using your calculator, determine $\lim\limits_{h \to 0} \dfrac{\cos(h) - 1}{h}$.

14. (a) What does the ratio $\dfrac{\ln(1 + h)}{h}$ represent on the graph of $y = \ln(x)$?

(It may help to recognize that $\dfrac{\ln(1 + h)}{h} = \dfrac{\ln(1 + h) - \ln(1)}{h}$.)

(b) Graphically and using your calculator, determine $\lim\limits_{h \to 0} \dfrac{\ln(1 + h)}{h}$.

15. Use your calculator (to generate a table of values) to help you estimate

(a) $\lim\limits_{h \to 0} \dfrac{e^h - 1}{h}$ (b) $\lim\limits_{c \to 0} \dfrac{\tan(1 + c) - \tan(1)}{c}$ (c) $\lim\limits_{t \to 0} \dfrac{g(2 + t) - g(2)}{t}$ when $g(t) = t^2 - 5$.

16. (a) For $h > 0$, find the slope of the line through the points $(h, |h|)$ and $(0, 0)$.

(b) For $h < 0$, find the slope of the line through the points $(h, |h|)$ and $(0, 0)$.

(c) Evaluate $\lim\limits_{h \to 0^-} \dfrac{|h|}{h}$, $\lim\limits_{h \to 0^+} \dfrac{|h|}{h}$, and $\lim\limits_{h \to 0} \dfrac{|h|}{h}$

17. Describe the behavior of the function $y = f(x)$ in Fig. 16 at each integer using one of the phrases:
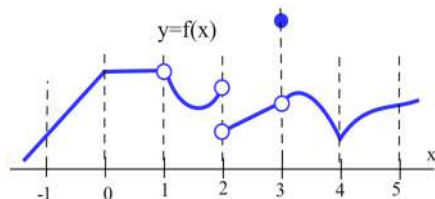
Fig. 16

(a) "connected and smooth", (b) "connected with a corner",

(c) "not connected because of a simple hole which could be

plugged by adding or moving one point", or

(d) "not connected because of a vertical jump which could

not be plugged by moving one point."

18. Describe the behavior of the function y = f(x) in Fig. 17 at each
    integer using one of the phrases: (a) "connected and smooth",
    (b) "connected with a corner", (c) "not connected because of a
    simple hole which could be plugged by adding or moving one
    point", or
    (d) "not connected because of a vertical jump which could not be
    plugged by moving one point."

y=f(x)
Fig. 17
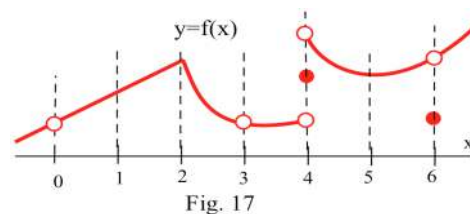
19. This problem outlines the steps of a proof that $\lim\limits_{\theta \to 0^+} \dfrac{\sin(\theta)}{\theta} = 1$. Statements (a) – (h)

    below refer to Fig. 18. Assume that $0 < \theta < \dfrac{\pi}{2}$ and justify why

    each statement is true.

$x^2 + y^2 = 1$
$P = (\cos\theta, \sin\theta)$
Fig. 18

(a) Area of $\triangle OPB = \dfrac{1}{2}$ (base)(height) $= \dfrac{1}{2} \sin(\theta)$.

(b) $\dfrac{\text{area of the sector (the pie shaped region) OPB}}{\text{area of the whole circle}}$

   $= \dfrac{\text{angle defining sector OPB}}{\text{angle of the whole circle}} = \dfrac{\theta}{2\pi}$

   so (area of the sector OPB) $= \dfrac{\theta\pi}{2\pi} = \dfrac{\theta}{2}$.

(c) The line L through the points (0,0) and P = (cos(θ),sin(θ)) has slope $m = \dfrac{\sin(\theta)}{\cos(\theta)}$, so

   $C = (1, \dfrac{\sin(\theta)}{\cos(\theta)})$ and   the area of $\triangle OCB = \dfrac{1}{2}$ (base)(height) $= \dfrac{1}{2} (1) \dfrac{\sin(\theta)}{\cos(\theta)}$.

(d) Area of $\triangle OPB$ < area of sector OPB < area of $\triangle OCB$.

(e) $\dfrac{1}{2} \sin(\theta) < \dfrac{\theta}{2} < \dfrac{1}{2} (1) \dfrac{\sin(\theta)}{\cos(\theta)}$ and $\sin(\theta) < \theta < \dfrac{\sin(\theta)}{\cos(\theta)}$.

(f) $1 < \dfrac{\theta}{\sin(\theta)} < \dfrac{1}{\cos(\theta)}$ and $1 > \dfrac{\sin(\theta)}{\theta} > \cos(\theta)$.

(g) $\lim\limits_{\theta \to 0^+} 1 = 1$ and $\lim\limits_{\theta \to 0^+} \cos(\theta) = 1$.

(h) $\lim\limits_{\theta \to 0^+} \dfrac{\sin(\theta)}{\theta} = 1$.
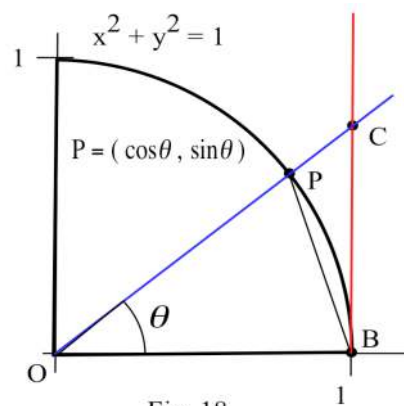
20.   Use the list method to show that $\lim\limits_{x \to 2} \dfrac{|x-2|}{x-2}$ does not exist .

21.   Show that $\lim\limits_{x \to 0} \sin(\frac{1}{x})$ does not exist.

(Suggestion: Let $f(x) = \sin(\frac{1}{x})$ and pick $a_n = \frac{1}{n\pi}$ so $f(a_n) = \sin(\frac{1}{a_n}) = \sin(n\pi) = 0$ for every n.

Then pick $b_n = \dfrac{1}{2n\pi + \pi/2}$ so $f(b_n) = \sin(\frac{1}{b_n}) = \sin(2n\pi + \frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$ for every n.)

Section 2.3   **PRACTICE Answers**

Practice 1:   (a)   **−10**   (b) **24**   (c) **3/2**   (d) **0**

                  (e) **0**   (f) **5/4**   (g) **−64**   (h) **2**

Practice 2:   (a) **39**   (b) **−3/5**   (c) **2/3**

Practice 3:   (a) **0**   (b) **2**   (c) **3**   (d) **1**

Practice 4:   (a) slope of the line tangent to the graph of g at the point (1, g(1)): estimated slope $\approx$ **−2**

                   (b) slope of the line tangent to the graph of g at the point (3, g(3)): estimated slope $\approx$ **0**

                   (c) slope of the line tangent to the graph of g at the point (0, g(0)): estimated slope $\approx$ **1**

Practice 5: $\lim\limits_{x \to 1} x^2 + 2 = 3$ and $\lim\limits_{x \to 1} 2x + 1 = 3$ so $\lim\limits_{x \to 1} f(x) = \mathbf{3}$.

Practice 6: $\lim\limits_{x \to 0} \cos(x) = 1$ and $\lim\limits_{x \to 0} 1 = 1$ so $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x} = \mathbf{1}$.

2.4 CONTINUOUS FUNCTIONS

In section 2.3 we saw a few "nice" functions whose limits as $x \rightarrow a$ simply involved substituting a into the function: $\lim_{x \to a} f(x) = f(a)$. Functions whose limits have this substitution property are called **continuous functions**, and they have a number of other useful properties and are very common in applications. We will examine what it means graphically for a function to be continuous or not continuous. Some properties of continuous functions will be given, and we will look at a few applications of these properties including a way to solve horrible equations such as $\sin(x) = \dfrac{2x + 1}{x - 2}$.

DEFINITION AND MEANING OF CONTINUOUS

> **Definition of Continuity at a Point**
>
> A function **f is continuous at x = a**   if and only if $\lim_{x \to a} \mathbf{f(x) = f(a)}$.

The graph in Fig. 1 illustrates some of the different ways a function can behave at and near a point, and Table 1 contains some numerical information about the function and its behavior. Based on the information in the table, we can conclude that f

is continuous at 1 since $\lim_{x \to 1} f(x) = 2 = f(1)$.

We can also conclude from the information in the table that f is not continuous at 2 or 3 or 4, because

$\lim_{x \to 2} f(x) \neq f(2)$, $\lim_{x \to 3} f(x) \neq f(3)$, and $\lim_{x \to 4} f(x) \neq f(4)$.

Fig. 1

Graphic Meaning of Continuity

When x is close to 1, the values of f(x) are close to the value f(1), and the graph of f in Fig. 1 does not have a hole or break at x=1 . The graph of f is connected at x=1 and can be drawn without lifting your pencil. At x=2 and x=4 the graph of f has holes, and at x=3 the graph has a break. The function f is also continuous at 1.7 (why?) , and at every point shown **except** at 2, 3, and 4.
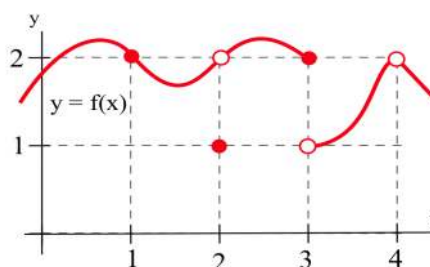
a	f(a)	$\lim_{x \to a} f(x)$
1	2	2
2	1	2
3	2	does not exist
4	undefined	2

Table 1

> **Informally:** A function is **continuous** at a point if the graph of the function is **connected** there.
>
>   A function is **not continuous** at a point if its graph **has a hole or break** at that point.

Sometimes the definition of continuous (the substitution condition for limits) is easier to use if we break it
into several smaller pieces and then check whether or not our function satisfies each piece.

$\{$ f is continuous at a $\}$ if and only if $\{$ $\lim\limits_{x \to a}$ f(x) = f(a) $\}$ if and only if

    (i)   f is defined at a,

    (ii)   the limit of f(x) , as x→ a , exists (so the left limit and right limits exist and are equal)

and (iii) the value of f at a equals the value of the limit as x→ a: $\lim\limits_{x \to a}$ f(x) = f(a) .

If f satisfies conditions (i), (ii) and (iii) , then f is continuous at a. If f does not satisfy one or more of
the three conditions at a , then f is not continuous at a.

For the function in Fig. 1, at a = 1, all 3 conditions are satisfied, and f is continuous at 1. At a = 2,
conditions (i) and (ii) are satisfied but not (iii), so f is not continuous at 2. At a = 3, condition (i) is
satisfied but (ii) is violated, so f is not continuous at 3. At a = 4, condition (i) is violated, so f is not
continuous at 4.

A function is **continuous on an interval** if it is continuous <u>at every point</u> in the interval. A function

f is **continuous from the left** at a if $\lim\limits_{x \to a^-}$ f(x) = f(a) , and is **continuous from the right**

if $\lim\limits_{x \to a^+}$ f(x) = f(a) .

Example 1:   Is   f(x) = $\begin{cases} x+1 & \text{if } x \le 1 \\ 2 & \text{if } 1 < x \le 2 \\ 1/(x-3) & \text{if } x > 2 \end{cases}$ continuous at 1, 2, 3 ?

Solution: We could answer these questions by examining the graph of f(x), but lets try them without the

    graph.   At a = **1**, f(**1**) = 2   and the left and right limits are equal,

    f(x) = $\lim\limits_{x \to 1^-}$ (x + 1) = 2   and $\lim\limits_{x \to 1^+}$ f(x) = $\lim\limits_{x \to 1^+}$ 2 = 2, so f is continuous at **1**.

  At a = **2**, f(**2**) = 2, but the left and right limits are not equal,

    $\lim\limits_{x \to 2^-}$ f(x) = $\lim\limits_{x \to 2^-}$ 2 = 2   and $\lim\limits_{x \to 2^+}$ f(x) = $\lim\limits_{x \to 2^+}$ 1/(x–3) = –1, so f fails condition (ii) and is

    not continuous at   **2**.   f is continuous from the left at **2**, but not from the right.

  At a = **3**, f(**3**) = 1/0 which is undefined so f is not continuous at **3** because it fails condition (i).

Example 2:   Where is f(x) = $3x^2 - 2x$ continuous?

Solution: At every point. By the Substitution Theorem for Polynomials, every polynomial

        is continuous everywhere.

Example 3:   Where are   g(x) = $\dfrac{x+5}{x-3}$ and h(x) = $\dfrac{x^2 + 4x - 5}{x^2 - 4x + 3}$ continuous?

Solution: g is a rational function so by the Substitution Theorem for Polynomials and Rational Functions it

is continuous everywhere except where its denominator is 0: g is continuous everywhere except 3. The

graph of g (Fig. 2) is connected everywhere except at 3 where it has a vertical asymptote.

$h(x) = \dfrac{(x-1)(x+5)}{(x-1)(x-3)}$ is also continuous everywhere except where its denominator is 0: h is

continuous everywhere except 3 and 1 . The graph of h (Fig. 3) is connected everywhere except at

3 where it has a vertical asymptote and at 1 where it has a hole: f(1) = 0/0 is undefined.

$y = \dfrac{x+5}{x-3}$

Fig. 2

$y = \dfrac{(x-1)(x+5)}{(x-1)(x-3)}$

hole
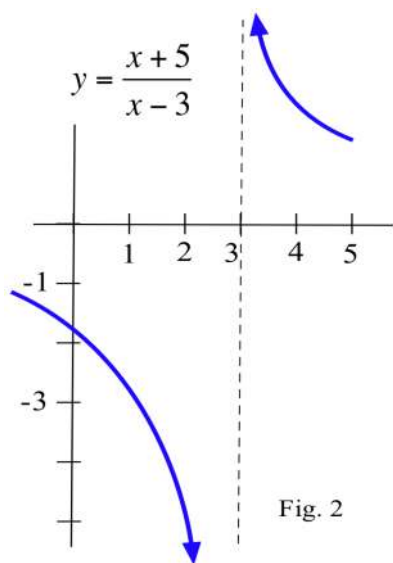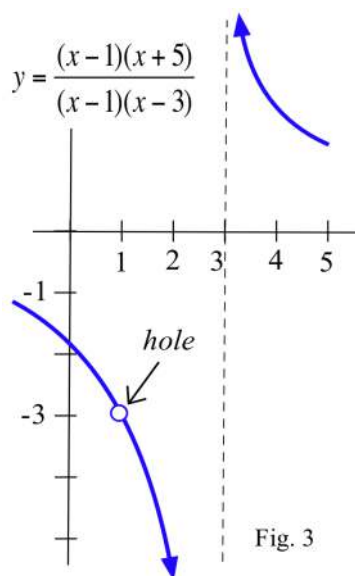
Fig. 3

Example 4:   Where is f(x) = INT(x) continuous?

Solution: The graph of y = INT(x) seems to be connected except at each integer, and at each integer there

is a "jump" (Fig. 4).

If a is an **integer**, then

$\lim\limits_{x \to a^-} \text{INT}(x) = a{-}1$, and $\lim\limits_{x \to a^+} \text{INT}(x) = a$, so $\lim\limits_{x \to a} \text{INT}(x)$ is

undefined, and INT(x) is not continuous.

y= [x] = INT(x)

If a is **not an integer**, then the left and right limits of INT(x) , as x→ a,

both equal INT(a) so

$\lim\limits_{x \to a} \text{INT}(x) = \text{INT}(a) = f(a)$ and INT(x) is continuous. f(x) = INT(x)

is continuous except at the integers.

Fig. 4

In fact, f(x) = INT(x) is continuous from the right everywhere and is

continuous from the left everywhere except at the integers.
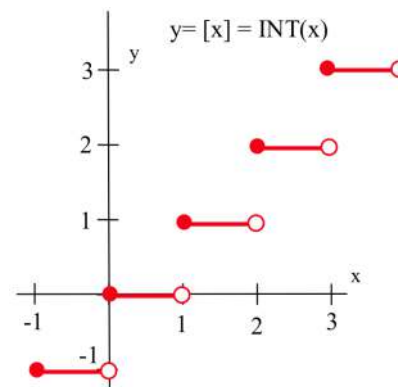
Practice 1:   Where is f(x) = |x|/x   continuous?

Why do we care whether a function is continuous?

There are several reasons for us to examine continuous functions and their properties:

- Most of the applications in engineering, the sciences and business are continuous and are modeled by continuous functions or by pieces of continuous functions.

- Continuous functions have a number of useful properties which are not necessarily true if the function is not continuous. If a result is true of all continuous functions and we have a continuous function, then the result is true for our function. This can save us from having to show, one by one, that each result is true for each particular function we use. Some of these properties are given in the rest of this section.

- Differential calculus has been called the study of **continuous** change, and many of the results of calculus are guaranteed to be true only for continuous functions. If you look ahead into Chapters 2 and 3, you will see that many of the theorems have the form "If f is **continuous** and (some additional hypothesis), then (some conclusion)".

Combinations of Continuous Functions

> **Theorem:**   If   f(x) and g(x) are continuous at a, and k is any constant,
>
> then   the elementary combinations of f and g
>
> ($k \cdot f(x)$, f(x) + g(x), f(x) − g(x), f(x)·g(x), and f(x)/g(x) (g(a)≠0))
>
> are continuous at a.

The continuity of a function is defined in terms of limits, and all of these results about simple combinations of continuous functions follow from the results about simple combinations of limits in the Main Limit Theorem. Our hypothesis is that f and g are both continuous at a, so we can assume that

$\lim\limits_{x \to a} f(x) = f(a)$ and $\lim\limits_{x \to a} g(x) = g(a)$ and then use the appropriate part of the Main Limit Theorem.

For example, $\lim\limits_{x \to a} \{ f(x) + g(x) \} = \left\{ \lim\limits_{x \to a} f(x) \right\} + \left\{ \lim\limits_{x \to a} g(x) \right\} = f(a) + g(a)$, so f + g is continuous at a.

Practice 2:   Prove: If f and g are continuous at a, then kf and f − g are continuous at a.   (k a constant.)

> **Composition of Continuous Functions**
>
> If   g is continuous at   a   and f is continuous at   g(a),
>
> then $\lim\limits_{x \to a} \{ f(\, g(x)\,) \} = f(\lim\limits_{x \to a} g(x)\,) = f(\, g(a)\,)$ so   f∘g(x) = f(\, g(x)\,) is continuous at a.

This result will not be proved here, but the proof just formalizes the following line of reasoning:

The hypothesis that "g is continuous at a" means that if x is close to a then g(x) will be close to g(a). Similarly, "f is continuous at g(a)" means that if g(x) is close to g(a) then f(g(x)) = f∘g(x) will be close to f(g(a)) =f∘g(a) . Finally, we can conclude that if x is close to a, then g(x) is close to g(a) so f∘g(x) is close to f∘g(a), and therefore f∘g is continuous at x = a.

The next theorem presents an alternate version of the limit condition for continuity, and we will use this alternate version occasionally in the future.

Theorem: $\lim\limits_{x \to a} f(x) = f(a)$ if and only if $\lim\limits_{h \to 0} f(a+h) = f(a)$.

Proof: Let's define a new variable h by h = x − a so x = a + h

(Fig. 5). Then x → a if and only if h = x − a → 0 , so

$\lim\limits_{x \to a} f(x) = \lim\limits_{h \to 0} f(a+h)$, and $\lim\limits_{x \to a} f(x) = f(a)$ if and only if

$\lim\limits_{h \to 0} f(a+h) = f(a)$.

Fig. 5

A function f is continuous at a if and only if $\lim\limits_{h \to 0} \mathbf{f(a+h) = f(a)}$.
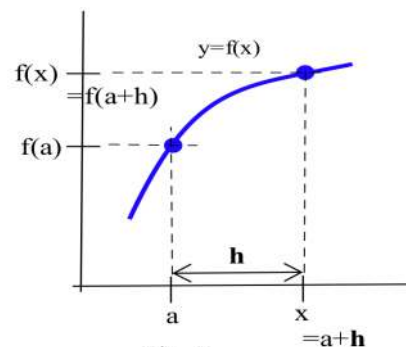
Which Functions Are Continuous?

Fortunately, the situations which we encounter most often in applications and the functions which model those situations are either continuous everywhere or continuous everywhere except at a few places, so any result which is true of all continuous functions will be true of most of the functions we commonly use.

Theorem: The following functions are continuous everywhere, at every value of x:

  (a) polynomials,   (b) sin(x) and cos(x),   and   (c) | x | .

Proof: (a) This follows from the Substitution Theorem for Polynomials and the definition of continuity.

  (b) The graph of y = sin(x) (Fig. 6) clearly indicates that sin(x) does not have any holes or breaks so sin(x) is continuous everywhere. Or we could justify that result analytically:
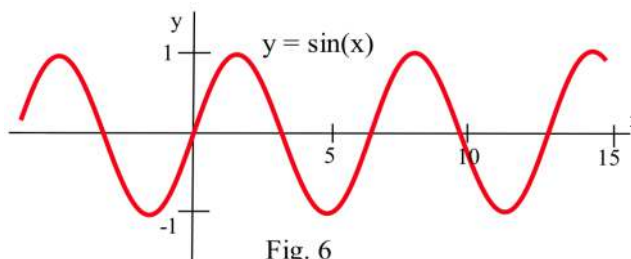
Fig. 6

for every real number a,

$$\lim_{h \to 0} \sin(a+h) \quad = \quad \lim_{h \to 0} \sin(a)\cos(h) + \cos(a)\sin(h)$$

$$= \quad \lim_{h \to 0} \sin(a) \cdot \lim_{h \to 0} \cos(h) + \lim_{h \to 0} \cos(a) \cdot \lim_{h \to 0} \sin(h)$$

(recall from section 1.2 that $\lim_{h \to 0} \cos(h) = 1$ and $\lim_{h \to 0} \sin(h) = 0$)

$$= \quad \lim_{h \to 0} \sin(a) \cdot \mathbf{1} + \lim_{h \to 0} \cos(a) \cdot \mathbf{0} = \sin(a),$$

so f(x) = sin(x) is continuous at every point. The justification of f(x) = cos(x) is similar.

(c) f(x) = |x| .   When x > 0, then |x| =x and its graph (Fig. 7) is a straight line and is continuous

since x is a polynomial function. When x <0, then |x| = −x and it is also continuous. The only

questionable point is the "corner" on the graph when x = 0, but the

graph there is only bent, not broken:

$$\lim_{h \to 0^+} |x| = \lim_{h \to 0^+} x = 0$$

and $\lim_{h \to 0^-} |x| = \lim_{h \to 0^-} -x = 0$ so $\lim_{h \to 0} |x| = 0 = |0|$,

and f(x) = |x| is also continuous at 0.
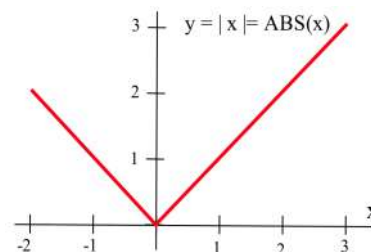
Fig. 7

A continuous function can have corners but not holes or breaks (jumps).

Several results about limits of functions can be written in terms of continuity of those functions. Even

functions which fail to be continuous at some points are often continuous most places.

Theorem: (a)   A rational function is continuous **except** where the denominator is 0.

(b)   Tangent, cotangent, secant and cosecant are continuous **except** where they are undefined.

(c)   The greatest integer function [x] = INT(x) is continuous **except** at each integer.

(d)   But the "holey" function h(x) = $\begin{cases} 2 \\ 1 \end{cases}$ $\quad$ if x is a **rational** number
$\quad\quad$ if x is an **irrational** number
is discontinuous **everywhere**.

INTERMEDIATE VALUE PROPERTY OF CONTINUOUS FUNCTIONS

Since the graph of a continuous function is connected and does not have any holes or

breaks in it, the values of the function can not "skip" or "jump over" a horizontal line.

(Fig. 8): if one value of the continuous function is below the line and another value of

the function is above the line, then **somewhere** the graph will cross the line. The next

theorem makes this statement more precise. The result seems obvious, but its proof is

technically difficult and is not given here.
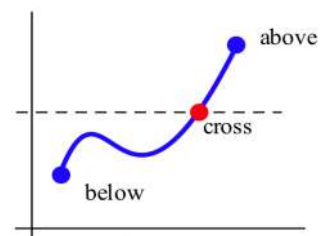
Fig. 8

<div style="background-color: #ffff99;">

Intermediate Value Theorem for Continuous Functions

If   f is continuous on the interval [a,b] and V is any value between f(a) and f(b),

then   there is a number c between a and b so that f(c) = V

(that is, f actually takes each intermediate value between f(a) and f(b).)

</div>

If the graph of f connects the points (a, f(a)) and (b, f(b)) and V is any
number between f(a) and f(b), then the graph of f must cross the horizontal
line y = V **somewhere between x = a and x = b** (Fig. 9). Since f is
continuous, its graph cannot "hop" over the line y = V.
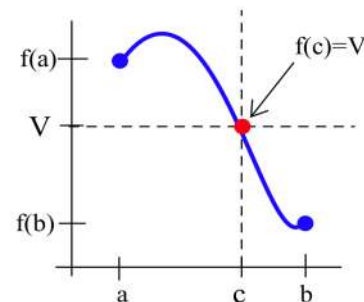
Most people take this theorem for granted in some common situations:

- If a child's temperature rose from 98.6° to 101.3° , then there was an instant
  when the child's temperature was exactly 100° . In fact, every temperature
  between 98.6° and 101.3° occurred at some instant.

- If you dove to pick up a shell 25 feet below the surface of a lagoon, then at some instant in time you
  were 17 feet below the surface. (Actually, you want to be at 17 feet twice. Why?)

- If you started driving from a stop (velocity = 0) and accelerated to a velocity of 30 kilometers per hour,
  then there was an instant when your velocity was exactly 10 kilometers per hour.

But we cannot apply the Intermediate Value Theorem if the function is not continuous:

- In 1987 it cost 22¢ to mail a letter first class inside the United States, and in 1990 it cost 25¢ to mail
  the same letter, but we cannot conclude that there was a time when it cost 23¢ or 24¢ to send the letter.
  Postal rates did not increase in a continuous fashion. They jumped directly from 22¢ to 25¢ .

- Prices, taxes and rates of pay change in jumps, discrete steps, without taking on the intermediate values.

The Intermediate Value Property can help us finds roots of functions and solve equations. If f is
continuous on [a,b] and f(a) and f(b) have opposite signs (one is positive and one is negative) , then 0
is an intermediate value between f(a) and f(b) so f will have a root between x = a and x= b.

Bisection Algorithm for Approximating Roots

The Intermediate Value Theorem is an example of an "existence theorem" because it concludes that
something exists: a number c so that f(c) = V. Many existence theorems do not tell us how to find the
number or object which exists and are of no use in actually finding those numbers or objects. However, the
Intermediate Value is the basis for a method commonly used to approximate the roots of continuous
functions, the Bisection Algorithm.

Bisection Algorithm for Finding a Root of f(x)

(i)   Find two values of x , say a and b, so that f(a) and f(b) have opposite signs

     (then   f(x) has a root between a and b, a root in the interval [a,b].)

(ii)   Calculate the midpoint (bisection point) of the interval [a,b], m = (a+b)/2, and evaluate f(m).

(iii)   (a) If f(m) = 0, then   m is a root of f , and we are done.

     (b) If f(m) ≠ 0, then f(m) has the sign opposite one of f(a) or f(b):

         if f(a) and f(m) have opposite signs, then f has a root in [a,m] so put b = m

         if f(b) and f(m) have opposite signs, then f has a root in [m,b] so put a = m

(iv)   Repeat steps (ii) and (iii) until a root is found exactly or is approximated closely enough.

The length of the interval known to contain a root is cut in half each time through steps (ii) and (iii) so the Bisection Algorithm quickly "squeezes" in on a root (Fig. 10).

Fig. 10

The steps of the Bisection Algorithm can be done "by hand", but it is tedious to do very many of them that way. Computers are very good with this type of tedious repetition, and the algorithm is simple to program.

Example 7:   Find a root of $f(x) = x - x^3 + 1$.

Solution: f(0) = 1 and f(1) = 1 so we cannot conclude that f has a root between 0 and 1. f(1) = 1 and f(2) = –5 have opposite signs, so by the Intermediate Value Property of continuous functions (this function is a polynomial so it is continuous everywhere) we know that there is a number c between 1 and 2 such that f(c) = 0 (Fig. 11). The midpoint of the interval [1,2] is m = (1+2)/2 = 3/2 = 1.5 , and f(3/2) = –7/8 so f changes sign between 1 and 1.5 and we can be sure that there is a root between 1 and 1.5 . If we repeat the operation for the interval [1, 1.5], the midpoint is   m = (1+1.5)/2 = 1.25, and f(1.25) = 19/64 > 0 so f changes sign between 1.25 and 1.5 and we know f has a root between 1.25 and 1.5 .
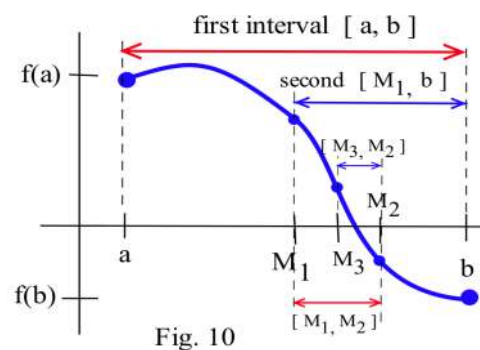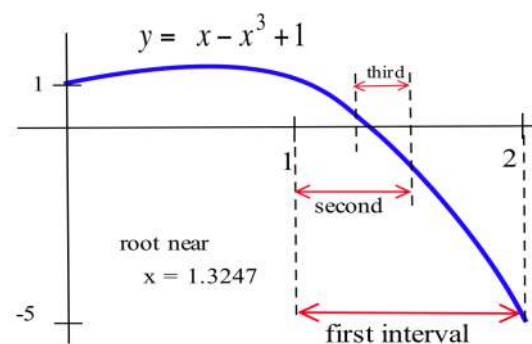
Fig. 11

Repeating this procedure a few more times, we get that

a	b	m = (b+a)/2	f(a)	f(b)	f(m)	root between	
1	2		1	–5		1	2
1	2	1.5	1	–5	–0.875	1	1.5
1	1.5	1.25	1	–0.875	0.2969	1.25	1.5
1.25	1.5	1.375	0.2969	–0.875	–0.2246	1.25	1.375
1.25	1.375	1.3125	0.2969	–0.2246	0.0515	1.3125	1.375
1.3125	1.375	1.34375					

If we continue the table, the interval containing the root will squeeze around the value 1.324718 .

The Bisection Algorithm has one major drawback — there are some roots it does not find. The algorithm
requires that the function be both positive and negative near the root so that the graph actually <u>crosses</u> the
x–axis. The function $f(x) = x^2 - 6x + 9 = (x - 3)^2$ has the root x = 3 but is never negative (Fig. 12).

We cannot find two starting points a and b so that f(a) and f(b) have
opposite signs, and we cannot use the Bisection Algorithm to find the root
x = 3. In Chapter 2 we will see another method, Newton's Method, which
does find roots of this type.
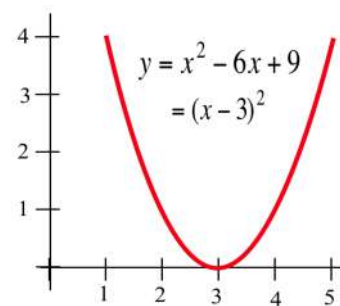
The Bisection Algorithm requires that we supply two starting points a and
b, two x–values at which the function has opposite signs. These values can
often be found with a little "trial and error", or we can examine the graph of
the function and use it to help pick the two values.

$$y = x^2 - 6x + 9$$
$$= (x - 3)^2$$

Fig. 12

Finally, the Bisection Algorithm can also be used to solve equations because the solution of any equation can
always be transformed into an equivalent problem of finding roots by moving everything to one side of the equal
sign. For example, the problem of solving the equation $x^3 = x + 1$ can be transformed into the equivalent
problem of solving $x + 1 - x^3 = 0$ or of finding the roots of $f(x) = x + 1 - x^3$ which we did in the previous
example.

Example 8:   Find all of the solutions of $\sin(x) = \dfrac{2x + 1}{x - 2}$. (x is in radians.)

Solution: We can convert this problem of solving an equation to the problem of finding the roots of

$f(x) = \sin(x) - \dfrac{2x + 1}{x - 2} = 0$. The function f(x) is continuous everywhere except at x = 2, and the

graph of f(x) in Fig. 13 can help us find two starting values for the Bisection Algorithm. The graph

shows that f(–1) is negative and f(0) is positive, and we know f(x) is continuous on the interval

[–1,0]. Using the algorithm with the starting interval [–1,0], we get that the root is contained in the

shrinking intervals:

$[-.5,0], [-.25,0], [-.25, -.125], \ldots ,$

$[-.238281, -.236328] , \ldots , [-.237176, -.237177]$

so the root is approximately $-.237177$.

We might also notice that $f(0) = 0.5$ is positive and

$f(\pi) = 0 - \dfrac{2\pi + 1}{\pi - 2} \approx -6.38$ is negative. Why is it

wrong to conclude that $f(x)$ has another root between

$x = 0$ and $x = \pi$?

$y = \sin(x) - \dfrac{2x + 1}{x - 2}$ root
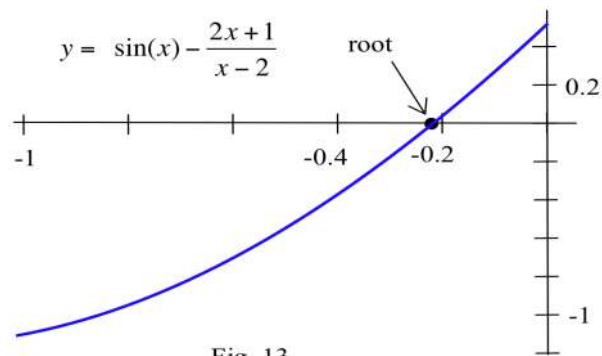
Fig. 13

PROBLEMS

1. At which points is the function in Fig. 14 discontinuous?

2. At which points is the function in Fig. 15 discontinuous?
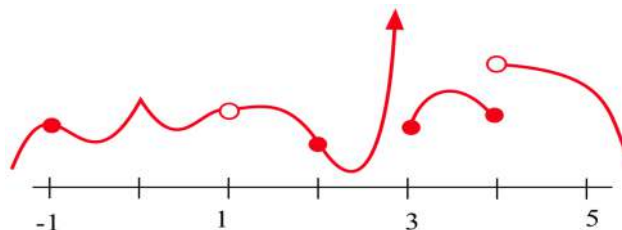
Fig. 14

3. Find at least one point at which each function is not continuous and state which of the 3 conditions in the definition of continuity is violated at that point.
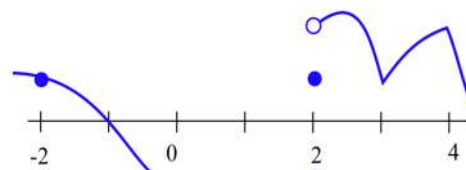
Fig. 15

(a) $\dfrac{x + 5}{x - 3}$

(b) $\dfrac{x^2 + x - 6}{x - 2}$

(c) $\sqrt{\cos(x)}$

(d) $INT(x^2)$

(e) $\dfrac{x}{\sin(x)}$

(f) $\dfrac{x}{x}$

(g) $\ln(x^2)$

(h) $\dfrac{\pi}{x^2 - 6x + 9}$

(i) $\tan(x)$

4. Which <u>three</u> of the following functions are not continuous. Use the appropriate theorems of this section to justify that each of the other functions is continuous.

(a) $\dfrac{7}{\sqrt{2 + \sin(x)}}$

(b) $\cos(x^5 - 7x + \pi)$

(c) $\dfrac{x^2 - 5}{1 + \cos^2(x)}$

(d) $\dfrac{x^2 - 5}{1 + \cos(x)}$

(e) $INT(3 + 0.5\sin(x))$

(f) $INT(0.3\sin(x) + 1.5)$

(g) $\sqrt{\cos(\sin(x))}$

(h) $\sqrt{x^2 - 6x + 10}$

(i) $\sqrt[3]{\cos(x)}$

(j) $2^{\sin(x)}$

(k) $\log(|x|)$

(l) $1 - 3^{-x}$

5. A continuous function f has the values given below:

x	0	1	2	3	4	5
f(x)	5	3	−2	−1	3	−2

(a) f has at least ___ roots between 0 and 5.   (b) f(x) = 4 at least ___ times between 0 and 5.

(c) f(x) = 2 at least ___ times between 0 and 5.   (d) f(x) = 3 at least ___ times between 0 and 5.

(e) Is it possible for f(x) to equal 7 for some x values between 0 and 5?

6. A continuous function g has the values given below:

x	1	2	3	4	5	6	7
g(x)	−3	1	4	−1	3	−2	−1

(a) g has at least ___ roots between 1 and 5.   (b) g(x) = 3.2 at least ___ times between 1 and 7.

(c) g(x) = −0.7 at least ___ times between 3 and 7.   (d) g(x) = 1.3 at least ___ times between 2 and 6.

(e) Is it possible for g(x) to equal π for some value(s) of x between 5 and 6?

7. This problem asks you to verify that the Intermediate Value Theorem is true for some particular functions,

intervals and intermediate values. In each problem you are given a function f , an interval [a,b] and a value

V. Verify that V is between f(a) and f(b) and find a value of c in the interval so that f(c) = V.

(a) $f(x) = x^2$ on [0,3], V = 2.   (b) $f(x) = x^2$ on [−1,2], V = 3.

(c)   f(x) = sin(x) on [0,π/2], V = 1/2 .   (d)   f(x) = x on [0,1], V = 1/3.

(e) $f(x) = x^2 − x$ on [2,5], V = 4.   (f)   f(x) = ln(x) on [1,10], V = 2.

8. Two students claim that they both started with the points x = 1
    and x = 9 and applied the Bisection Algorithm to the function in
    Fig. 16. The first student says that the algorithm converged to
    the root near x = 8, but the second claims that the algorithm will
    converge to the root near x = 4. Who is right?
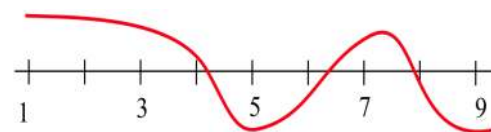
Fig. 16

9. Two students claim that they both started with the points x = 0
    and x = 5 and applied the Bisection Algorithm to the function
    in Fig. 17. The first student says that the algorithm converged to
    the root labeled A, but the second claims that the algorithm will
    converge to the root labeled B. Who is right?

Fig. 17

10. If you apply the Bisection Algorithm to the function in

    Fig. 18 and use the given starting points, which root does

    the algorithm find?   (a) starting points 0 and 9.

    (b) starting points 1 and 5.   (c) starting points 3 and 5.

Fig. 18

11. If you apply the Bisection Algorithm to the function in Fig. 19 and

    use the given starting points, which root does the algorithm find?

    (a) starting points 3 and 7.   (b) starting points 4 and 6.

    (c) starting points 1 and 6.

Fig. 19

In problems 12 – 17 , use the Intermediate Value Theorem to verify that each function has a root in the

given interval(s). Then use the Bisection Algorithm to narrow the location of that root to an interval of

length less than or equal to 0.1 .

12. $f(x) = x^2 - 2$ on $[\,0, 3\,]$.

13. $g(x) = x^3 - 3x^2 + 3$ on $[\,-1, 0\,]$, $[\,1, 2\,], [\,2, 4\,]$.

14. $h(t) = t^5 - 3t + 1$ on $[\,1, 3\,]$.

15. $r(x) = 5 - 2^x$ on $[\,1, 3\,]$.

16. $s(x) = \sin(2x) - \cos(x)$ on $[\,0, \pi]$

17. $p(t) = t^3 + 3t + 1$ on $[\,-1, 1]$

18. What is wrong with this reasoning: "If $f(x) = 1/x$ then $f(-1) = -1$ and $f(1) = 1$. Because $f(-1)$ and

    $f(1)$ have opposite signs, f has a root between $x = -1$ and $x = 1$."

19. Each of the following statements is false for some functions. For each statement, sketch the graph of

    a counterexample.

    a)   If $f(3) = 5$ and $f(7) = -3$, then f has a root between $x = 3$ and $x = 7$.

    b)   If f has a root between $x = 2$ and $x = 5$, then $f(2)$ and $f(5)$ have opposite signs.

    c)   If the graph of a function has a sharp corner, then the function is not continuous there.

20. Define $A(x)$ to be the **area** bounded by the x and y axes, the curve $y = f(x)$,

    and the vertical line at x (Fig. 20). From the figure, it

    is clear that $A(1) < 3$ and $A(3) > 3$.

    Do you think there is a value of x, between 1 and 3,

    so $A(x) = 3$? If so, justify your conclusion and estimate

    the location of the value of x so $A(x) = 3$.

    If not, justify your conclusion.

Fig. 20

21. Define A(x) to be the **area** bounded by the x and y axes, the curve

y = f(x), and the vertical line at x (Fig. 21).

a) Shade the part of the graph represented

by A(2.1) – A(2) and

estimate the value of $\dfrac{A(2.1) - A(2)}{0.1}$.

b) Shade the part of the graph represented by

A(4.1) – A(4) and

estimate the value of $\dfrac{A(4.1) - A(4)}{0.1}$.
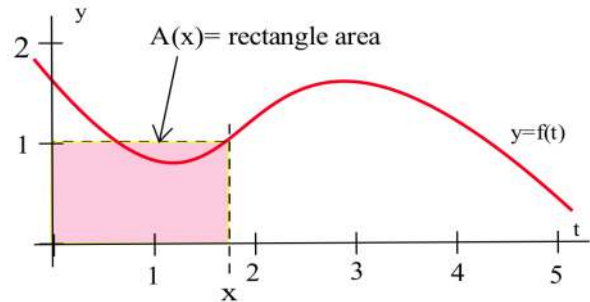
Fig. 21

22. (a) A square sheet of paper has a straight line drawn on it from the lower left corner to the upper right

corner. Is it possible for you to start on the left edge of the sheet and draw a connected line to the

right edge that does not cross the diagonal line?

(b) Prove: If f is continuous on the interval [0,1] and $0 \le f(x) \le 1$ for all x, then there is a number

$c, 0 \le c \le 1$, such that f(c) = c. (The number c is called a "fixed point" of f because the image

of c is the same as c — f does not move c.)

Hint: Define a new function g(x) = f(x) – x and start by considering the values g(0) and   g(1).

(c) What has part (b) got to do with part (a) of this problem?

(d) Is the theorem in part (b) true if we replace the closed interval [0,1] with the open interval (0,1)?

True/False: "If f is continuous on the interval (0,1) and $0 < f(x) < 1$ for all x, then there is a

number $c, 0 < c < 1$, such that f(c) = c."

23. A piece of string is tied in a loop and tossed onto quadrant I enclosing

a single region (Fig. 22).

(a) Is it always possible to find a line L which goes through the origin

so that L divides the region into two equal areas? (Justify your

answer.)

(b) Is it always possible to find a line L which is parallel to the

x–axis so that L divides the region into two equal areas?

(Justify your answer.)

(c) Is it always possible to find 2 lines, L parallel to the x–axis and

M parallel to the y–axis, so L and M divide the region into 4 equal areas? (Justify your answer.)
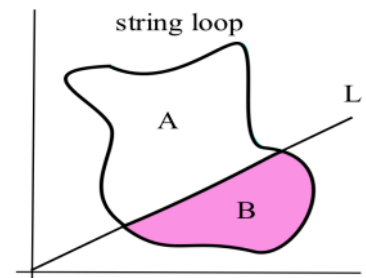
Fig. 22

Section 2.4   **PRACTICE Answers**

Practice 1: $f(x) = \dfrac{|x|}{x}$ (Fig. 23) is continuous everywhere **except at x = 0** where this function

is not defined.

If $a > 0$, then $\lim\limits_{x \to a} \dfrac{|x|}{x} = 1 = f(a)$ so f is continuous at a.

If $a < 0$, then $\lim\limits_{x \to a} \dfrac{|x|}{x} = -1 = f(a)$ so f is continuous at a.

f(0) is not defined., $\lim\limits_{x \to 0^-} \dfrac{|x|}{x} = -1$ and $\lim\limits_{x \to 0^+} \dfrac{|x|}{x} = +1$ so

$\lim\limits_{x \to 0} \dfrac{|x|}{x}$ does not exist.
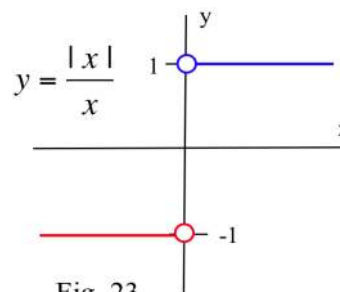
Fig. 23

Practice 2:   (a)   To prove that kf is continuous at a, we need to prove that kf satisfies

the definition of continuity at a: $\lim\limits_{x \to a} kf(x) = kf(a)$.

Using results about limits, we know

$\lim\limits_{x \to a} kf(x) = k \lim\limits_{x \to a} f(x) = k \, f(a)$ (since f is assumed to be

continuous at a) so   kf is continuous at a.

(b)   To prove that f – g is continuous at a, we need to prove that f – g satisfies

the definition of continuity at a: $\lim\limits_{x \to a} (f(x) - g(x)) = f(a) - g(a)$.

Again using information about limits,

$\lim\limits_{x \to a} (f(x) - g(x)) = \lim\limits_{x \to a} f(x) - \lim\limits_{x \to a} g(x) = f(a) - g(a)$ (since f and g are

both continuous at a) so f – g is continuous at a.

2.5   DEFINITION OF LIMIT

It may seem strange that we have been using and calculating the values of limits for awhile without having a precise definition of limit, but the history of mathematics shows that many concepts, including limits, were successfully used before they were precisely defined or even fully understood. We have chosen to follow the historical sequence in this chapter and to emphasize the intuitive and graphical meaning of limit because most students find these ideas and calculations easier than the definition. Also, this intuitive and graphical understanding of limit was sufficient for the first hundred years of the development of calculus (from Newton and Leibniz in the late 1600's to Cauchy in the early 1800's), and it is sufficient for using and understanding the results in beginning calculus.

Mathematics, however, is more than a collection of useful tools, and part of its power and beauty comes from the fact that in mathematics terms are precisely defined and results are rigorously proved. Mathematical tastes (what is mathematically beautiful, interesting, useful) change over time, but because of these careful definitions and proofs, the results remain true, everywhere and forever. Textbooks seldom give all of the definitions and proofs, but it is important to mathematics that such definitions and proofs exist.

The goal of this section is to provide a precise definition of the limit of a function. The definition will not help you calculate the values of limits, but it provides a precise statement of what a limit is. The definition of limit is then used to verify the limits of some functions, and some general results are proved.

The Intuitive Approach

The precise ("formal") definition of limit carefully defines the ideas that we have already been using graphically and intuitively. The following side–by–side columns show some of the phrases we have been using to describe limits, and those phrases, particularly the last ones, provide the basis to building the definition of limit.

<u>A Particular Limit</u>

$$\lim_{x \to 3} 2x - 1 = 5$$

<u>General Limit</u>

$$\lim_{x \to a} f(x) = L$$

"as the values of x approach 3, the values of 2x–1 approach (are arbitrarily close to) 5"	"as the values of x approach a, the values of f(x) approach (are arbitrarily close to) L"
"when x is close to 3 (but not equal to 3), the value of 2x–1 is close to 5"	"when x is close to a (but not equal to a), the value of f(x) is close to L"
"we can guarantee that the values of f(x) = 2x–1 are as close to 5 as we want by starting with values of x sufficiently close to 3 (but not equal to 3)"	"we can guarantee that the values of f(x) are as close to L as we want by starting with values of x sufficiently close to a (but not equal to a)"

Let's examine what the last phrase ("we can ...") means for the Particular Limit.

Example 1: We know $\lim_{x \to 3} 2x - 1 = 5$. Show that we can guarantee that the values of $f(x) = 2x - 1$ are as close to 5 as we want by starting with values of x sufficiently close to 3.

    (a)   What values of x guarantee that $f(x) = 2x - 1$ is within 1 unit of 5 ? (Fig. 1a)

Solution: "within 1 unit of 5" means between $5-1 = 4$ and $5+1 = 6$, so the question can be rephrased as "for what values of x is $y = 2x - 1$ between 4 and 6: $4 < 2x - 1 < 6$?" We want to know which values of x put the values of $y = 2x - 1$ into the shaded band in Fig. 1a. The algebraic process is straightforward: solve $4 < 2x - 1 < 6$ for x to get $5 < 2x < 7$ and $2.5 < x < 3.5$. We can restate this result as follows: "If x is within **0.5** units of 3, then $y = 2x-1$ is within 1 unit of 5." (Fig. 1b)

Any smaller distance also satisfies the guarantee: e.g., "If x is within **0.4** units of 3, then $y = 2x-1$ is within 1 unit of 5." (Fig. 1c)
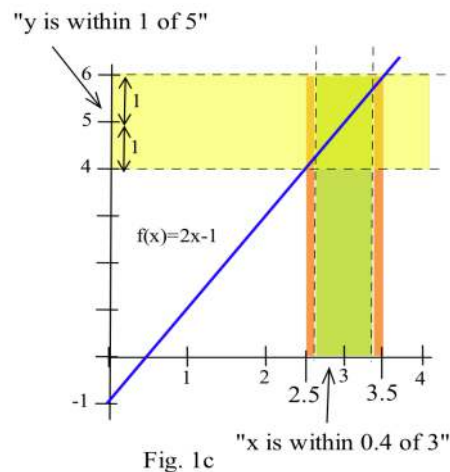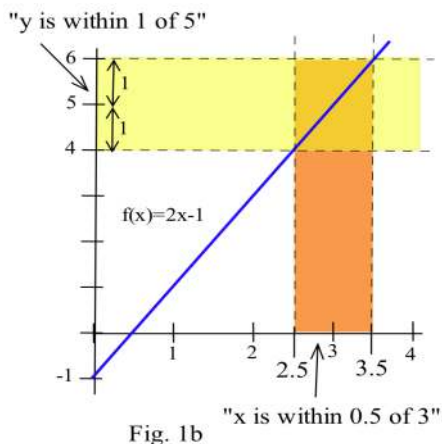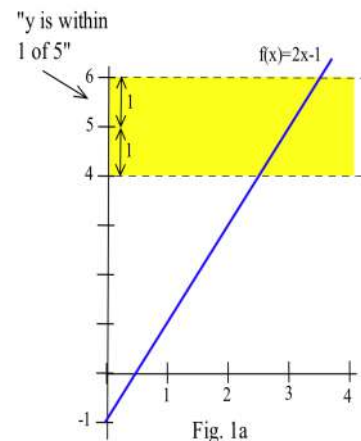
(b)   What values of x guarantee the f(x) = 2x – 1 is within
       0.2 units of 5? (Fig. 2a)

Solution: "within 0.2 units of 5" means between 5–0.2 = 4.8 and
   5+0.2 = 5.2, so the question can be rephrased as "for what values of
   x is y = 2x – 1 between 4.8 and 5.2: 4.8 < 2x – 1 < 5.2?"
   Solving for x, we get 5.8 < 2x < 6.2 and 2.9 < x < 3.1. "If x is
   within 0.1 units of 3, then y = 2x–1 is within 0.2 units of 5."
   (Fig. 2b) Any smaller distance also satisfies the guarantee.

Rather than redoing these calculations for every possible distance from 5,
we can do the work once, generally:

(c)   What values of x guarantee that f(x) = 2x – 1 is within
       E units of 5? (Fig. 3a)

Solution: "within E unit of 5" means between 5–E and 5+E , so the
   question is "for what values of x is y = 2x – 1 between 5–E and
   5+ε: 5–E < 2x – 1 < 5+E?" Solving 5–E < 2x – 1 < 5+E for x, we
   get 6–E < 2x < 6+E and 3 – E/2 < x < 3 + E/2. "If x is within E/2
   units of 3, then y = 2x–1 is within E units of 5." (Fig. 3b) Any
   smaller distance also satisfies the guarantee.

Part (c) of Example 1 illustrates a little of the power of general solutions
in mathematics. Rather than doing a new set of similar calculations every
time someone demands that f(x) = 2x – 1 be within some given distance
of 5, we did the calculations once. And then we can respond for any given
distance. For the question "What values of x guarantee that f(x) = 2x – 1
is within 0.4, 0.1 and 0.006 units of 5?", we can answer "If x is within
0.2 (= 0.4/2), 0.05 (=0.1/2) and 0.003 (=0.006/2) units of 3."

Practice 1: $\lim_{x \to 2} 4x - 5 = 3$. What

values of x guarantee that

f(x) = 4x – 5 is within

(a) 1 unit of 3?

(b) 0.08 units of 3?

(c) E units of 3? (Fig. 4)

The same ideas work even if the graphs of the functions are not straight lines, but the calculations are more complicated.

Example 2: $\lim\limits_{x \to 2} x^2 = 4$. (a) What values of x   guarantee that

$f(x) = x^2$ is within 1 unit of 4? (b) Within 0.2 units of 4?

(Fig. 5a) State each answer in the form "If x is within _____

units   of 2, then f(x) is within 1 (or 0.2) unit of 4."

"y is within 1 of 4" $f(x) = x^2$

Fig. 5a   2
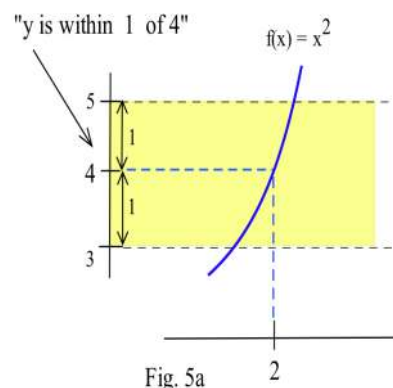
Solution: (a)   If x^2 is within 1 unit of 4, then $3 < x^2 < 5$ so $\sqrt{3} < x < \sqrt{5}$

or 1.732 < x < 2.236. The interval containing these x values

extends from $2 - \sqrt{3} \approx 0.268$ units to the left of 2 to

$\sqrt{5} - 2 \approx 0.236$ units to the right of 2. Since we want to

specify a single distance on each side of 2, we can pick the

smaller of the two distances, 0.236 . (Fig. 5b)
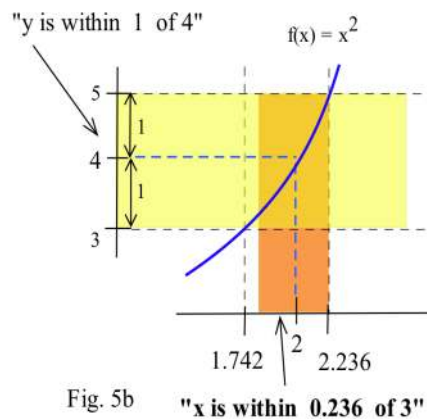
    "If x is within __0.236__ units of 2,

    then f(x) is within 1 unit of 4."

"y is within 1 of 4" $f(x) = x^2$

1.742   2   2.236

Fig. 5b   **"x is within 0.236 of 3"**

(b) Similarly, if x^2 is within 0.2 units of 4, then $3.8 < x^2 < 4.2$

so $\sqrt{3.8} < x < \sqrt{4.2}$ or 1.949 < x < 2.049. The interval

containing these x values extends from $2 - \sqrt{3.8} \approx 0.051$ units to the left of 2 to

$\sqrt{4.2} - 2 \approx 0.049$ units to the right of 2. Again picking the smaller of the two distances, "If x is

within __0.049__ units of 2, then f(x) is within 1 unit of 4."

The situation in Example 2 of different distances on the left and right sides is very common, and we **always** pick our single distance to be the **smaller** of the distances to the left and right. By using the smaller distance, we can be certain that if x is within that smaller distance on either side, then the value of f(x) is within the specified distance of the value of the limit.

Practice 2: $\lim\limits_{x \to 9} \sqrt{x} = 3$. What values of x guarantee that f(x)

$= \sqrt{x}$ is within 1 unit of 3? Within 0.2 units of 3?

(Fig. 6) State each answer in the form

"If x is within _____ units of 2, then f(x)

is within 1 (or 0.2) unit of 4."

"y is within 1 of 3" $f(x) = \sqrt{x}$

Fig. 6   9
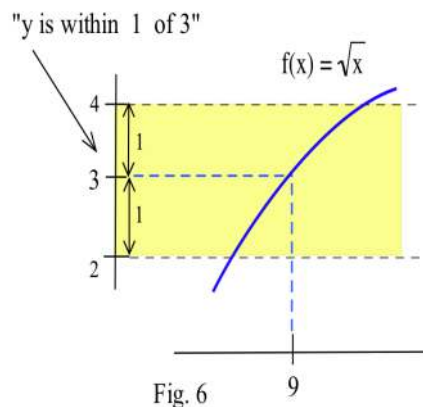
The same ideas can also be used when the function and the specified distance are given graphically, and in that case we can give the answer graphically.

Example 3:   In Fig. 7, $\lim\limits_{x \to 2} f(x) = 3$. What values of x guarantee that

y = f(x) is within E units (given graphically) of 3? State your

answer in the form "If x is within _____ (show a distance D

graphically) of 2, then f(x) is within E units of 3."

Fig. 7
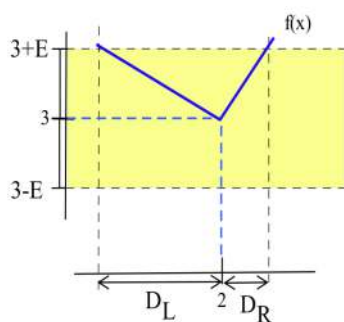
Solution: The solution process requires several steps as illustrated in Fig. 8:

i.   Use the given distance E to find the values 3 – E and 3 + E on the y–axis.

ii.   Sketch the horizontal band which has its lower edge at y = 3 – E and

its upper edge at y = 3 + E.

iii. Find the first locations to the right and left of x = 2 where the

graph of y = f(x) crosses the lines y = 3 – E and y = 3 + E,

and at these locations draw vertical lines to the x–axis.

iv.   On the x–axis, graphically determine the distance from 2 to

the vertical line on the left (labeled D_L) and from 2 to the vertical

line on the right (labeled D_R).

v. Let the length D be the smaller of the lengths D_L and D_R .

Fig. 8a: steps i and ii

Fig. 8b: steps iii and iv

$D_L = $ ⟵——⟶
$D_R = $ ⟷
so D = ⟷
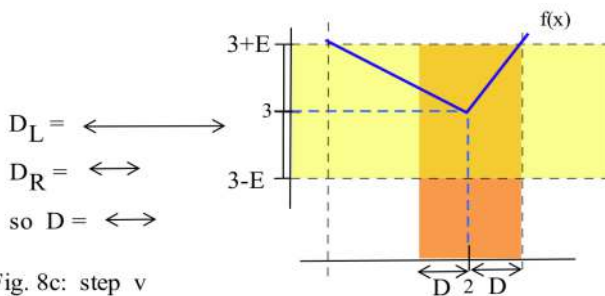
Fig. 8c: step v

Practice 3:   In Fig. 9, $\lim\limits_{x \to 3} f(x) = 1.8$. What values

of x guarantee that y = f(x) is within

E units of 1.8?
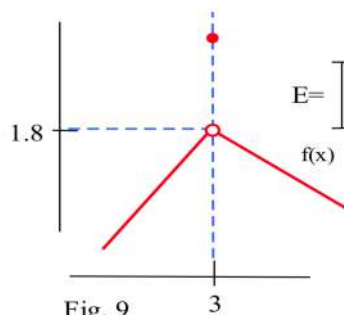
Fig. 9

The Formal Definition of Limit

The ideas of the previous examples and practice problems can be stated for general functions and limits, and they provide the basis for the definition of limit which is given in the box. The use of the lower case Greek letters **ε** (epsilon) and **δ** (delta) in the definition is standard, and this definition is sometimes called the "epsilon–delta" definition of limit.

> **Definition of** $\lim\limits_{x \to a} f(x) = L$**:**
>
> $\lim\limits_{x \to a} f(x) = L$ means
>
> for every given $\varepsilon > 0$ there is a $\delta > 0$ so that (Fig. 10)
>
> if   x   is within δ units of a (and $x \neq a$)
>
> then   f(x)   is within ε units of L.
>
> (Equivalently: $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.)

In this definition, ε represents the given distance on either side of the limiting value $y = L$, and δ is the distance on each side of the point $x = a$ on the x–axis that we have been finding in the previous examples. This definition has the form of a a "challenge and reponse:" for any positive challenge ε (make f(c) within ε of L), there is a positive response δ (start with x within δ of a and $x \neq a$).
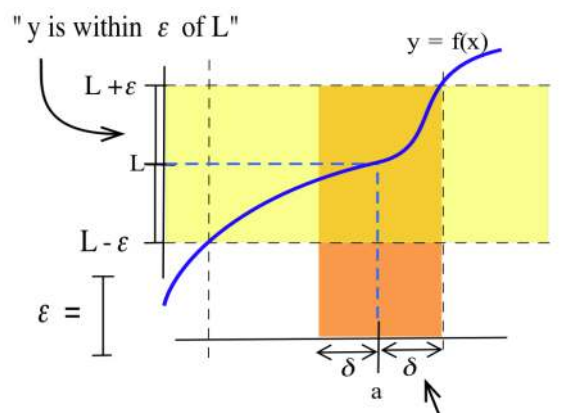
Fig. 10

Example 4:   In Fig. 11a, $\lim\limits_{x \to a} f(x) = L$, and a value for ε is given graphically as a length. Find a length for δ that satisfies the definition of limit (so "if x is within δ of a (and $x \neq a$), then f(x) is within ε of L").
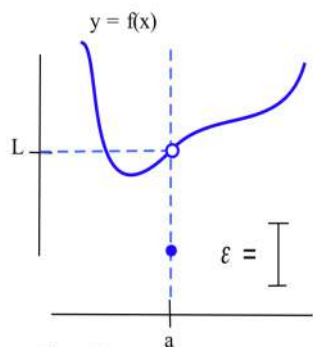
Solution: Follow the steps outlined in Example 3. The length for δ is shown in Fig. 11b, and any shorter length for δ also satisfies the definition.

Fig. 11a

Fig. 11b   "x is within δ of a"
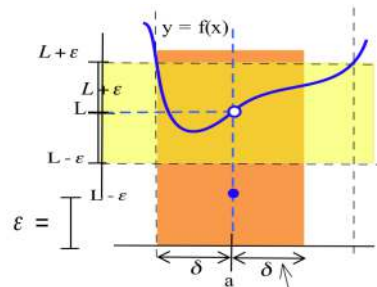
Practice 4:   In Fig. 12, $\displaystyle\lim_{x \to a} f(x) = L$, and a value for ε is given

graphically as a length. Find a length for δ that

satisfies the definition of limit
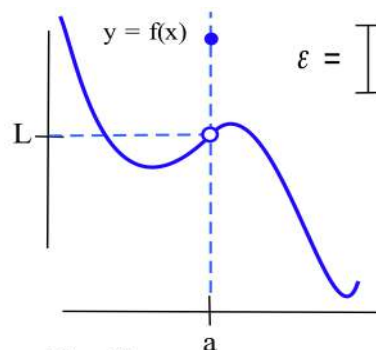
Example 5:   Prove that $\displaystyle\lim_{x \to 3} 4x - 5 = 7$.

Solution: We need to show that

"for every given $\varepsilon > 0$ there is a $\delta > 0$ so that

if   x   is within δ units of 3 (and x ≠ 3)

then   4x – 5   is within ε units of 7."

Actually there are two things we need to do. First, we need to find a value for δ (typically depending on

ε), and, second, we need to show that our δ really does satisfy the "if – then" part of the definition.

i.   Finding δ is similar to part (c) in Example 1 and Practice 1: assume 4x – 5 is within ε units of

7 and solve for x. If $7 - \varepsilon < 4x - 5 < 7 + \varepsilon$, then $12 - \varepsilon < 4x < 12 + \varepsilon$ and

$3 - \varepsilon/4 < x < 3 + \varepsilon/4$, so x is within $\varepsilon/4$ units of 3. Put $\delta = \varepsilon/4$.

ii.   To show that $\delta = \varepsilon/4$ satisfies the definition, we merely reverse the order of the steps in part i.

Assume that x is within δ units of 3. Then $3 - \delta < x < 3 + \delta$ so

$3 - \varepsilon/4 < x < 3 + \varepsilon/4$ (replacing δ with $\varepsilon/4$),

$12 - \varepsilon < 4x < 12 + \varepsilon$ (multiplying by 4), and

$7 - \varepsilon < 4x - 5 < 7 + \varepsilon$ (subtracting 5), so

we can conclude that f(x) = 4x – 5 is within ε units of 7. This formally verifies that $\displaystyle\lim_{x \to 3} 4x - 5 = 7$.

Practice 5:   Prove that $\displaystyle\lim_{x \to 4} 5x + 3 = 23$.

The method used to prove the values of the limits for these particular linear functions can also be used to

prove the following general result about the limits of linear functions.

Theorem: $\displaystyle\lim_{x \to a} mx + b = ma + b$

Proof: Let f(x) = mx + b.

Case 1: m = 0. Then f(x) = 0x + b = b is simply a constant function, and any value for $\delta > 0$ satisfies

the definition. Given any value of $\varepsilon > 0$, let $\delta = 1$ (any positive value for δ works). If x is

is within 1 unit of a, then f(x) – f(a) = b – b = 0 < e, so we have shown that for any

$\varepsilon > 0$, there is a $\delta > 0$ which satisfies the definition.

Case 2: m ≠ 0. Then f(x) = mx + b. For any $\varepsilon > 0$, put $\delta = \frac{\varepsilon}{|m|} > 0$. If x is within $\delta = \frac{\varepsilon}{|m|}$ of a, then

$a - \frac{\varepsilon}{|m|} < x < a + \frac{\varepsilon}{|m|}$ so $-\frac{\varepsilon}{|m|} < x - a < \frac{\varepsilon}{|m|}$ and $|x - a| < \frac{\varepsilon}{|m|}$.

Then the distance between f(x) and L = ma + b is

$$| f(x) - L | = | (mx + b) - (ma + b) | = |m| \cdot | x - a | < |m| \frac{\varepsilon}{|m|} = \varepsilon$$

so f(x) is within ε

of L = ma + b. (Fig. 13)

In each case, we have shown that "given any ε > 0, there is a

δ > 0" that satisfies the rest of the definition is satisfied.
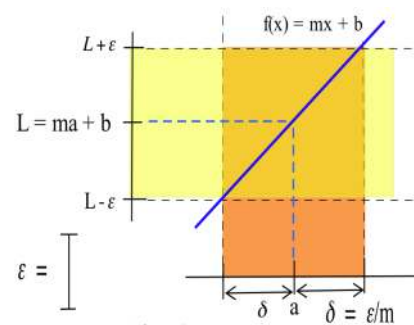
Fig. 13

If there is even a single value of ε for which there is no δ, then the function

does not satisfy the difinition, and we say that the limit "does not exist.:

Example 6:   Let $f(x) = \begin{cases} 2 & \text{if } x < 1 \\ 4 & \text{if } x > 1 \end{cases}$ as is shown in Fig. 14 .

Use the definition to prove that $\lim_{x \to 1} f(x)$ does not exist.

Fig. 14

Solution: One common proof technique in mathematics is called "proof by

contradiction," and that is the method we use here. Using that method in this

case, (i) we assume that the limit does exist and equals some number L, (ii) we show that this assumption

leads to a contradiction, and (iii) we conclude that the assumption must have been false. Therefore, we

conclude that the limit does not exist.
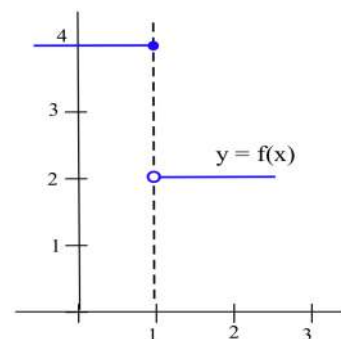
(i) Assume that the limit exists: $\lim_{x \to 1} f(x) = L$ for some value for L. Let $\varepsilon = \frac{1}{2}$. (The definition says

"for every ε" so we can pick this value. Why we chose this value for ε shows up later in the proof.)

Then, since we are assuming that the limit exists, there is a δ > 0 so that if x is within δ of 1 then

f(x) is within ε of L.

(ii) Let x_1 be between 1 and 1 + δ. Then $x_1 > 1$ so $f(x_1) = 4$. Also, x_1 is within δ of 1 so $f(x_1) =$

4 is within $\frac{1}{2}$ of L, and L is between 3.5 and 4.5: **3.5 < L < 4.5**.

Let x_2 be between 1 and 1 − δ. Then $x_2 < 1$ so $f(x_2) = 2$. Also, x_2 is within δ of 1 so $f(x_2) =$

2 is within $\frac{1}{2}$ of L, and L is between 1.5 and 2.5: **1.5 < L < 2.5**.

(iii)   The two inequalities in bold print provide the contradiction we were hoping

to find. There is no value L that simultaneously satisfies 3.5 < L < 4.5 **and** 1.5 < L < 2.5, so we

can conclude that our assumption was false and that f(x) does not have a limit as x → 1.

Practice 6:   Use the definition to prove that $\lim\limits_{x \to 0} \dfrac{1}{x}$ does not exist (Fig. 15).
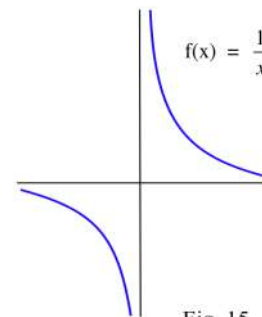
Proofs of Two Limit Theorems

The theorems and their proofs are included here so you can see how such
proofs proceed — you have already used these theorems to evaluate limits
of functions.. There are rigorous proofs of all of the other limit properties,
but they are somewhat more complicated than the proofs given here.

f(x) = $\dfrac{1}{x}$

Fig. 15

Theorem: If $\lim\limits_{x \to a} f(x) = L$, then $\lim\limits_{x \to a} k{\cdot}f(x) = k{\cdot}L$.

Proof: Case k = 0: The Theorem is true but not very interesting: $\lim\limits_{x \to a} 0{\cdot}f(x) = \lim\limits_{x \to a} 0 = 0{\cdot}L$.

Case $k \neq 0$: Since $\lim\limits_{x \to a} f(x) = L$, then, by the definition, for every $\varepsilon > 0$ there is a $\delta > 0$ so that

$|\,f(x) - L\,| < \varepsilon$ whenever $|\,x - a\,| < \delta$. For any $\varepsilon > 0$, we know $\dfrac{\varepsilon}{|k|} > 0$ and pick a value of

δ that satisfies $|\,f(x) - L\,| < \dfrac{\varepsilon}{|k|}$ whenever $|\,x - a\,| < \delta$. When

$|\,x - a\,| < \delta$ ("x is within δ of a") then

$|\,f(x) - L\,| < \dfrac{\varepsilon}{|k|}$ ("f(x) is within $\dfrac{\varepsilon}{|k|}$ of L")   so

$|k|{\cdot}|\,f(x) - L\,| < \varepsilon$ (multiplying each side by $|k| > 0$) and

$|\,k{\cdot}f(x) - k{\cdot}L\,| < \varepsilon$ (k${\cdot}$f(x) is within ε of k${\cdot}$L).

Theorem: If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$, then $\lim\limits_{x \to a} f(x) + g(x) = L + M$.

Proof: Assume that $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$. Then, given any $\varepsilon > 0$, we know $\varepsilon/2 > 0$ and

that there are deltas for f and g , δ_f and δ_g , so that

if $|\,x - a\,| < \delta_f$, then $|\,f(x) - L\,| < \varepsilon/2$ ("if x is within δ_f of a, then f(x) is within $\varepsilon/2$ of L", and

if $|\,x - a\,| < \delta_g$, then $|\,g(x) - M\,| < \varepsilon/2$ ("if x is within δ_g of a, then g(x) is within $\varepsilon/2$ of M").

Let δ be the smaller of δ_f and δ_g. If $|\,x - a\,| < \delta$, then $|\,f(x) - L\,| < \varepsilon/2$ and $|\,g(x) - M\,| < \varepsilon/2$ so

$|\,(f(x) + g(x)) - (L + M))\,| = |\,(f(x) - L) + (g(x) - M)\,|$ (rearranging the terms)

$\leq |\,f(x) - L\,| + |\,g(x) - M\,|$ (by the Triangle Inequality for absolute values)

$< \dfrac{\varepsilon}{2} + \dfrac{\varepsilon}{2} = \varepsilon$.   (by the definition of the limits for f and g).

Problems

In problems 1–4, state each answer in the form "If x is within _____ units of . . . "

1. $\lim\limits_{x \to 3} 2x + 1 = 7$. What values of x guarantee that $f(x) = 2x + 1$ is (a) within 1 unit of 7?

    (b) within 0.6 units of 7?   (c) within 0.04 units of 7?   (d) within ε units of 7?

2. $\lim\limits_{x \to 1} 3x + 2 = 5$. What values of x guarantee that $f(x) = 3x + 2$ is within 1 unit of 5 ?

    (b) within 0.6 units of 5?   (c) within 0.09 units of 5?   (d) within ε units of 5?

3. $\lim\limits_{x \to 2} 4x - 3 = 5$. What values of x guarantee that $f(x) = 4x - 3$ is within 1 unit of 5 ?

    (b) within 0.4 units of 5?   (c) within 0.08 units of 5?   (d) within ε units of 5?

4. $\lim\limits_{x \to 1} 5x - 3 = 2$. What values of x guarantee that $f(x) = 5x - 3$ is within 1 unit of 2 ?

    (b) within 0.5 units of 5?   (c) within 0.01 units of 5?   (d) within ε units of 5?

5.   For problems 1 – 4, list the slope of each function f and the δ (as a function of ε). For these linear

    functions f, how is   δ related to the slope?

6.   You have been asked to cut two boards (exactly the same length after the cut) and place them end to end. If

    the combined length must be within 0.06 inches of 30 inches, then each board must be within how many

    inches of 15?

7.   You have been asked to cut three boards (exactly the same length after the cut) and place them end to end. If

    the combined length must be within 0.06 inches of 30 inches, then each board must be within how many

    inches of 10?

8. $\lim\limits_{x \to 3} x^2 = 9$. What values of x guarantee that $f(x) = x^2$ is within 1 unit of 9? within 0.2 units?

9. $\lim\limits_{x \to 2} x^3 = 8$. What values of x guarantee that $f(x) = x^3$ is within 0.5 unit of 8?   within 0.05 units?

10. $\lim\limits_{x \to 16} \sqrt{x} = 4$. What values of x guarantee that $f(x) = \sqrt{x}$ is within 1 unit of 4? Within 0.1 units?

11. $\lim\limits_{x \to 3} \sqrt{1 + x} = 2$. What values of x guarantee that $f(x) = \sqrt{1+x}$ is within 1 unit of 2? Within 0.0002 units?

12.   You have been asked to cut four pieces of wire (exactly the same length after the cut) and form them into a

    square. If the area of the square must be within 0.06 inches of 100 inches, then each piece of wire must be

    within how many inches of 10?

13. You have been asked to cut four pieces of wire (exactly the same length after the cut) and form them into a square. If the area of the square must be within 0.06 inches of 25 inches, then each piece of wire must be within how many inches of 5?

In problems 14 – 17, $\lim\limits_{x \to a} f(x) = L$ and the function f and a value for ε are given graphically. Find a

length for δ that satisfies the definition of limit for the given function and value of ε.

14. f and ε as shown in Fig. 16.

Fig. 16   a

15. f and ε as shown in Fig. 17.
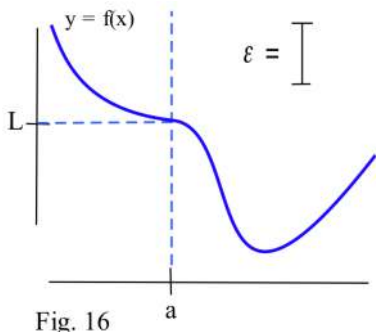
Fig. 17   a
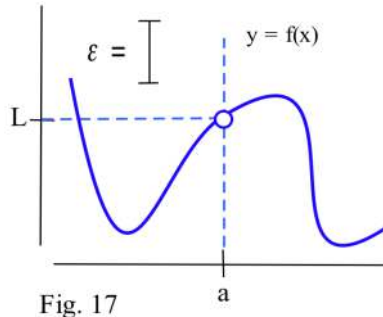
16. f and ε as shown in Fig. 18.

Fig. 18   a
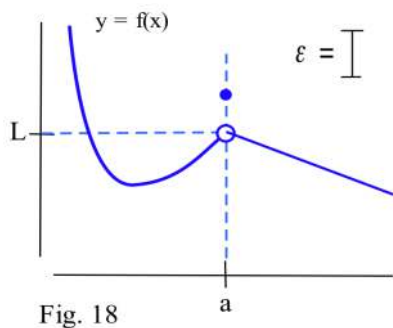
17. f and ε as shown in Fig. 19.

Fig. 19   a
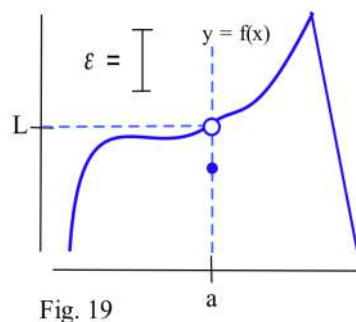
18. Redo each of problems 14 – 17 taking a new value of ε to be half the value of ε given in the problem.

In problems 19–22, use the definition to prove that the given limit does not exist.

(Find a value for ε > 0 for which there is no δ that satisfies the definition.)

19. $f(x) = \begin{cases} 4 & \text{if } x < 2 \\ 3 & \text{if } x > 2 \end{cases}$ as is shown in Fig. 20.

Show $\lim\limits_{x \to 2} f(x)$ does not exist.

Fig. 20
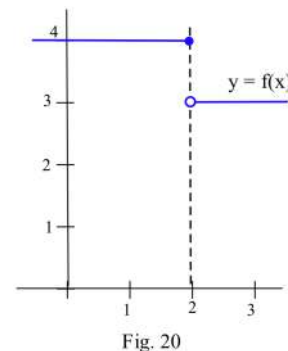
20. $f(x) = INT(x)$ as is shown in Fig. 21.

Show $\lim\limits_{x \to 3} f(x)$ does not exist.

y = INT(x)
Fig. 21

21. $f(x) = \begin{cases} x & \text{if } x < 2 \\ 6-x & \text{if } x > 2 \end{cases}$. Show $\lim\limits_{x \to 2} f(x)$ does not exist.

22. $f(x) = \begin{cases} x+1 & \text{if } x < 2 \\ x^2 & \text{if } x > 1 \end{cases}$. Show $\lim\limits_{x \to 2} f(x)$ does not exist.
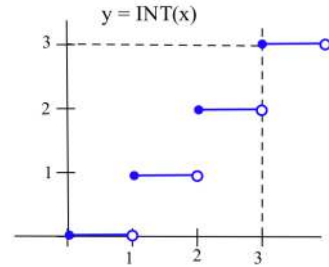
23. Prove: If $\lim\limits_{x \to a} f(x) = L$ and $\lim\limits_{x \to a} g(x) = M$, then $\lim\limits_{x \to a} f(x) - g(x) = L - M$.

Section 2.5   **PRACTICE Answers**
€   €   €

Practice 1:   (a) $3 - 1 < 4x - 5 < 3 + 1$ so $7 < 4x < 9$ and $1.75 < x < 2.25$: "x within 1/4 unit of 2."

   (b) $3 - 0.08 < 4x - 5 < 3 + 0.08$ so $7.92 < 4x < 8.08$ and $1.98 < x < 2.02$: " x within 0.02 units of 2."

   (c) $3 - E < 4x - 5 < 3 + E$ so $8 \_ E < 4x < 8 + E$ and $2 - \dfrac{E}{4} < x < 2 + \dfrac{E}{4}$: "x within E/4 units of 2."

Practice 2:   "within 1 unit of 3": If $2 < \sqrt{x} < 4$, then $4 < x < 16$ which extends from 5 units to

the left of 9 to 7 units to right of 9. Using the **smaller** of these two distances from 9,

"If x is within 5 units of 9, then $\sqrt{x}$ is within 1 unit of 3."

"within 0.2 units of 3": If $2.8 < \sqrt{x} < 3.2$, then $7.84 < x < 10.24$ which extends from

1.16 units to the left of 9 to 1.24 units to the right of 9. "If x is within 1.16 units of 9,

then $\sqrt{x}$ is wqithin 0.2 units of 3.

Practice 3:   See Fig. 22.   **Practice 4:**   See Fig. 23
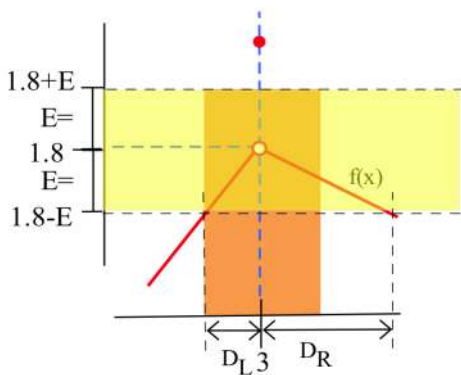
D = smaller of D_L and D_R
D = ⟷
Fig. 22 (based on Fig. 9)
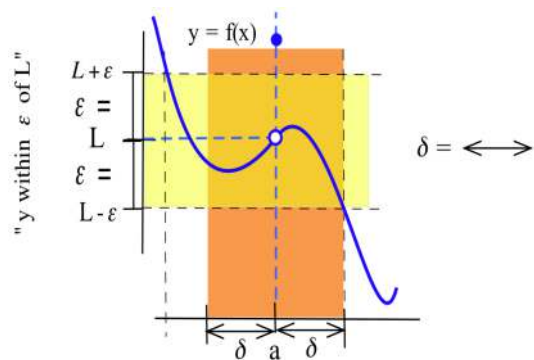
"x within δ of a"
Fig. 23 (based on Fig. 12)

Practice 5:   Given any $\varepsilon > 0$, take $\delta = \varepsilon/5$.

If x is within $\delta = \varepsilon/5$ of 4, then

$4 - \varepsilon/5 < x < 4 + \varepsilon/5$ so

$-\varepsilon/5 < x - 4 < \varepsilon/5$ (subtracting 4)

$-\varepsilon < 5x - 20 < \varepsilon$ (multiplying by 5)

$-\varepsilon < (5x + 3) - 23 < \varepsilon$ (rearranging to get the form we want)

so, finally, $f(x) = 5x + 3$ is within ε of $L = 23$.

We have shown that "for any $\varepsilon > 0$, there is a $\delta > 0$ (namely $\delta = \varepsilon/5$)" so that the rest of the definition is satisfied.

Practice 6:   This is a much more sophisticated (= harder) problem.

Using "proof by contradiction" as outlined in the solution to Example 6.

(i) Assume that the limit exists: $\lim\limits_{x \to 0} \dfrac{1}{x} = L$ for some value for L. Let $\varepsilon = 1$. (The definition says "for every ε" so we can pick this value. For this limit, the definition fails for every $\varepsilon > 0$.) Then, since we are assuming that the limit exists, there is a $\delta > 0$ so that if x is within δ of 0 then $f(x) = \dfrac{1}{x}$ is within $\varepsilon = 1$ of L.

(ii) (See Fig. 24) Let x_1 be between 0 and $0 + \delta$ and also require that $x_1 < \dfrac{1}{2}$. Then

$0 < x_1 < \dfrac{1}{2}$ so $f(x_1) = \dfrac{1}{x_1} > 2$. Since x_1 is within δ of 0, $f(x_1) > 2$ is within $\varepsilon = 1$ of L,

so L is greater than $2 - \varepsilon = 1$: $\mathbf{1 < L}$.

Let x_2 be between 0 and $0 - \delta$ and also require that $x_2 > -\dfrac{1}{2}$. Then $0 > x_2 > \dfrac{1}{2}$ so

$f(x_2) = \dfrac{1}{x_2} < -2$. Since x_2 is within δ of 0, $f(x_2) < -2$ is within

$\varepsilon = 1$ of L, so L is less than $-2 + \varepsilon = -1$: $\mathbf{-1 > L}$.

(iii)   The two inequalities in bold print provide the contradiction we were hoping to find. There is no value L that satisfies

   BOTH $\mathbf{1 < L}$ and $\mathbf{L < -1}$,

so we can conclude that our assumption was false and that

$f(x) = \dfrac{1}{x}$ does not have a limit as $x \to 0$.
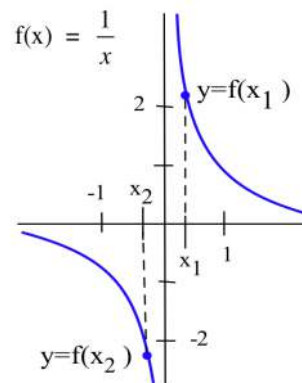
Fig. 23 (based on Fig. 15)

Section Two

Section 2.1

1. $m = \dfrac{y - 9}{x - 3}$.   If x = 2.97, then $m = \dfrac{-0.1791}{-0.03} = 5.97$. If x = 3.001, then $m = \dfrac{0.00601}{0.001} = 6.001$.

   If x = 3 + h, then $m = \dfrac{(3+h)^2 - 9}{(3+h) - 3} = \dfrac{9 + 6h + h^2 - 9}{h} = 6 + h$. When h is very small (close to 0), 6 + h is very close to 6.

3. $m = \dfrac{y - 4}{x - 2}$.   If x = 1.99, then $m = \dfrac{-0.0499}{-0.01} = 4.99$. If x = 2.004, then $m = \dfrac{0.020016}{0.004} = 5.004$.

   If x = 2 + h, then $m = \dfrac{\{(2+h)^2 + (2+h) - 2\} - 4}{(2+h) - 2} = \dfrac{4 + 4h + h^2 + 2 + h - 2 - 4}{h} = 5 + h$. When h is very small, 5 + h is very close to 5.

5. All of these answers are **approximate**. Your answers should be close to these numbers.

   (a) average rate of temperature change $\approx \dfrac{80^\mathrm{o} - 64^\mathrm{o}}{1\ \mathrm{pm} - 9\ \mathrm{am}} = \dfrac{16^\mathrm{o}}{4\ \mathrm{hours}} = 4^\mathrm{o}$ per hour.

   (b) at 10 am, temperature was rising about 5° per hour.

   at 7 pm, temperature was rising about -10° per hour (**falling** about 10° per hour).

7. All of these answers are **approximate**. Your answers should be close to these numbers.

   (a) average velocity $\approx \dfrac{300\ \mathrm{m} - 0\ \mathrm{m}}{20\ \mathrm{sec.} - 0\ \mathrm{sec.}} = 15$ meters per second.

   (b) average velocity $\approx \dfrac{100\ \mathrm{m} - 250\ \mathrm{m}}{30\ \mathrm{sec.} - 10\ \mathrm{sec.}} = -7.5$ meters per second.

   (c) at t = 10 seconds, velocity ≈ 30 meters per second (between 20 and 35 m/s).

   at t = 20 seconds, velocity ≈ -1 meters per second.

   at t = 30 seconds, velocity ≈ -40 meters per second.

9. (a) A(0) = 0, A(1) = 3, A(2) = 6, A(2.5) = 7.5, A(3) = 9.
   (b) the area of the rectangle bounded below by the x–axis, above by the line y = 3, on the left by the vertical line x = 1, and on the right by the vertical line x = 4.
   (c) Graph of y = A(x) = 3x.

Section 2.2

1. (a) 2   (b) 1   (c) DNE (does not exist)   (d) 1

3. (a) 1   (b) –1   (c) –1 (d) DNE (does not exist)

5. (a) –7   (b) (13/0) DNE

7. (a) 0.54 (remember, we are using radian mode)   (b) –0.318   (c) –0.54

9. (a) 0   (b) 0   (c) 0   10. (a) –1   (b) +1   (c) DNE

11. (a) 0   (b) –1   (c) DNE

13. $\displaystyle\lim_{h \to 0^-} g(x) = 1$ $\displaystyle\lim_{x \to 0^+} g(x) = 1$ $\displaystyle\lim_{h \to 0} g(x) = 1$

    $\displaystyle\lim_{h \to 2^-} g(x) = 1$ $\displaystyle\lim_{x \to 2^+} g(x) = 4$ $\displaystyle\lim_{h \to 2} g(x)$ does not exist

$$\lim_{h \to 4^-} g(x) = 2 \qquad\qquad \lim_{x \to 4^+} g(x) = 2 \qquad\qquad \lim_{h \to 4} g(x) = 2$$

$$\lim_{h \to 5^-} g(x) = 1 \qquad\qquad \lim_{x \to 5^+} g(x) = 1 \qquad\qquad \lim_{h \to 5} g(x) = 1$$

€　15.　　(a) 1.0986　　　　(b) 1　　　17. (a) 0.125　　(b) 3.5

€　19.　　(a) A(0) = 0, A(1) = 2.25, A(2) = 5, A(3) = 8.25
　　　　　(b) A(x) = 2x + x^2/4
　　　　　(c) the area of the region bounded below by the x–axis, above by the line y = x/2 + 2, on the left
　　　　　　　by the vertical line x = 1, and on the right by the vertical line x = 3.

Section 2.3

1.　　(a) 2　　　(b) 0　　(c) DNE (does not exist)　(d) 1.5

3.　　(a) 1　　　(b) 3　　(c) 1　　(d) ≈ 0.8

5.　　See Fig. 1.2P5 .

7.　　(a) 2　(b) –1　(c) DNE　(d) 2
　　　(e) 2　(f) 2　(g) 1　(h) 2　(i) DNE

9.　　(a) When v = 0, L = A .

$$\text{(b)} \quad \lim_{v \to c^-} A \sqrt{1 - \frac{v^2}{c^2}} = 0$$

11.　　(a) 4　(b) 1　(c) 2　(d) 0　(e) 1　(f) 1

slope of secant line = $\dfrac{f(2+h) - f(2)}{(2+h) - 2}$

y=f(x)
h = ⊢——⊣
f(2+h)
f(2+h)-f(2)
1
f(2)
x
secant
line
2
2+h

Fig. 1.2P5

€13.　　(a) Slope of the line tangent to the graph of y = cos(x) at the point (0,1). (b) Slope = 0.

15.　　(a) ≈ 1　　(b) ≈ 3.43　　(c) ≈ 4

17.　　at x = –1: a　　　　　　at x = 0: b　　　　　at x = 1: c　　　　　at x = 2: d
　　　at x = 3: c at x = 4: b　　at x = 5: a

19.　　Verify each step.

21.　　Several different lists will work. Here is one example.

　　　Put a_n = 1/(nπ) for n = 1, 2, 3, ... so a_n approaches 0 and sin(a_n) = sin($\frac{1}{1/(n\pi)}$) = sin(nπ) = 0 for all n.

　　　Put bn = $\dfrac{1}{2n\pi + \pi/2}$ for n = 1, 2, 3, ... so bn approaches 0 and
　　　　sin(bn) = sin(2nπ + π/2) = sin(π/2) = 1 for all n.
　　　Therefore, $\lim\limits_{h \to 0}$ sin(1/x) does not exist.

€
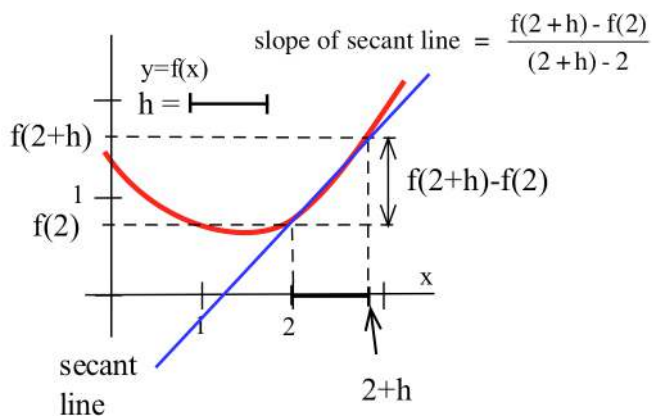
Section 2.4

1. Discontinuous at 1, 3, and 4.
3. (a) Discontinuous at x = 3. Fails condition (i) there.
    (b) Discontinuous at x = 2. Fails condition (i) there.
    (c) Discontinuous where cos(x) is negative, (e.g., at x = π). Fails condition (i) there.
    (d) Discontinuous where x^2 is an integer (e.g., at x = 1 or $\sqrt{2}$). Fails condition (ii) there.
    (e) Discontinuous where sin(x) = 0 (e.g., at x = 0, $\pm\pi$, $\pm 2\pi$, ...). Fails condition (i) there.
    (f) Discontinuous at x = 0. Fails condition (i) there.
    (g) Discontinuous at x = 0. Fails condition (i) there.
    (h) Discontinuous at x = 3. Fails condition (i) there.
    (i) Discontinuous at x = π/2. Fails condition (i) there.

5. (a) f(x) = 0 for at least 3 values of x , $0 \le x \le 5$.
    (b) 1   (c) 3   (d) 2   (e) Yes. It does not have to happen, but it is possible.

7. (a) f(0) = 0, f(3) = 9 and $0 \le 2 \le 9$. c = $\sqrt{2}$ $\approx$ 1.414
    (b) f(–1) = 1, f(2) = 4 and $1 \le 3 \le 4$. c = $\sqrt{3}$ $\approx$ 1.732
    (c) f(0) = 0, f(π/2) = 1 and $0 \le 1/2 \le 1$. c = (inverse sine of 1/2) ≈ 0.524
    (d) f(0) = 0, f(1) = 1 and $0 \le 1/3 \le 1$. c = 1/3
    (e) f(2) = 2, f(5) = 20 and $2 \le 4 \le 20$. c = (1 + $\sqrt{17}$)/2 $\approx$ 2.561 .
    (f) f(1) = 0, f(10) $\approx$ 2.30 and $0 \le 2 \le 2.30$. c = (inverse of ln(2)) = e^2 ≈ 7.389 .

9. Neither student is correct. The bisection algorithm converges to the root labeled C.

11. (a) D
    (b) D
    (c) hits B

13. [–0.9375, –0.875], $\approx$ –0.879
    [1.3125, 1.375], $\approx$ 1.347
    [2.5, 2.5625], $\approx$ 2.532

15. [2.3125, 2.375], $\approx$ 2.32 .

17. [–0.375, –0.3125], $\approx$ –0.32 .
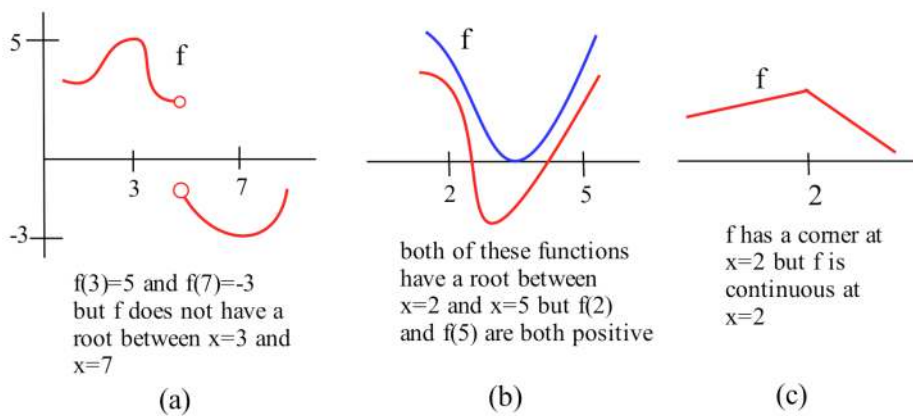
19. See the three
    graphs in Fig. 1.3P19.

f(3)=5 and f(7)=-3
but f does not have a
root between x=3 and
x=7

(a)

both of these functions
have a root between
x=2 and x=5 but f(2)
and f(5) are both positive

(b)

f has a corner at
x=2 but f is
continuous at
x=2

(c)

Fig. 1.3P19

21. (a) A(2.1) – A(2) is the area of the region bounded below by the x–axis, above by the graph of f, on the
    left by the vertical line x = 2, and on the right by the vertical line x = 2.1.

    $$\frac{A(2.1) - A(2)}{0.1} \approx f(2) \text{ or } f(2.1) \text{ so } \frac{A(2.1) - A(2)}{0.1} \approx 1.$$

    (b) A(4.1) – A(4) is the area of the region bounded below by the x–axis, above by the graph of f, on the left
    by the vertical line x = 4, and on the right by the vertical line x = 4.1. $\frac{A(4.1) - A(4)}{0.1} \approx f(4) \approx 2.$

23. (a) Yes. You supply the justification.   (b) Yes   (c) Try it.

3.1 INTRODUCTION TO DERIVATIVES

PREVIEW

The two previous chapters have laid the foundation for the study of calculus.   They provided a review of some material you will need, and they started to emphasize the various ways we will need to view and use functions: functions given by graphs, equations, and tables of values.

This section will focus on the idea of tangent lines. We will get a definition for the derivative of a function and calculate the derivatives of some functions using this definition. Then we will examine some of the properties of derivatives, see some relatively easy ways to calculate the derivatives, and begin to look at some ways we can use derivatives. Here we will emphasize what derivatives are, how to calculate them, and some of their applications.

This section begins with a very graphical approach to slopes of tangent lines. It then examines the problem of finding the slopes of the tangent lines for a single function, $y = x^2$, in some detail, and illustrates how these slopes can help us solve fairly sophisticated problems.

Slopes of Tangent Lines: Graphically

Fig. 1 is the graph of a function $y = f(x)$. We can use the information in the graph to fill in the table:

Fig. 1

x	y = f(x)	m(x) =	the **estimated** SLOPE of the tangent line to y=f(x) at the point (x,y)
0	**0**	1	
1	**1**	0	
2	**0**	− 1	
3	**− 1**	0	
4	**1**	1	
5	**2**	1/2	

We can estimate the values of m(x) at some non-integer values of x, m(.5) ≈ 0.5 and m(1.3) ≈ –0.3, and even over entire intervals, if $0 < x < 1$, then m(x) is positive.

Fig. 2

The values of m(x) definitely depend on the values of x , and m(x) is a function of x. We can use the results in the table to help sketch the graph of m(x) in Fig. 2.

Practice 1:   The graph of $y = f(x)$ is given in Fig. 3. Set up a table of values for x and m(x) (the slope of the line tangent to the graph of y=f(x) at the point (x,y)) and then graph the function m(x).
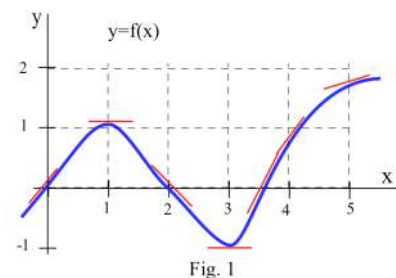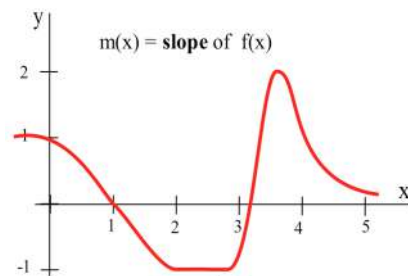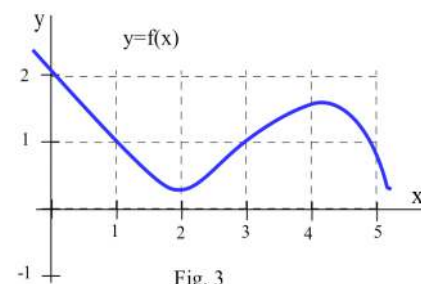
Fig. 3

In some applications, we need to know where the graph of a function f(x) has horizontal tangent lines (slopes = 0).

In Fig. 3, the slopes of the tangent lines to graph of y = f(x) are 0 when x = 2 or x ≈ 4.5 .

Practice 2:   At what values of x does the graph of y = g(x)

in Fig. 4 have horizontal tangent lines?
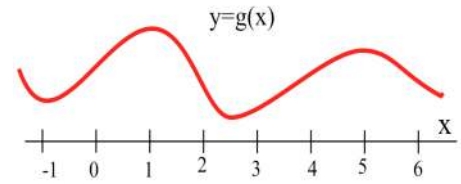
Fig. 4
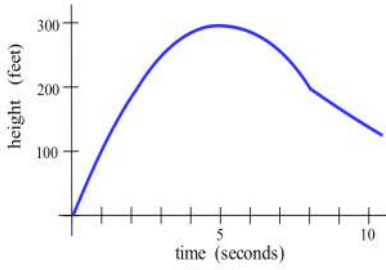
Example 1:   Fig. 5 is the graph of the

height of a rocket at time t. Sketch the graph of the **velocity** of the rocket

at time t. (Velocity is the **slope of the tangent** to the graph of position

or height.)

Solution: The lower graph in Fig. 6 shows the velocity of the rocket.
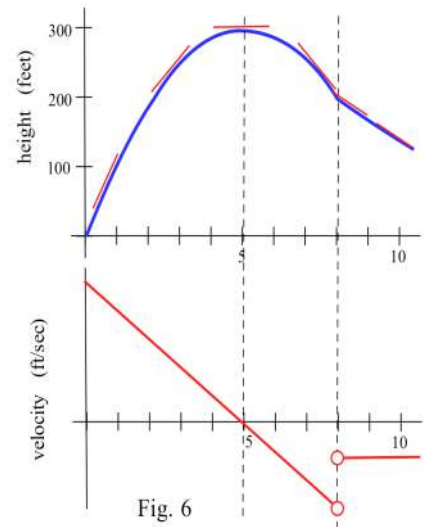
Fig. 6

Practice 3:   Fig. 7 shows the temperature during a summer day

in Chicago. Sketch the graph of the **rate** at which the temperature

is changing. (This is just the graph of the **slopes** of the lines which

are tangent to the temperature graph.)

The function m(x) , the slope of the line tangent to
the graph of f(x), is called the   **derivative of f(x)** .
We have used the idea of the slope of the tangent
line throughout Chapter 1. In the Section 2.1, we
will formally define the derivative of a function
and begin to examine some of the properties of the
derivative function, but first lets see what we can
do when we have a formula for f(x).
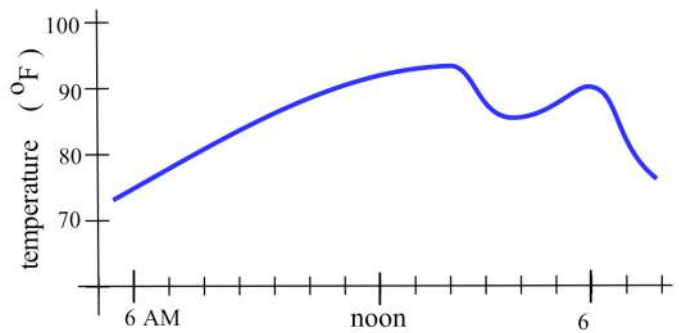
Fig. 7

Tangents to $y = x^2$

When we have a formula for a function, we can determine the slope of the tangent line at a point $(x, f(x))$ by calculating the slope of the secant line through the points $(x, f(x))$ and $(x+h, f(x+h))$,

$m_{sec} = \dfrac{f(x+h) - f(x)}{(x+h) - (x)}$, and then taking the limit of m_{sec}

as h approaches 0 (Fig. 8) :

$$m_{tan} = \lim_{h \to 0} m_{sec} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - (x)} .$$
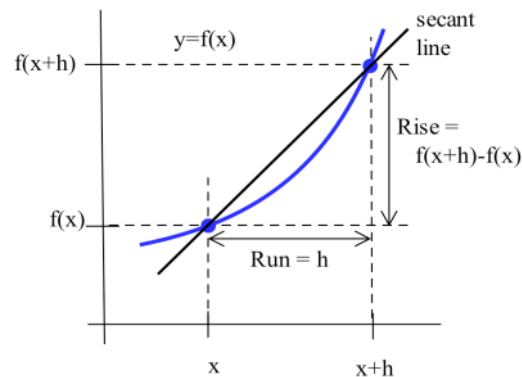
Fig. 8

Example 2:   Find the **slope** of the line tangent to the graph of $y = f(x) = x^2$ at the point (2,4). (Fig. 9).
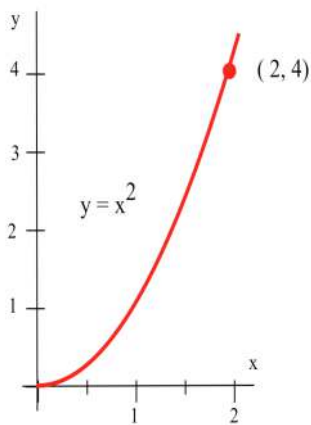
Fig. 9

Solution: In this example $x = 2$, so $x + h = 2 + h$ and $f(x + h) = f(2+h) = (2+h)^2$.

The slope of the tangent line at (2,4) is

$$m_{tan} = \lim_{h \to 0} m_{sec} = \lim_{h \to 0} \frac{f(2+h) - f(2)}{(2+h) - (2)}$$

$$= \lim_{h \to 0} \frac{(2+h)^2 - (2)^2}{(2+h) - (2)} = \lim_{h \to 0} \frac{4 + 4h + h^2 - 4}{h}$$

$$= \lim_{h \to 0} \frac{4h + h^2}{h} = \lim_{h \to 0} \frac{h(4+h)}{h} = \lim_{h \to 0} (4 + h) = 4 .$$

The tangent line to the graph of $y = x^2$ at the point (2,4) has slope 4 .

We can use the point–slope formula for a line to find the equation of the tangent line:
$$y - y_o = m(x - x_o) \text{ so } y - 4 = 4(x - 2) \text{ and } y = 4x - 4.$$

Practice 4:   Use the method of Example 2 to show that the **slope** of the line tangent to the graph of $y = f(x) = x^2$ at the point (1,1) is $m_{tan} = 2$. Also find the values of m_{tan} at (0,0) and (–1,1).

It is possible to find the slopes of the tangent lines one point at a time, but that is not very efficient. You should have noticed in the Practice 4 that the algebra for each point was very similar, so let's do all the work once for an arbitrary point $(x, f(x)) = (x, x^2)$ and then use the general result for our particular problems. The slope of the line tangent to the graph of $y = f(x) = x^2$ at the arbitrary point (x, x^2) is

$$m_{tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{(x+h) - (x)} = \lim_{h \to 0} \frac{(x+h)^2 - (x)^2}{(x+h) - (x)} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$

$$= \lim_{h \to 0} \frac{2xh + h^2}{h} = \lim_{h \to 0} \frac{h(2x+h)}{h} = \lim_{h \to 0} (2x + h) = 2x .$$

The **slope** of the line tangent to the graph of $y = f(x) = x^2$ at the point (x, x^2) is $m_{tan} = \mathbf{2x}$. We can use this general result at any value of x without going through all of the calculations again. The slope of the line tangent to $y = f(x) = x^2$ at the point $(4, 16)$ is $m_{tan} = 2(4) = 8$ and the slope at (π, π^2) is $m_{tan} = 2(\pi) = 2\pi$. The value of x determines where we are on the curve (at $y = x^2$) as well as the slope of the tangent line, $m_{tan} = 2x$, at that point. The slope $m_{tan} = 2x$ is a **function** of x and is called the **derivative of y = x^2** .

Simply knowing that the slope of the line tangent to the graph of $y = x^2$ is $m_{tan} = 2x$ at a point (x,y) can help us quickly find the equation of the line tangent to the graph of $y = x^2$ at any point and answer a number of difficult–sounding questions.

Example 3:   Find the equations of the lines tangent to $y = x^2$ at $(3, 9)$ and (p, p^2).

Solution: At $(3, 9)$, the slope of the tangent line is $2x = 2(3) = 6$, and the equation of the line is
$$y - y_o = m(x - x_o) \text{ so } y - 9 = 6(x - 3) \text{ and } y = 6x - 9.$$

At (p, p^2), the slope of the tangent line is $2x = 2(p) = 2p$, and the equation of the line is
$$y - y_o = m(x - x_o) \text{ so } y - p^2 = 2p(x - p) \text{ and } y = 2px - p^2.$$
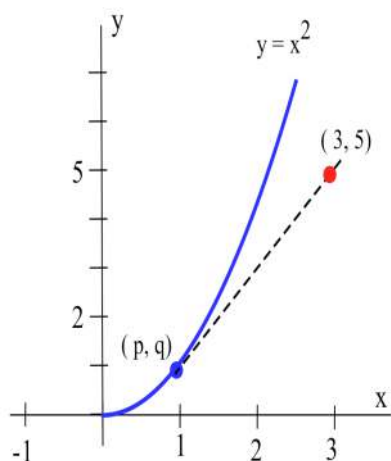
Fig. 10

Example 4:   A rocket has been programmed to follow the path $y = x^2$ in space (from left to right along the curve), but an emergency has arisen and the crew must return to their base which is located at coordinates $(3,5)$. At what point on the path $y = x^2$ should the captain turn off the engines so the ship will coast along the tangent to the curve to return to the base? (Fig. 10)

Solution: You might spend a few minutes trying to solve this problem without using the relation $m_{tan} = 2x$, but the problem is much easier if we do use that result.

Lets assume that the captain turns off the engine at the point (p,q) on the curve $y = x^2$, and then try to determine what values p and q must have so that the resulting tangent line to the curve will go through the point $(3,5)$. The point (p,q) is on the curve $y = x^2$, so $q = p^2$, and the equation of the tangent line, found in Example 3, is $y = 2px - p^2$.

To find the value of p so that the tangent line will go through the point $(3,5)$, we can substitute the values $x = 3$ and $y = 5$ into
the equation of the tangent line and solve for p :
$$y = 2px - p^2 \text{ so } 5 = 2p(3) - p^2 \text{ and } p^2 - 6p + 5 = 0.$$

The only solutions of $p^2 - 6p + 5 = (p-1)(p-5) = 0$ are

$p = 1$ and $p = 5$, so the only possible points are $(1,1)$ and

$(5,25)$. You can verify that the tangent lines to $y = x^2$ at $(1,1)$

and $(5,25)$ go through the base at the point $(3,5)$ (Fig. 11).

Since the ship is moving from left to right along the curve, the

captain should turn off the engines at the point $(1,1)$. Why not

at $(5,25)$?

Fig. 11

Practice 5:   Verify that if the rocket engines in Example 4 are

shut off at $(2,4)$, then the rocket will go through the

point $(3,8)$.
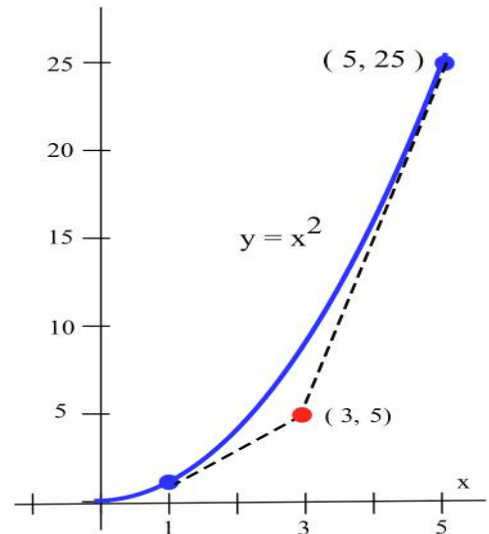
PROBLEMS

1.   Use the function in Fig. 12 to fill in the table and then graph $m(x)$.

x	y = f(x)	m(x) = the estimated slope of the tangent line to y=f(x) at the point (x,y)
0		
0.5		
1.0		
1.5		
2.0		
2.5		
3.0		
3.5		
4.0		

Fig. 12

2.   Use the function in Fig. 13 to fill in the table and then graph $m(x)$.

x	y = g(x)	m(x) = the estimated slope of the tangent line to y=g(x) at the point (x,y)
0		
0.5		
1.0		
1.5		
2.0		
2.5		
3.0		
3.5		
4.0		

Fig. 13

3.   (a) At what values of x does the graph of f in Fig. 14 have a
horizontal tangent line?

(b) At what value(s) of x is the value of f the largest? smallest?

(c) Sketch the graph of $m(x) = $ the slope of the line tangent to the graph
of f at the point (x,y).
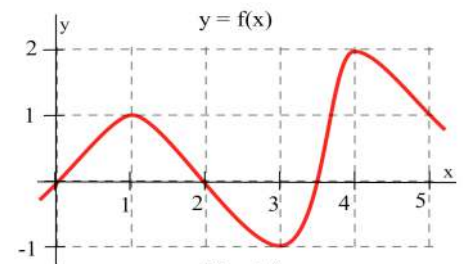
Fig. 14
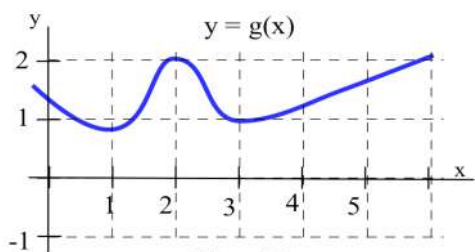
Fig. 15

4. (a) At what values of x does the graph of g in Fig. 15 have a horizontal tangent line?

    (b) At what value(s) of x is the value of g the largest? smallest?

    (c) Sketch the graph of m(x) = the slope of the line tangent to the graph of g at the point (x,y).

5. (a) Sketch the graph of f(x) = sin(x) for $-3 \le x \le 10$.

    (b) Sketch the graph of m(x) = slope of the line tangent to the graph of sin(x) at the point (x, sin(x)).

    (c) Your graph in part (b) should look familiar. What function is it?

6. Match the situation descriptions with the corresponding **time–velocity** graphs in Fig. 16.

    (a) A car quickly leaving from a stop sign.

    (b) A car sedately leaving from a stop sign.

    (c) A student bouncing on a trampoline.

    (d) A ball thrown straight up.

    (e) A student confidently striding across campus to take a calculus test.

    (f) An unprepared student walking across campus to take a calculus test.
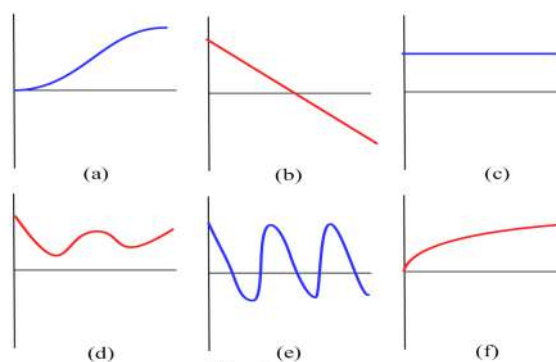
Fig. 16

Problems 7 – 10 assume that a rocket is following the path $y = x^2$, from left to right.

7. At what point should the engine be turned off in order to coast along the tangent line to a base at (5,16)?

8. At what point should the engine be turned off in order to coast along the tangent line to a base at (3,–7)?

9. At what point should the engine be turned off in order to coast along the tangent line to a base at (1,3)?

10. Which points in the plane can not be reached by the rocket? Why not?

For each function f(x) in problems 11 – 16, perform steps (a) – (d):

(a) calculate $m_{sec} = \dfrac{f(x+h) - f(x)}{(x+h) - (x)}$ and simplify   (b) determine $m_{tan} = \lim\limits_{h \to 0} m_{sec}$

(c) evaluate m_{tan} at x = 2,   (d) find the equation of the line tangent to the graph of f at (2, f(2))

11. f(x) = 3x – 7   12. f(x) = 2 – 7x   13. f(x) = ax + b where a and b are constants

14. f(x) = x^2 + 3x   15. f(x) = 8 – 3x^2 16. f(x) = ax^2 + bx + c where a, b and c are constants

In problems 17 and 18, use the result that if $f(x) = ax^2 + bx + c$ then $m_{tan} = 2ax + b$.

17. $f(x) = x^2 + 2x$. At which point(s) $(p, f(p))$ does the line tangent to the graph at that point also go through the point $(3, 6)$?

18. (a) If $a \neq 0$, then what is the shape of the graph of $y = f(x) = ax^2 + bx + c$?
    (b) At what value(s) of x is the line tangent to the graph of $f(x)$ horizontal?

Section 3.1   **PRACTICE Answers**

Practice 1: Approximate values of m(x) are in the table below. Fig. 17 is a graph of m(x).

x	y = f(x)	m(x) =	the **estimated** SLOPE of the tangent line to y=f(x) at the point (x,y)
0	**2**	**−1**	
1	**1**	**−1**	
2	**1/3**	**0**	
3	**1**	**1**	
4	**3/2**	**1/2**	
5	**1**	**−2**	

approximate graph of
m(x) = slopes of f(x)
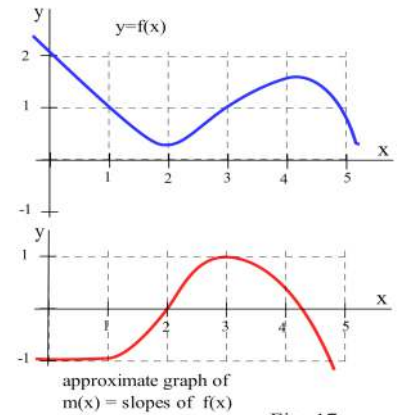
Fig. 17

Practice 2: The tangent lines to the graph of g are horizontal (slope = 0) when $x \approx -1, 1, 2.5$, and 5.

Practice 3: Fig. 18 is a graph of the approximate **rate** of temperature change (slope).
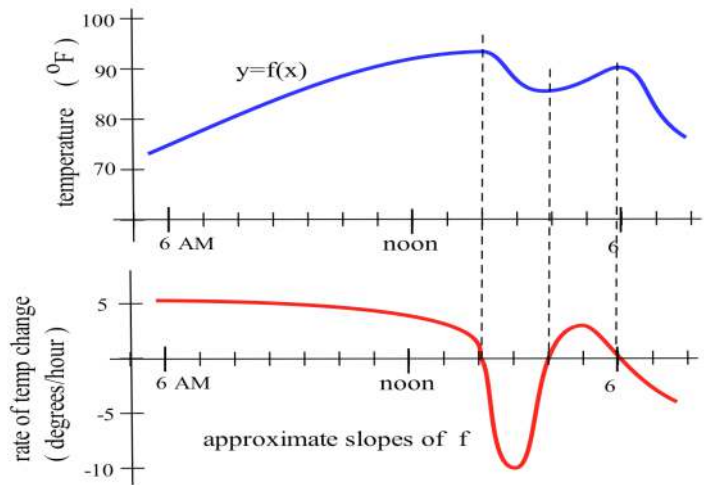
Fig. 18

Practice 4: $y = x^2$.

At (1,1), m_{tan} $= \lim\limits_{h \to 0} \dfrac{f(1+h) - f(1)}{(1+h) - (1)}$ $= \lim\limits_{h \to 0} \dfrac{(1+h)^2 - (1)^2}{h}$ $= \lim\limits_{h \to 0} \dfrac{\{1 + 2h + h^2\} - 1}{h}$

$= \lim\limits_{h \to 0} \dfrac{2h + h^2}{h}$ $= \lim\limits_{h \to 0} \dfrac{h(1+h)}{h}$ $= \lim\limits_{h \to 0} (2 + h)$ $= 2$

At (0,0), m_{tan} $= \lim\limits_{h \to 0} \dfrac{f(0+h) - f(0)}{(0+h) - (0)}$ $= \lim\limits_{h \to 0} \dfrac{(0+h)^2 - (0)^2}{h}$ $= \lim\limits_{h \to 0} \dfrac{h^2}{h}$ $= \lim\limits_{h \to 0} h = \mathbf{0}$.

At (−1,1), m_{tan} $= \lim\limits_{h \to 0} \dfrac{f(-1+h) - f(-1)}{(-1+h) - (-1)}$ $= \lim\limits_{h \to 0} \dfrac{\{1 - 2h + h^2\} - 1}{h}$ $= \lim\limits_{h \to 0} \dfrac{-2h + h^2}{h}$ $= \mathbf{-2}$.

Practice 5:   From Example 4 we know the slope of the tangent line is $m_{tan} = 2x$ so the slope of the

tangent line at (2,4) is $m_{tan} = 2x = 2(2) = 4$. The tangent line has slope 4 and goes through the point

(2,4) so the equation of the tangent line (using $y - y_o = m(x - x_o)$) is $y - 4 = 4(x - 2)$ or $y = 4x - 4$.

The point (3,8) satisfies the equation $y = 4x - 4$ so the point (3,8) lies on the tangent line.

3.2   THE DEFINITION OF DERIVATIVE

The graphical idea of a **slope of a tangent line** is very useful, but for some uses we need a more algebraic definition of the **derivative of a function.** We will use this definition to calculate the derivatives of several functions and see that the results from the definition agree with our graphical understanding. We will also look at several different interpretations for the derivative, and derive a theorem which will allow us to easily and quickly determine the derivative of any fixed power of x.

In the last section we found the slope of the tangent line
to the graph of the function $f(x) = x^2$ at an arbitrary
point $(x, f(x))$ by calculating the slope of the secant
line through the points $(x, f(x))$ and $(x+h, f(x+h))$,

$$m_{sec} \ = \ \frac{f(x+h) - f(x)}{(x+h) - (x)} \ ,$$

and then by taking the limit of m_{sec} as h approached 0 (Fig.
1).   That approach to calculating slopes of tangent lines is the
definition of the derivative of a function.
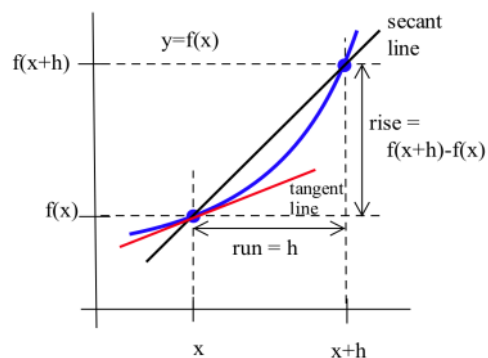
Definition of the Derivative:

The **derivative** of a function f is a new function, **f '** (pronounced "eff prime"),

whose value at x is $\mathbf{f\ '(x)} = \lim_{h \to 0} \dfrac{f(x + h) - f(x)}{h}$ if the limit exists and is finite.

This is <u>the</u> definition of differential calculus, and you must know it and understand what it says. The rest of this chapter and all of Chapter 3 are built on this definition as is much of what appears in later chapters. It is remarkable that such a simple idea (the slope of a tangent line) and such a simple definition (for the derivative f ') will lead to so many important ideas and applications.

> **Notation:**   There are three commonly used notations for the **derivative of y = f(x)**:
>
> **f '(x)**   emphasizes that the derivative is a **function** related to f
>
> **D(f)**   emphasizes that we perform an **operation** on f to get the derivative of f
>
> $\dfrac{\mathbf{df}}{\mathbf{dx}}$ emphasizes that the derivative is the **limit of** $\dfrac{\mathbf{\Delta f}}{\mathbf{\Delta x}} \ = \ \dfrac{f(x+h) - f(x)}{h} \ \ .$

We will use all three notations so you can get used to working with each of them.

f '(x) represents the <u>slope of the tangent line</u> to the graph of

y = f(x) at the point (x, f(x)) or the <u>instantaneous rate of</u>

<u>change</u> of the function f at the point (x, f(x)).

If, in Fig. 2, we let x be the point a+h, then

h = x – a. As h→0, we see that x→a and

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \; = \; \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \; \text{so}$$

f '(a) = $\lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h} \; = \; \lim_{x \to a} \dfrac{f(x) - f(a)}{x - a}$.

We will use whichever of these two forms is more convenient algebraically.
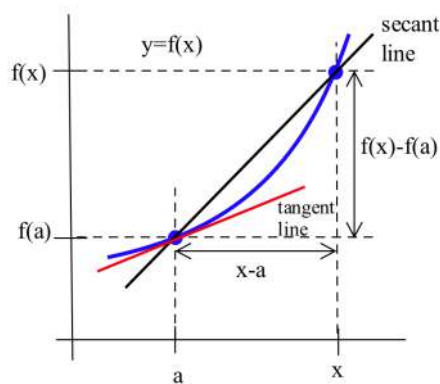
Calculating Some Derivatives Using The Definition

Fortunately, we will soon have some quick and easy ways to calculate most derivatives, but first we will have

to use the definition to determine the derivatives of a few basic functions. In Section 2.2 we will use those

results and some properties of derivatives to calculate derivatives of combinations of the basic functions. Let's

begin by using the graphs and then the definition to find a few derivatives.

Example 1: Graph **y = f(x) = 5** and estimate the **slope** of the tangent line at each point on the graph. Then

use the definition of the derivative to calculate the exact slope of the tangent line at each point. Your

graphic estimate and the exact result from the definition should agree.
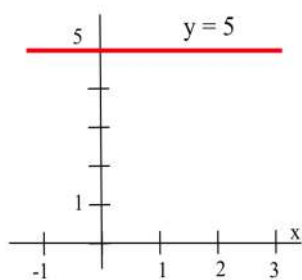
Solution: The graph of y = f(x) = 5 is a horizontal line (Fig. 3) which has

slope 0 so we should expect that its tangent line will also have slope 0.

<u>Using the definition</u>: Since f(x) = 5, then f(x+h) = 5, so

$$\mathbf{D}(\, f(x) \,) \;\; \equiv \;\; \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \;=\; \lim_{h \to 0} \frac{5 - 5}{h} \;=\; \lim_{h \to 0} \frac{0}{h} \;=\; 0 \, .$$

Using similar steps, it is easy to show that the derivative of any constant function is 0.

Theorem: If f(x) = k, then f '(x) = 0 .

Practice 1:   Graph **y = f(x) = 7x** and estimate the slope of the tangent line at each point on the graph.

Then use the definition of the derivative to calculate the exact slope of the tangent line at each point.

Example 2:   Determine the derivative of **y = f(x) = 5x^3** graphically and using the definition. Find

the equation of the line tangent to y = 5x^3 at the point (1,5).

Solution: It appears that the graph of $y = f(x) = 5x^3$ (Fig. 4) is increasing so

the slopes of the tangent lines are positive except perhaps at $x = 0$

where the graph seems to flatten out.

Using the definition: Since $f(x) = 5x^3$, then

$f(x+h) = 5(x+h)^3 = 5(x^3 + 3x^2h + 3xh^2 + h^3$) so

$\mathbf{f\,'(x)} \quad \equiv \quad \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$ the definition

$= \lim_{h \to 0} \dfrac{5(x^3 + 3x^2h + 3xh^2 + h^3) - 5(x^3)}{h}$ eliminate $5x^3 - 5x^3$

$= \lim_{h \to 0} \dfrac{15x^2h + 15xh^2 + 5h^3}{h}$

$= \lim_{h \to 0} \dfrac{h(15x^2 + 15xh + 5h^2)}{h}$ divide by h

$= \lim_{h \to 0} (15x^2 + 15xh + 5h^2) \;=\; 15x^2 + 0 + 0 \;=\; \mathbf{15x^2}$

so $\mathbf{D(\,5x^3\,)} = 15x^2$ which is positive except when x=0, and then $15x^2 = 0$.

$f\,'(x) = 15x^2$ is the slope of the line tangent to the graph of f at the point (x , f(x)). At the point (1,5),

the slope of the tangent line is $f\,'(1) = 15(1)^2 = 15$. From the point–slope formula, the equation of the

tangent line to f is $y - 5 = 15(x - 1)$ or $y = 15x - 10$.

Practice 2:   Use the definition to show that the derivative of $y = x^3$ is $\dfrac{dy}{dx} = 3x^2$. Find the equation of

the line tangent to the graph of $y = x^3$ at the point (2, 8).

If f has a derivative at x, we say that f is **differentiable** at x. If we have a point on the graph of a

differentiable function and a slope (the derivative evaluated at the point), it is easy to write the equation of the

tangent line.

Tangent Line Formula

If   f is differentiable at **a**

then   the equation of the tangent line to f at the point (a ,f(a)) is $\mathbf{y = f(a) + f\,'(a)(x - a)}$.

Proof: The tangent line goes through the point (a , f(a)) with slope f '(a) so, using the point–slope

formula, $y - f(a) = f\,'(a) (x - a)$ or $y = f(a) + f\,'(a) (x - a)$.
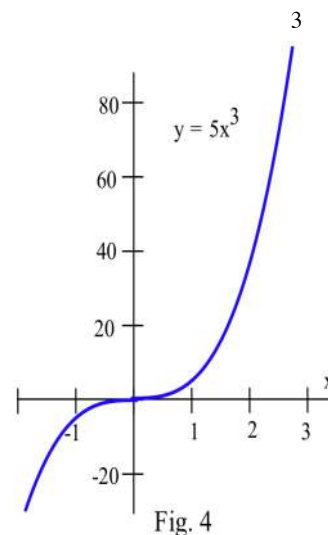
Practice 3: The derivatives $D(x) = 1$, $D(x^2) = 2x$, $D(x^3) = 3x^2$ exhibit the start of a pattern. <u>Without</u> using the definition of the derivative, what do you <u>think</u> the following derivatives will be? $D(x^4)$, $D(x^5)$, $D(x^{43})$, $D(\sqrt{x}) = D(x^{1/2})$ and $D(x^\pi)$.

(Just make an intelligent "guess" based on the pattern of the previous examples.)

Before going on to the "pattern" for the derivatives of powers of x and the general properties of derivatives, let's try the derivatives of two functions which are not powers of x: sin(x) and | x | .

Theorem: $D(\sin(x)) = \cos(x)$.

The graph of $y = f(x) = \sin(x)$ is well–known (Fig. 5). The graph has horizontal tangent lines (slope = 0) when $x = \pm \frac{\pi}{2}$ and $x = \pm \frac{3\pi}{2}$ and so on. If $0 < x < \frac{\pi}{2}$, then the slopes of the
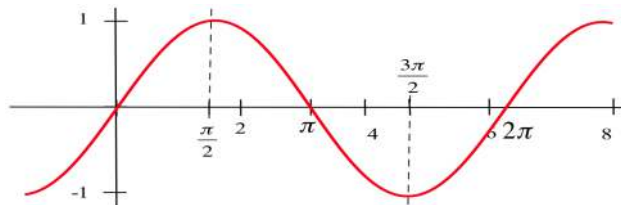
Fig. 5

tangent lines to the graph of $y = \sin(x)$ are positive. Similarly, if $\frac{\pi}{2} < x < \frac{3\pi}{2}$, then the slopes of the tangent lines are negative. Finally, since the graph of $y = \sin(x)$ is periodic, we expect that the derivative of $y = \sin(x)$ will also be periodic.

Proof of the theorem: Since $f(x) = \sin(x)$, $f(x+h) = \sin(x+h) = \sin(x)\cos(h) + \cos(x)\sin(h)$ so

$$f'(x) \quad \equiv \quad \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{\{\sin(x)\cos(h) + \cos(x)\sin(h)\} - \{\sin(x)\}}{h}$$

this limit looks formidable, but if we just collect the terms containing $\sin(x)$ and then those containing $\cos(x)$ we get

$$= \lim_{h \to 0} \left\{ \sin(x) \cdot \frac{\cos(h) - 1}{h} + \cos(x) \cdot \frac{\sin(h)}{h} \right\}$$

now calculate the limits separately

$$= \left\{ \lim_{h \to 0} \sin(x) \right\} \cdot \left\{ \lim_{h \to 0} \frac{\cos(h) - 1}{h} \right\} + \left\{ \lim_{h \to 0} \cos(x) \right\} \cdot \left\{ \lim_{h \to 0} \frac{\sin(h)}{h} \right\}$$

the first and third limits do not depend on h, and we calculated the second and fourth limits in Section 1.2

$$= \sin(x) \cdot (0) + \cos(x) \cdot (1) = \textbf{cos (x)}.$$

So $D(\sin(x)) = \cos(x)$, and the various properties we expected of the derivative of $y = \sin(x)$ by examining its graph are true of $\cos(x)$.

Practice 4:   Use the definition to show that $D(\cos(x)) = - \sin(x)$. (This is similar to the situation for $f(x) =$

sin(x). You will need the formula $\cos(x+h) = \cos(x) \cdot \cos(h) - \sin(x) \cdot \sin(h)$. Then collect all the terms

containing cos(x) and all the terms with sin(x). At that point you should recognize and be able to evaluate the

limits.)

Example 3:   For $y = |x|$ find dy/dx .

Solution: The graph of $y = f(x) = |x|$ (Fig. 6) is a "V" with its vertex

at the origin. When $x > 0$, the graph is just $y = |x| = x$ which is a line

with slope $+1$ so we should expect the derivative of |x| to be $+1$. When

$x < 0$, the graph is $y = |x| = -x$ which is a line with slope -1, so we

expect the derivative of |x| to be -1. When $x = 0$, the graph has a corner,

and we should expect the derivative of |x| to be undefined at $x = 0$.
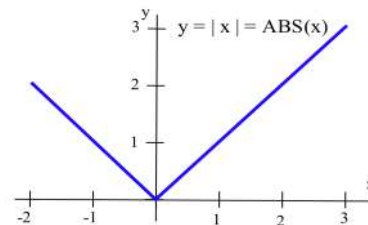
Fig. 6

Using the definition: It is easiest to consider 3 cases in the definition of $|x|$: $x > 0$, $x < 0$ and $x = 0$.

If $x > 0$, then, for small values of h, $x + h > 0$ so $Df(x) \equiv \lim_{h \to 0} \dfrac{|x + h| - |x|}{h} = \lim_{h \to 0} \dfrac{h}{h} = 1.$

If $x < 0$, then, for small values of h, we also know that $x + h < 0$ so $Df(x) = \lim_{h \to 0} \dfrac{-h}{h} = -1.$
When $x = 0$, the situation is a bit more complicated and

$$Df(x) \equiv \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0} \frac{|h|}{h} \quad \text{which is undefined}$$

since $\lim_{h \to 0^+} \dfrac{|h|}{h} = +1$ and $\lim_{h \to 0^-} \dfrac{|h|}{h} = -1$. $D(|x|) = \begin{cases} +1 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$.

Practice 5:   Graph $y = |x - 2|$ and $y = |2x|$ and use the graphs to determine $D(|x - 2|)$ and $D(|2x|)$.

INTERPRETATIONS OF THE DERIVATIVE

So far we have emphasized the derivative as the slope of the line tangent to a graph . That interpretation is very

visual and useful when examining the graph of a function, and we will continue to use it. Derivatives, however,

are used in a wide variety of fields and applications, and some of these fields use other interpretations. The

following are a few interpretations of the derivative which are commonly used.

General

Rate of Change $f'(x)$ is the **rate of change** of the function at x. If the units for x are years and the units for

f(x) are people, then the units for $\dfrac{df}{dx}$ are $\dfrac{people}{year}$, a rate of change in population.

Graphical

Slope $f'(x)$ is the **slope of the line tangent to the graph of f at the point (x, f(x)).**

Physical

Velocity   If f(x) is the position of an object at time x , then f '(x) is the **velocity** of the object at time x.

If the units for x are hours and f(x) is distance measured in miles, then the units for

$\mathbf{f}'(x) = \dfrac{df}{dx}$ are $\dfrac{miles}{hour}$, miles per hour, which is a measure of velocity.

Acceleration   If f(x) is the velocity of an object at time x , then f '(x) is the **acceleration** of the object at time

x. If the units are for x are hours and f(x) has the units $\dfrac{miles}{hour}$, then the units for the

acceleration $\mathbf{f}'(\mathbf{x}) = \dfrac{df}{dx}$ are $\dfrac{miles/hour}{hour} = \dfrac{miles}{hour^2}$, miles per hour per hour.

Magnification $\mathbf{f}'(x)$ is the **magnification factor** of the function f for points which are close to x.

If a and b are two points very close to x , then the distance between f(a) and f(b) will be

close to f '(x) times the original distance between a and b: $f(b) - f(a) \approx f'(x)(b-a)$.

Business

Marginal Cost   If f(x) is the total cost of x objects, then $\mathbf{f}'(x)$ is the **marginal cost**, at a production level of

x. This marginal cost is approximately the additional cost of making one more object once we

have already made x objects. If the units for x are bicycles and the units for f(x) are

dollars, then the units for $f'(x) = \dfrac{df}{dx}$ are $\dfrac{dollars}{bicycle}$, the cost per bicycle.

Marginal Profit   If f(x) is the total profit from producing and selling x objects, then $\mathbf{f}'(x)$ is the **marginal**

profit, the profit to be made from producing and selling one more object.

If the units for x are bicycles and the units for f(x) are dollars, then the units for

$f'(x) = \dfrac{df}{dx}$ are $\dfrac{dollars}{bicycle}$, dollars per bicycle, which is the profit per bicycle.

In business contexts, the word "marginal" usually means the derivative or rate of change of some quantity.

One of the strengths of calculus is that it provides a unity and economy of ideas among diverse applications. The

vocabulary and problems may be different, but the ideas and even the notations of calculus are still useful.

h(x) = sin(x)
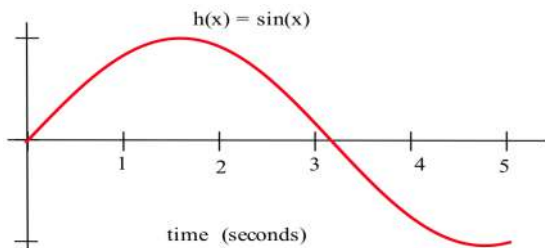
time (seconds)

Fig. 7

Example 4:   A small cork is bobbing up and down, and at time t seconds it is $h(t) = \sin(t)$ feet above the mean water level (Fig. 7). Find the height, velocity and acceleration of the cork when $t = 2$ seconds. (Include the proper units for each answer.)

Solution: $h(t) = \sin(t)$ represents the height of the cork at any time t , so the height of the cork when

$t = 2$ is $h(2) = \sin(2) \approx 0.91$ feet.

The velocity is the derivative of the position, so $v(t) = \dfrac{d\,h(t)}{dt} = \dfrac{d\,\sin(t)}{dt} = \cos(t)$. The derivative of

position is the limit of $(\Delta h)/(\Delta t)$, so the units are (feet)/(seconds). After 2 seconds the velocity is $v(2) = \cos(2) \approx -0.42$ feet per second $= -0.42$ ft/s .

The acceleration is the derivative of the velocity, so $a(t) = \dfrac{d\,v(t)}{dt} = \dfrac{d\,\cos(t)}{dt} = -\sin(t)$. The derivative

of velocity is the limit of $(\Delta v)/(\Delta t)$, so the units are (feet/second) / (seconds) or feet/second^2. After 2

seconds the acceleration is $a(2) = -\sin(2) \approx -0.91$ ft/s^2 .

Practice 6:   Find the height, velocity and acceleration of the cork in the previous example after 1 second?

A MOST USEFUL FORMULA: D(x^n)

Functions which include powers of x are very common (every polynomial is a sum of terms which include powers of x), and, fortunately, it is easy to calculate the derivatives of such powers. The "pattern" emerging from the first few examples in this section is, in fact, true for all powers of x. We will only state and prove the "pattern" here for positive integer powers of x, but it is also true for other powers as we will prove later.

> **Theorem: If n is a positive integer, then D(x^n) = n·x^{n-1} .**

This theorem is an example of the power of <u>generality</u> and <u>proof</u> in mathematics. Rather than resorting to the definition when we encounter a new power of x (imagine using the definition to calculate the derivative of x^{307}), we can justify the pattern for all positive integer exponents n , and then simply apply the result for whatever exponent we have. We know, from the first examples in this section, that the theorem is true for n= 1, 2 and 3 , but no number of <u>examples</u> would guarantee that the pattern is true for all exponents. We need a proof that what we think is true really is true.

Proof of the theorem: Since $f(x) = x^n$, then $f(x+h) = (x+h)^n$, and in order to simplify

$f(x+h) - f(x) = (x+h)^n - x^n$, we will need to expand $(x+h)^n$. However, we really only need to know the

first two terms of the expansion and to know that all of the other terms of the expansion contain a power

of h of at least 2. The Binomial Theorem from algebra says (for $n > 3$) that

$(x+h)^n = x^n + n{\cdot}x^{n-1}h + a{\cdot}x^{n-2}h^2 + b{\cdot}x^{n-3}h^3 + ... + h^n$ where a and b   represent numerical coefficients.

(Expand $(x+h)^n$ for at least a few different values of n to convince yourself of this result.)

Then $\mathbf{D}(\,f(x)\,)\ \equiv\ \lim\limits_{h \to 0} \dfrac{f(x+h)-f(x)}{h}\ =\ \lim\limits_{h \to 0} \dfrac{(x+h)^n - x^n}{h}$ then expand to get

$= \lim\limits_{h \to 0} \dfrac{\left\{ x^n + n\cdot x^{n-1}h + a\cdot x^{n-2}h^2 + b\cdot x^{n-3}h^3 + ... + h^n \right\} - x^n}{h}$ eliminate $x^n - x^n$

$= \lim\limits_{h \to 0} \dfrac{\left\{ n\cdot x^{n-1}h + a\cdot x^{n-2}h^2 + b\cdot x^{n-3}h^3 + ... + h^n \right\}}{h}$ factor h out of the numerator

$= \lim\limits_{h \to 0} \dfrac{h\cdot\left\{ n\cdot x^{n-1} + a\cdot x^{n-2}h + b\cdot x^{n-3}h^2 + ... + h^{n-1} \right\}}{h}$ divide by the factor h

$= \lim\limits_{h \to 0} \{\, n{\cdot}x^{n-1} + a{\cdot}x^{n-2}h + b{\cdot}x^{n-3}h^2 + ... + h^{n-1} \,\}$ separate the limits

$= n{\cdot}x^{n-1} + \lim\limits_{h \to 0} \{\, a{\cdot}x^{n-2}\mathbf{h} + b{\cdot}x^{n-3}\mathbf{h^2} + ... + \mathbf{h^{n-1}} \,\}$ each term has a factor of h, and $h{\to}0$

$= n{\cdot}x^{n-1} + \mathbf{0}\ =\ \mathbf{n{\cdot}x^{n-1}}$ so $\mathbf{D}(\,x^n\,) = \mathbf{n{\cdot}x^{n-1}}$.

Practice 7:   Use the theorem to calculate $\mathbf{D}(\,x^5\,)$, $\dfrac{\mathbf{d}}{\mathbf{dx}}(\,x^2\,)$, $\mathbf{D}(\,x^{100}\,)$, $\dfrac{\mathbf{d}}{\mathbf{dt}}(\,t^{31}\,)$, and $\mathbf{D}(\,x^0\,)$.

We will occasionally use the result of the theorem for the derivatives of **all** constant powers of x even though it

has only been proven for positive integer powers, so far. The result for all constant powers of x is proved in

Section 2.9

Example 5: Find $\mathbf{D}(\,1/x\,)$ and $\dfrac{\mathbf{d}}{\mathbf{dx}}(\,\sqrt{x}\,)$.

Solution: $\mathbf{D}(\tfrac{1}{x}\,) = \mathbf{D}(x^{-1}) = -1x^{(-1)-1} = -1x^{-2} = \dfrac{-1}{x^2}$. $\dfrac{\mathbf{d}}{\mathbf{dx}}(\,\sqrt{x}\,)\ = \mathbf{D}(\,x^{1/2}\,) = (1/2)x^{-1/2} = \dfrac{1}{2\sqrt{x}}$.

These results can be obtained by using the definition of the derivative, but the algebra is slightly awkward.

Practice 8:   Use the pattern of the theorem to find $\mathbf{D}(\,x^{3/2}\,)$, $\dfrac{\mathbf{d}}{\mathbf{dx}}(\,x^{1/3}\,)$, $\mathbf{D}(\dfrac{1}{\sqrt{x}}\,)$ and $\dfrac{\mathbf{d}}{\mathbf{dt}}(\,t^{\pi}\,)$.

Example 6: It costs $\sqrt{x}$ hundred dollars to run a training program for x employees.

  (a) How much does it cost to train 100 employees? 101 employees? If you already need to train 100

   employees, how much additional will it cost to add 1 more employee to those being trained?

  (b) For $f(x) = \sqrt{x}$, calculate f '(x) and evaluate f ' at x = 100. How does f '(100) compare with the

   last answer in part (a)?

Solution: (a) Put $f(x) = \sqrt{x} = x^{1/2}$ hundred dollars, the cost to train x employees. Then $f(100) = \$1000$

   and $f(101) = \$1004.99$, so it costs \$4.99 additional to train the 101st employee.

  (b) $f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$ so $f'(100) = \frac{1}{2\sqrt{100}} = \frac{1}{20}$ hundred dollars $= \$5.00$.

   Clearly f '(100) is very close to the actual additional cost of training the 101st employee.

IMPORTANT DEFINITIONS AND RESULTS

Definition of Derivative: $f'(x) \equiv \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ if the limit exists and is finite.

Notations For The Derivative: $f'(x), Df(x), \frac{d\ f(x)}{dx}$

Tangent Line Equation: The line $y = f(a) + f'(a) \cdot (x - a)$ is tangent to the graph of f at (a , f(a)) .

Formulas: $D(\text{constant}) = 0$

     $D(x^n) = n \cdot x^{n-1}$ (proven for n = positive integer: true for all constants n)

     $D(\sin(x)) = \cos(x)$ and $D(\cos(x)) = -\sin(x)$

     $D(|x|) = \begin{cases} +1 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$.

Interpretations of f '(x):   Slope of a line tangent to a graph

       Instantaneous rate of change of a function at a point

       Velocity or acceleration

       Magnification factor

       Marginal change

PROBLEMS

1. Match the graphs of the three functions in Fig. 8 with the graphs of their derivatives.
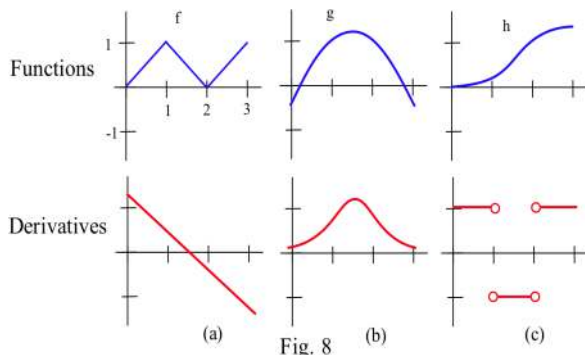
2. Fig. 9 shows six graphs, three of which are derivatives of the other three. Match the functions with their derivatives.

Fig. 8

Fig. 9
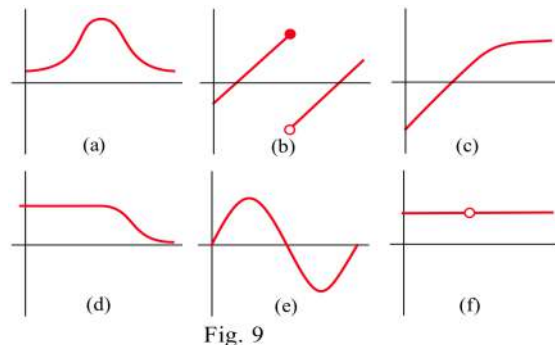
In problems 3 – 6, find the slope m_{sec} of the secant line through the two given points and then calculate $m_{tan} = \lim\limits_{h \to 0} m_{sec}$.

3. $f(x) = x^2$ (a) $(-2,4), (-2+h, (-2+h)^2)$ (b) $(0.5, 0.25), (0.5+h, (0.5+h)^2)$

4. $f(x) = 3 + x^2$ (a) $(1,4), (1+h, 3+(1+h)^2)$ (b) $(x, 3 + x^2), (x+h, 3 + (x+h)^2)$

5. $f(x) = 7x - x^2$ (a) $(1, 6), (1+h, 7(1+h) - (1+h)^2)$ (b) $(x, 7x - x^2), (x+h, 7(x+h) - (x+h)^2)$

6. $f(x) = x^3 + 4x$ (a) $(1, 5), (1+h, (1+h)^3 + 4(1+h))$ (b) $(x, x^3 + 4x), (x+h, (x+h)^3 + 4(x+h))$
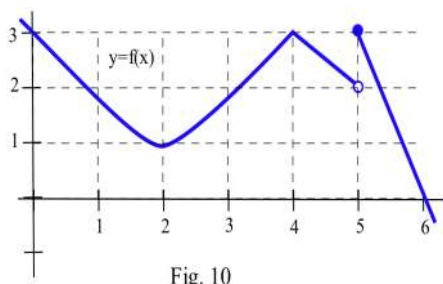
Fig. 10

7. Use the graph in Fig. 10 to estimate the values of these limits. (It helps to recognize what the limit represents.)

(a) $\lim\limits_{h \to 0} \dfrac{f(0 + h) - f(0)}{h}$ (b) $\lim\limits_{h \to 0} \dfrac{f(1 + h) - f(1)}{h}$ (c) $\lim\limits_{h \to 0} \dfrac{f(2 + h) - 1}{h}$

(d) $\lim\limits_{w \to 0} \dfrac{f(3 + w) - f(3)}{w}$ (e) $\lim\limits_{h \to 0} \dfrac{f(4 + h) - f(4)}{h}$ (f) $\lim\limits_{s \to 0} \dfrac{f(5 + s) - f(5)}{s}$
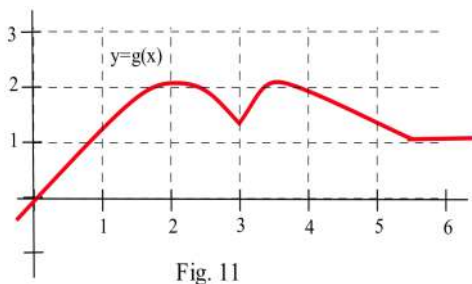
Fig. 11

8. Use the graph in Fig. 11 to estimate the values of these limits.

(a) $\lim\limits_{h \to 0} \dfrac{g(h) - g(0)}{h}$ (b) $\lim\limits_{h \to 0} \dfrac{g(1 + h) - g(1)}{h}$ (c) $\lim\limits_{h \to 0} \dfrac{g(2 + h) - 2}{h}$

(d) $\lim\limits_{w \to 0} \dfrac{g(3 + w) - g(3)}{w}$ (e) $\lim\limits_{h \to 0} \dfrac{g(4 + h) - g(4)}{h}$ (f) $\lim\limits_{s \to 0} \dfrac{g(5 + s) - g(5)}{s}$

In problems 9 – 12 , use the Definition of the derivative to calculate $f'(x)$ and then evaluate $f'(3)$.

9. $f(x) = x^2 + 8$

10. $f(x) = 5x^2 - 2x$

11. $f(x) = 2x^3 - 5x$

12. $f(x) = 7x^3 + x$

13. Graph $f(x) = x^2$, $g(x) = x^2 + 3$ and $h(x) = x^2 - 5$. Calculate the derivatives of f, g, and h.

14. Graph $f(x) = 5x$, $g(x) = 5x + 2$ and $h(x) = 5x - 7$. Calculate the derivatives of f, g, and h.

In problems 15 – 18, find the slopes and equations of the lines tangent to $y = f(x)$ at the given points.

15. $f(x) = x^2 + 8$ at (1,9) and (–2,12)

16. $f(x) = 5x^2 - 2x$ at (2, 16) and (0,0)

17. $f(x) = \sin(x)$ at $(\pi, 0)$ and $(\pi/2, 1)$

18. $f(x) = |x + 3|$ at (0,3) and (–3,0)

19. (a) Find the equation of the line tangent to the graph of $y = x^2 + 1$ at the point (2,5).

   (b) Find the equation of the line perpendicular to the graph of $y = x^2 + 1$ at (2,5).

   (c) Where is the tangent to the graph of $y = x^2 + 1$ horizontal?

   (d) Find the equation of the line tangent to the graph of $y = x^2 + 1$ at the point (p,q).

   (e) Find the point(s) (p,q) on the graph of $y = x^2 + 1$ so the tangent line to the curve at (p,q) goes

        through the point (1, –7).

20. (a) Find the equation of the line tangent to the graph of $y = x^3$ at the point (2,8).

   (b) Where, if ever, is the tangent to the graph of $y = x^3$ horizontal?

   (c) Find the equation of the line tangent to the graph of $y = x^3$ at the point (p,q).

   (d) Find the point(s) (p,q) on the graph of $y = x^3$ so the tangent line to the curve at (p,q) goes

        through the point (16,0).

21. (a) Find the angle that the tangent line to $y = x^2$ at (1,1) makes with the x–axis.

   (b) Find the angle that the tangent line to $y = x^3$ at (1,1) makes with the x–axis.

   (c) The curves $y = x^2$ and $y = x^3$ intersect at the point (1,1). Find the angle of intersection of the

        two curves (actually the angle between their tangent lines) at the point (1,1) .

22.   Fig. 12 shows the graph of $y = f(x)$. Sketch the graph of $y = f'(x)$.

23.   Fig. 13 shows the graph of the height of an object at time t. Sketch the graph of the

      object's upward velocity. What are the units for each axis on the velocity graph?

Fig. 12

Fig. 13
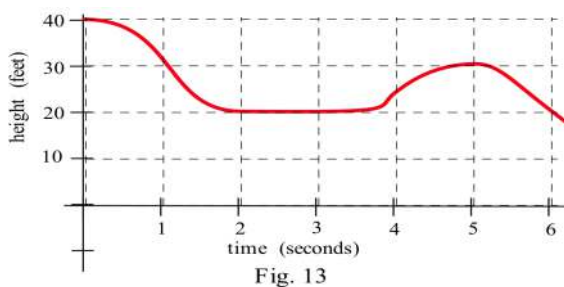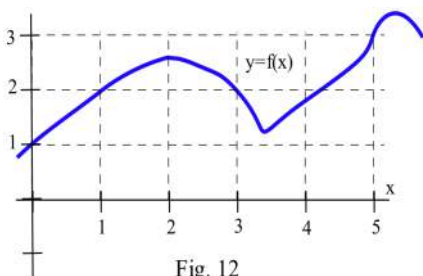
24.   Fill in the table with the appropriate units for f '(x).

units for x	units for f(x)	units for f '(x)
hours	miles	
people	automobiles	
dollars	pancakes	
days	trout	
seconds	miles per second	
seconds	gallons	
study hours	test points	

25. A rock dropped into a deep hole will drop $d(x) = 16x^2$ feet in x seconds.

   (a) How far into the hole will the rock be after 4 seconds? 5 seconds?

   (b) How fast will it be falling at exactly 4 seconds? 5 seconds? x seconds?

26. It takes $T(x) = x^2$ hours to weave x small rugs. What is the marginal production time to weave
      a rug? (Be sure to include the units with your answer.)

27. It costs $C(x) = \sqrt{x}$ dollars to produce x golf balls. What is the marginal production cost to make
      a golf ball? What is the marginal production cost when x = 25? when x= 100? (Include units.)

28. Define A(x) to be the **area** bounded by the x and y axes, the
      line y = 5, and a vertical line at x (Fig. 14).

(a) Evaluate A(0), A(1), A(2) and A(3).

   (b) Find a formula for A(x) for x ≥ 0: A(x) = ?

   (c) Determine $\dfrac{d\,A(x)}{dx}$.   (d) What does $\dfrac{d\,A(x)}{dx}$ represent?

Fig. 14
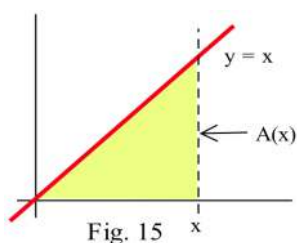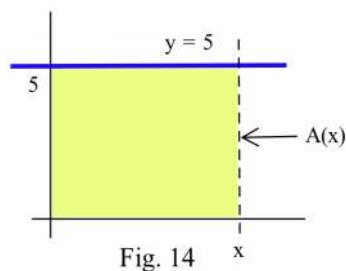
29. Define A(x) to be the **area** bounded by the x–axis,
      the line y = x, and a vertical line at x (Fig. 15).

   (a) Evaluate A(0), A(1), A(2) and A(3).

   (b) Find a formula which represents A(x) for all x ≥ 0: A(x) = ?

   (c) Determine $\dfrac{d\,A(x)}{dx}$.   (d) What does $\dfrac{d\,A(x)}{dx}$ represent?

Fig. 15

30. Find (a) $\mathbf{D}(x^{12})$ (b) $\dfrac{\mathbf{d}}{\mathbf{dx}}(\sqrt[7]{x})$ (c) $\mathbf{D}(\dfrac{1}{x^3})$ (d) $\dfrac{\mathbf{d}\,x^e}{\mathbf{dx}}$ (e) $\mathbf{D}(\,|\,x{-}2\,|\,)$

31. Find (a) $\mathbf{D}(x^9)$ (b) $\dfrac{\mathbf{d}\,x^{2/3}}{\mathbf{dx}}$ (c) $\mathbf{D}(\dfrac{1}{x^4})$ (d) $\mathbf{D}(x^{\pi})$ (e) $\dfrac{\mathbf{d}\,|\,x{+}5\,|}{\mathbf{dx}}$

In problems 32 – 37, find a function f which has the given derivative. (Each problem has several correct answers, just find one of them.)

32. $f'(x) = 4x + 3$

33. $f'(x) = 3x^2 + 8x$

34. $D(f(x)) = 12x^2 - 7$

35. $\dfrac{d\, f(t)}{dt} = 5\cos(t)$

36. $\dfrac{d\, f(x)}{dx} = 2x - \sin(x)$

37. $D(f(x)) = x + x^2$

Section 3.2   PRACTICE Answers

Practice 1:   The graph of $f(x) = 7x$ is a line through the origin. The slope of the line is 7.

$$\text{For all x, } m_{tan} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{7(x+h) - 7x}{h} = \lim_{h \to 0} \frac{7h}{h} = \lim_{h \to 0} 7 = \textbf{7}\ .$$

Practice 2: $f(x) = x^3$ so $f(x+h) = (x+h)^3 = x^3 + 3x^2 h + 3xh^2 + h^3$.

$$\frac{dy}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\left\{ x^3 + 3x^2 h + 3xh^2 + h^3 \right\} - x^n}{h}$$

$$= \lim_{h \to 0} \frac{3x^2 h + 3xh^2 + h^3}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2) = \textbf{3x}^2\ .$$

At the point (2,8), the slope of the tangent line is $3(2)^2 = 12$ so the equation of the tangent line is $y - 8 = 12(x - 2)$ or $y = 12x - 16$.

Practice 3: $D(x^4) = 4x^3$, $D(x^5) = 5x^4$, $D(x^{43}) = 43x^{42}$, $D(x^{1/2}) = \dfrac{1}{2}\, x^{-1/2}$, $D(x^{\pi}) = \pi x^{\pi - 1}$

Practice 4: $D(\cos(x)) = \lim_{h \to 0} \dfrac{\cos(x+h) - \cos(x)}{h} = \lim_{h \to 0} \dfrac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h}$

$$= \lim_{h \to 0} \cos(x)\frac{\cos(h) - 1}{h} - \sin(x)\frac{\sin(h)}{h} \longrightarrow \cos(x)\cdot(0) - \sin(x)\cdot(1) = \textbf{- sin(x)}\ .$$

y = | x-2 |

y = | 2x |
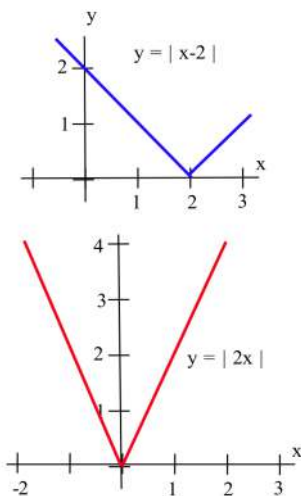
Fig. 16

Practice 5:   See Fig. 16 for the graphs of $y = |x - 2|$ and $y = |2x|$.

$$D(|x - 2|) = \begin{cases} +1 & \text{if } x > 2 \\ \text{undefined} & \text{if } x = 2 \\ -1 & \text{if } x < 2 \end{cases}$$

$$D(|2x|) = \begin{cases} +2 & \text{if } x > 0 \\ \text{undefined} & \text{if } x = 0 \\ -2 & \text{if } x < 0 \end{cases}$$

Practice 6: $h(t) = \sin(t)$ so $h(1) = \sin(1) \approx 0.84$ feet,

$v(t) = \cos(t)$ so $v(1) = \cos(1) \approx 0.54$ feet/second.

$a(t) = -\sin(t)$ so $a(1) = -\sin(1) \approx -0.84$ feet/second^2 .

Practice 7: $\mathbf{D}(x^5) = 5x^4$, $\dfrac{\mathbf{d}\,x^2}{\mathbf{dx}} = 2x^1 = 2x$, $\dfrac{\mathbf{d}\,x^{100}}{\mathbf{dx}} = 100x^{99}$, $\dfrac{\mathbf{d}\,t^{31}}{\mathbf{dt}} = 31t^{30}$,

$\mathbf{D}(x^0) = 0x^{-1} = 0$ or $\mathbf{D}(x^0) = \mathbf{D}(1) = 0$.

Practice 8: $\mathbf{D}(x^{3/2}) = \dfrac{3}{2}\,x^{1/2}$, $\dfrac{\mathbf{d}\,x^{1/3}}{\mathbf{dx}} = \dfrac{1}{3}\,x^{-2/3}$, $\mathbf{D}(x^{-1/2}) = \dfrac{-1}{2}\,x^{-3/2}$, $\dfrac{\mathbf{d}\,t^{\pi}}{\mathbf{dt}} = \pi\,t^{\pi-1}$.

3.3 DERIVATIVES: PROPERTIES AND FORMULAS

This section begins with a look at which functions have derivatives. Then we'll examine how to calculate derivatives of elementary combinations of basic functions. By knowing the derivatives of some basic functions and just a few differentiation patterns, you will be able to calculate the derivatives of a tremendous variety of functions. This section contains most, but not quite all, of the general differentiation patterns you will ever need.

WHICH FUNCTIONS HAVE DERIVATIVES?

Theorem: If   a function is **differentiable** at a point,

then   it is **continuous** at that point.

The contrapositive form of this theorem tells about some functions which do not have derivatives:

Contrapositive Form of the Theorem:

If   f is **not continuous** at a point,

then   f is **not differentiable** at that point.

Proof of the Theorem: We assume that the hypothesis , f is differentiable at the point c , is true so

$\lim_{h \to 0} \dfrac{f(c+h) - f(c)}{h}$ exists and equals f '(c). We want to show that f must necessarily be continuous at c : $\lim_{h \to 0} f(c+h) = f(c)$.

Since f(c + h) can be written as $f(c+h) = f(c) + \left\{ \dfrac{f(c+h) - f(c)}{h} \right\} \cdot h$, we have

$$\lim_{h \to 0} f(c+h) = \lim_{h \to 0} \left(f(c) + \left\{ \frac{f(c+h) - f(c)}{h} \right\} \cdot h \right)$$

$$= \lim_{h \to 0} f(c) + \lim_{h \to 0} \left\{ \frac{f(c+h) - f(c)}{h} \right\} \cdot \lim_{h \to 0}(h) = \text{f(c)} + \text{f '(c)} \cdot 0 = \text{f(c)} .$$

Therefore f is continuous at c .

It is important to clearly understand what is meant by this theorem and what is not meant: If the function is differentiable at a point, then the function is automatically continuous at that point. If the function is continuous at a point, then the function **may** or **may not** have a derivative at that point.

If the function is not continuous at a point, then the function **is not** differentiable at that point.

Example 1:   Show that f(x) = [x] = INT(x) is not continuous and not differentiable at 2 (Fig. 1).

Solution: The one–sided limits, $\lim\limits_{x\to 2^+}$ INT(x) = 2 and

$\lim\limits_{x\to 2^-}$ INT(x) = 1 , have different values so $\lim\limits_{x\to 2}$ INT(x) does not

exist, and   INT(x) is not continuous at 2. Since f(x) = INT(x) is

not continuous at 2, it is not differentiable there.
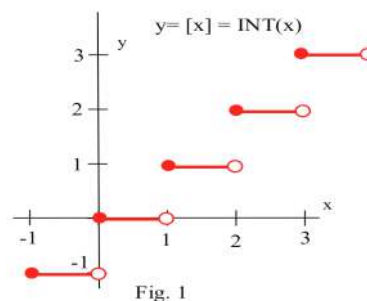
Fig. 1

Lack of continuity is enough to imply lack of differentiability, but the

next two examples show that continuity is **not** enough to guarantee

differentiability.

Example 2: Show that f(x) = | x | is continuous but **not** differentiable
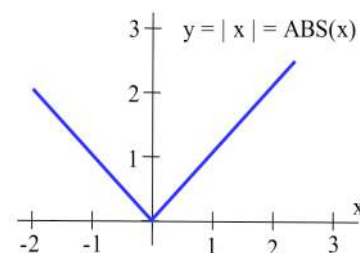
at x = 0 (Fig. 2)

Solution: $\lim\limits_{x\to 0}$ |x| = 0 = |0| so f is continuous at 0, but we showed in

Section 2.1 that the absolute value function was not differentiable at x = 0.

Fig. 2

A function is not differentiable at a cusp or a"corner."

Example 3: Show that f(x) = $\sqrt[3]{x}$ = x$^{1/3}$ is continuous but **not** differentiable at x = 0 (Fig. 3)
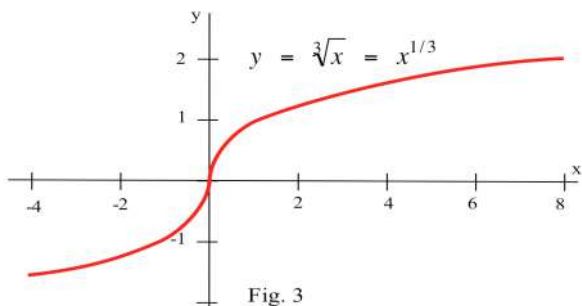
Fig. 3

Solution: $\lim\limits_{x\to 0^-} \sqrt[3]{x} = \lim\limits_{x\to 0^+} \sqrt[3]{x} = 0$ so

$\lim\limits_{x\to 0} \sqrt[3]{x} = 0 = \sqrt{0}$, and f is continuous at 0.

f '(x) = $\frac{1}{3}$ x$^{-(2/3)}$ = $\frac{1}{3\,x^{2/3}}$ which is undefined

at x = 0 so f is not differentiable at 0.

A function is not differentiable where its tangent line is vertical.

Practice 1:   At which integer values of x is the graph of f in Fig. 4 continuous?   differentiable?
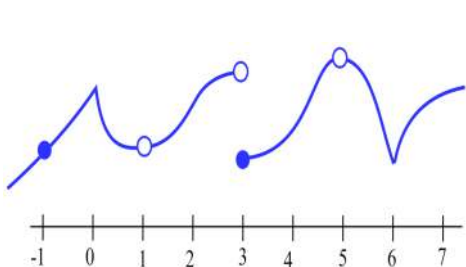
Fig. 4

Graphically, a function is **continuous** if and only if its graph

is **connected** and does not have any holes or breaks.

Graphically, a function is **differentiable** if and only if it is

continuous and its graph is **smooth** with no

corners or vertical tangent lines.

DERIVATIVES OF ELEMENTARY COMBINATIONS OF FUNCTIONS

Example 4: The derivative of $f(x) = x$ is $\mathbf{D}f(x) = 1$, and the derivative of $g(x) = 5$ is $\mathbf{D}g(x) = 0$.

What are the derivatives of their elementary combinations: $3f, f + g, f - g, f\cdot g$ and f/g ?

Solution: $\mathbf{D}(\,3\,f(x)\,) = \mathbf{D}(\,3x\,) = 3 = 3\,\mathbf{D}(\,f(x)\,)$**.**

$\mathbf{D}(\,f(x) + g(x)\,) = \mathbf{D}(\,x + 5\,) = 1 = \mathbf{D}(\,f(x)\,) + \mathbf{D}(\,g(x)\,)$.

$\mathbf{D}(\,f(x) - g(x)\,) = \mathbf{D}(\,x - 5\,) = 1 = \mathbf{D}(\,f(x)\,) - \mathbf{D}(\,g(x)\,)$.

Unfortunately, the derivatives of $f\cdot g$ and f/g don't follow the same easy patterns:

$\mathbf{D}(\,f(x)\cdot g(x)\,) = \mathbf{D}(\,5x\,) = 5$ but $\mathbf{D}(\,f(x)\,)\cdot \mathbf{D}(\,g(x)\,) = (1)\cdot(0) = 0$, and

$\mathbf{D}(\,f(x)\,/\,g(x)\,) = \mathbf{D}(\,x\,/\,5\,) = 1/5$ but $\mathbf{D}(\,f(x)\,)\,/\,\mathbf{D}(\,g(x)\,)$ is undefined.

These two very simple functions show that, in general, $\mathbf{D}(\,f\cdot g\,) \neq \mathbf{D}(\,f\,)\cdot \mathbf{D}(\,g\,)$ and $\mathbf{D}(\,f/g\,) \neq \mathbf{D}(\,f\,)\,/\,\mathbf{D}(\,g\,)$.

The Main Differentiation Theorem below states the correct patterns for differentiating products and quotients.

Practice 2:   For $f(x) = 6x + 8$ and $g(x) = 2$, what are the derivatives of $3f, f+g, f-g, f\cdot g$ and f/g ?

The following theorem says that the simple patterns in the example for constant multiples of functions and sums and differences of functions are true for all differentiable functions. It also includes the correct patterns for derivatives of products and quotients of differentiable functions.

Main Differentiation Theorem: If f and g are differentiable at x , then

(a) **Constant Multiple Rule:** $\mathbf{D}(\,kf(x)\,) = k\mathbf{D}(\,f(x)\,) = k\mathbf{D}\mathbf{f}$

or $(\,kf(x)\,)\,' = k\,f\,'(x)$

(b) **Sum Rule:** $\mathbf{D}(\,f(x) + g(x)\,) = \mathbf{D}(\,f(x)\,) + \mathbf{D}(\,g(x)\,) = \mathbf{D}\mathbf{f} + \mathbf{D}\mathbf{g}$

or $(\,f(x) + g(x)\,)\,' = f\,'(x) + g\,'(x)$

(c) **Difference Rule:** $\mathbf{D}(\,f(x) - g(x)\,) = \mathbf{D}(\,f(x)\,) - \mathbf{D}(\,g(x)\,) = \mathbf{D}\mathbf{f} - \mathbf{D}\mathbf{g}$

or $(\,f(x) - g(x)\,)\,' = f\,'(x) - g\,'(x)$

(d) **Product Rule:** $\mathbf{D}(\,f(x)\cdot g(x)\,) = f(x)\cdot \mathbf{D}(\,g(x)\,) + g(x)\cdot \mathbf{D}(\,f(x)\,) = f\,\mathbf{D}\mathbf{g} + g\,\mathbf{D}\mathbf{f}$

or $(\,f(x)\cdot g(x)\,)\,' = f(x)\cdot \mathbf{g}\,'(\mathbf{x}) + g(x)\cdot \mathbf{f}\,'(\mathbf{x})$

(e) **Quotient Rule:** $\mathbf{D}(\dfrac{f(x)}{g(x)}\,) = \dfrac{g(x)\cdot \mathbf{D}(\,f(x)\,) - f(x)\cdot \mathbf{D}(\,g(x)\,)}{(g(x))^2}$

$= \dfrac{g\mathbf{D}\mathbf{f} - f\mathbf{D}\mathbf{g}}{g^2}$ or $\dfrac{g(x)\cdot \mathbf{f}\,'(\mathbf{x}) - f(x)\cdot \mathbf{g}\,'(\mathbf{x})}{g^2(x)}$

(provided $g(x) \neq 0$)

The proofs of parts (a), (b), and (c) of this theorem are straightforward, but parts (d) and (e) require some clever algebraic manipulations. Lets look at an example first.

Example 5:   Recall that $D(x^2) = 2x$ and $D(\sin(x)) = \cos(x)$. Find $D(3\sin(x))$ and $\dfrac{d}{dx}(5x^2 - 7\sin(x))$.

Solution: $D(3\sin(x))$ is an application of part (a) of the theorem with $k = 3$ and $f(x) = \sin(x)$ so

   $D(3\sin(x)) = 3\, D(\sin(x)) = 3 \cos(x)$.

   $\dfrac{d}{dx}(5x^2 - 7\sin(x))$ uses part (c) of the theorem with $f(x) = 5x^2$ and $g(x) = 7\sin(x)$ so

$$\frac{d}{dx}(5x^2 - 7\sin(x)) = \frac{d}{dx}(5x^2) - \frac{d}{dx}(7\sin(x)) \qquad = 5\frac{d}{dx}(x^2) - 7\frac{d}{dx}(\sin(x))$$

$$= 5(2x) - 7(\cos(x)) = 10x - 7\cos(x) .$$

Practice 3:   Find $D(x^3 - 5\sin(x))$ and $\dfrac{d}{dx}(\sin(x) - 4x^3)$.

Practice 4:   Fill in the values in the table for $D(3f(x))$, $D(2f(x)+g(x))$, and $D(3g(x) - f(x))$

x	f(x)	f '(x)	g(x)	g '(x)	D(3f(x))	D(2f(x) + g(x))	D(3g(x) − f(x))
0	3	−2	−4	3			
1	2	−1	1	0			
2	4	2	3	1			

Proof of the Main Derivative Theorem (a) and (c): The only general fact we have about derivatives is the definition as a limit, so our proofs here will have to recast derivatives as limits and then use some results about limits. The proofs are applications of the definition of the derivative and results about limits.

(a) $D(kf(x)) \equiv \lim\limits_{h \to 0} \dfrac{k \cdot f(x + h) - k \cdot f(x)}{h} = \lim\limits_{h \to 0} k \cdot \dfrac{f(x + h) - f(x)}{h} = k \cdot \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h} = k \cdot D(f(x)).$

(c) $D(f(x) - g(x)) \quad = \lim\limits_{h \to 0} \dfrac{\{f(x + h) - g(x + h)\} - \{f(x) - g(x)\}}{h}$

$= \lim\limits_{h \to 0} \dfrac{\{f(x + h) - f(x)\} - \{g(x + h) - g(x)\}}{h} = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h} - \lim\limits_{h \to 0} \dfrac{g(x + h) - g(x)}{h}$

   $= D(f(x)) - D(g(x))$.

   The proof of part (b) is very similar to these two proofs, and is left for you as the next Practice Problem.

   The proof for the Product Rule and Quotient Rules will be given later.

Practice 5:   Prove part (b) of the theorem, the Sum Rule: $D(f(x) + g(x)) = D(f(x)) + D(g(x))$.

Practice 6:   Use the Main Differentiation Theorem and the values in the table to fill in the rest of the table.

x	f(x)	f '(x)	g(x)	g '(x)	D(f(x)·g(x))	D(f(x)/g(x))	D(g(x)/f(x))
0	3	−2	−4	3			
1	2	−1	1	0			
2	4	2	3	1			

Example 6:   Determine $D(x^2 \cdot \sin(x))$ and $\dfrac{d}{dx}(\dfrac{x^3}{\sin(x)})$.

Solution: (a)   We can use the Product Rule with $f(x) = x^2$ and $g(x) = \sin(x)$:

$$D(x^2 \cdot \sin(x)) = D(f(x) \cdot g(x)) \quad\quad = f(x) \cdot D(g(x)) + g(x) \cdot D(f(x))$$
$$= (x^2) \cdot D(\sin(x)) + \sin(x) \cdot D(x^2)$$
$$= (x^2) \cdot (\mathbf{cos(x)}) + \sin(x) \cdot (\mathbf{2x}) = x^2 \cdot \cos(x) + 2x \cdot \sin(x)$$

(b)   We can use the Quotient Rule with $f(x) = x^3$ and $g(x) = \sin(x)$:

$$\frac{d}{dx}(\frac{x^3}{\sin(x)}) = \frac{g(x) \cdot D(f(x)) - f(x) \cdot D(g(x))}{g^2(x)} = \frac{\sin(x) \cdot D(x^3) - x^3 \cdot D(\sin(x))}{(\sin(x))^2}$$

$$= \frac{\sin(x) \cdot (\mathbf{3x^2}) - x^3 \cdot \mathbf{cos(x)}}{\sin^2(x)} = \frac{3x^2 \sin(x) - x^3 \cos(x)}{\sin^2(x)}$$

Practice 7:   Determine $D((x^2 + 1)(7x - 3))$, $\dfrac{d}{dt}(\dfrac{3t - 2}{5t + 1})$ and $D(\dfrac{\cos(x)}{x})$.

Proof of the Product Rule: The proofs of parts (d) and (e) of the theorem are complicated but only
involve elementary techniques, used in just the right way. Sometimes we will omit such computational
proofs, but the Product and Quotient Rules are fundamental techniques you will need hundreds of times.

By the hypothesis, f and g are differentiable so $\displaystyle\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = f'(x)$ and $\displaystyle\lim_{h \to 0} \frac{g(x + h) - g(x)}{h} = g'(x)$.
Also, both f and g are continuous (why?) so $\displaystyle\lim_{h \to 0} f(x+h) = f(x)$ and $\displaystyle\lim_{h \to 0} g(x+h) = g(x)$.

(d) Product Rule: Let $P(x) = f(x) \cdot g(x)$. Then $P(x+h) = f(x+h) \cdot g(x+h)$.

$$D(f(x) \cdot g(x)) = D(P(x)) = \lim_{h \to 0} \frac{P(x + h) - P(x)}{h} = \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x + h)g(x + h) + \{ -f(x)g(x + h) + f(x)g(x + h) \} - f(x)g(x)}{h} \quad \text{adding and subtracting}$$
$$\text{f(x)g(x+h)}$$

$$= \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x + h)}{h} + \lim_{h \to 0} \frac{f(x)g(x + h) - f(x)g(x)}{h} \quad \text{regrouping the terms}$$

$$= \lim_{h \to 0} (g(x + h)) \cdot (\frac{f(x+h) - f(x)}{h}) + (f(x)) \cdot (\frac{g(x+h) - g(x)}{h}) \quad \text{finding common factors}$$
$$\downarrow \quad\quad\quad\quad \downarrow \quad\quad\quad\quad \downarrow \quad\quad\quad\quad \downarrow \quad\quad\quad \text{taking limit as } h \to 0$$
$$(g(x)) \cdot (f'(x)) \quad + \quad (f(x)) \cdot (g'(x)) = g \, Df + f \, Dg .$$

(e) The steps for a proof of the Quotient Rule are shown in Problem 55.

USING THE DIFFERENTIATION RULES

You definitely need to memorize the differentiation rules, but it is vitally important that you also know how to use them. Sometimes it is clear that the function we want to differentiate is a sum or product of two obvious functions, but we commonly need to differentiate functions which involve several operations and functions. Memorizing the differentiation rules is <u>only</u> the first step in learning to use them.

Example 7:   Calculate $\mathbf{D}(x^5 + x \cdot \sin(x))$.

Solution: This function is more difficult because it involves both an addition and a multiplication. Which rule(s) should we use, or, more importantly, which rule should we use <u>first</u>?

$$\mathbf{D}(x^5 + x \cdot \sin(x)) \quad = \mathbf{D}(x^5) + \mathbf{D}(x \cdot \sin(x)) \qquad \text{applying the Sum Rule and trading}$$
one derivative for two easier ones

$$= 5x^4 + \{ x \cdot \mathbf{D}(\sin(x)) + \sin(x) \cdot \mathbf{D}(x) \} \qquad \text{applying the product rule to } \mathbf{D}(x \cdot \sin(x))$$

$$= 5x^4 + x \cdot \cos(x) + \sin(x) \qquad \text{this expression has no more derivatives so we are done.}$$

If you were evaluating the function $x^5 + x \sin(x)$ for some particular value of x, you would (1) raise x to the 5th power, (2) calculate sin(x), (3) multiply sin(x) by x, and (4) your FINAL evaluation step, SUM the values of x^5 and x sin(x).

> **The FINAL step of your evaluation of f indicates**
> **the FIRST rule to use to calculate the derivative of f.**

Practice 8:   Which differentiation rule should you apply FIRST for each of the following:

(a) $x \cdot \cos(x) - x^3 \cdot \sin(x)$ (b) $(2x - 3) \cdot \cos(x)$ (c) $2\cos(x) - 7x^2$ (d) $\dfrac{\cos(x) + 3x}{\sqrt{x}}$

Practice 9:   Calculate $\mathbf{D}\left(\dfrac{x^2 - 5}{\sin(x)} \right)$ and $\dfrac{\mathbf{d}}{\mathbf{dt}}\left(\dfrac{t^2 - 5}{t \cdot \sin(t)} \right)$.
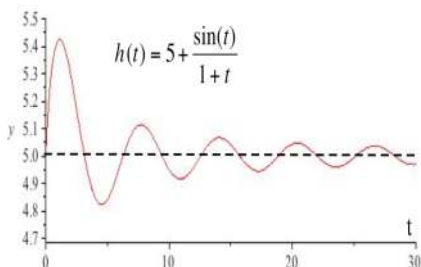
Fig. 5

Example 8:   A weight attached to a spring is oscillating up and down. Over a period of time, the motion becomes "damped" because of friction and air resistance (Fig. 5), and its height at time t seconds is $h(t) = 5 + \dfrac{\sin(t)}{1 + t}$ feet.

What are the height and velocity of the weight after 2 seconds?

Solution: The height is

$$h(2) = 5 + \frac{\sin(2)}{1 + 2} = 5 + \frac{.909}{3} = 5.303 \text{ feet above the ground.}$$

The velocity is h '(2) .

$$h\ '(t) = 0 + \frac{(1+t) \cdot D(\ \sin(t)\ ) - \sin(t) \cdot D(\ 1+t\ )}{(1+t)^2} = \frac{(1+t) \cdot \cos(t) - \sin(t)}{(1+t)^2}$$

so $h\ '(2) = \dfrac{3\cos(2) - \sin(2)}{9} = \dfrac{-2.158}{9} \approx -0.24$ feet per second .

Practice 10:   What are the height and velocity of the weight in the previous example after 5

seconds? What are the height and velocity of the weight be after a "long time" ?

Example 9:   Calculate $D(\ x \cdot \sin(x) \cdot \cos(x)\ )$.

Solution: Clearly we need to use the Product Rule since the only operation in this function is multiplication,

but the Product Rule deals with a product of <u>two</u> functions and we have the product of <u>three</u> ; x and

sin(x) and cos(x) . However, if we think of our two functions as $f(x) = \mathbf{x \cdot sin(x)}$ and $g(x) = \mathbf{cos(x)}$, then

we do have the product of two functions and

$$\mathbf{D}(\ x \cdot \sin(x) \cdot \cos(x)\ )\quad = \mathbf{D}(\ f(x) \cdot g(x)\ ) = f(x) \cdot \mathbf{D}(\ g(x)\ ) + g(x) \cdot \mathbf{D}(\ f(x)\ )$$

$$= x\ \sin(x) \cdot \mathbf{D}(\ \cos(x)\ ) + \cos(x) \cdot \mathbf{D}(\ x\ \sin(x)\ )$$

We are not done, but we have traded one hard derivative for two easier ones. We know that

$\mathbf{D}(\ \cos(x)\ ) = -\sin(x)$, and we can use the Product Rule (again) to calculate $\mathbf{D}(\ x\ \sin(x)\ )$. Then the

last line of our calculation becomes

$$= x\ \sin(x) \cdot (-\sin(x)\ ) + \cos(x) \cdot \left\{\ x\ \mathbf{D}(\ \sin(x)\ ) + \sin(x)\ \mathbf{D}(\ x\ )\ \right\}$$

$$= -x\ \sin^2(x) + \cos(x)\ \left\{\ x\ \mathbf{cos(x)} + \sin(x)\ \mathbf{(1)}\right\} = -x\ \sin^2(x) + x\ \cos^2(x) + \cos(x)\ \sin(x)\ .$$

EVALUATING A DERIVATIVE AT A POINT

The derivative of a function f is a new **function f '(x)** which gives the slope of the line tangent to the

graph of f at each point x. To find the slope of the tangent line at a particular point (c, f(c)) on the graph

of f, we should <u>first</u> calculate the derivative f '(x) and <u>then</u> evaluate the **function f '(x)** at the point

x = c to get the **number f '(c)**. If you mistakenly evaluate f first, you get a number f(c), and the derivative

of a constant is always equal to 0.

Example 10:   Determine the slope of the line tangent to $f(x) = 3x + \sin(x)$ at $(0, f(0)\ )$ and $(1, f(1\ ))$:

 Solution: $f\ '(x) = \mathbf{D}(\ 3x + \sin(x)\ ) = \mathbf{D}(3x) + \mathbf{D}(\ \sin(x)\ ) = 3 + \cos(x)$. When x = 0, the graph of

y = 3x + sin(x) goes through the point $(\ 0, 3(0) + \sin(0)\ )$ with slope $f\ '(0) = 3 + \cos(0) = 4$. When

x = 1, the graph goes through the point $(\ 1, 3(1) + \sin(1)\ ) = (1, 3.84)$ with slope

$f\ '(1) = 3 + \cos(1) \approx 3.54$.

Practice 11:   Where do $f(x) = x^2 - 10x + 3$ and $g(x) = x^3 - 12x$ have a horizontal tangent lines ?

IMPORTANT RESULTS OF THIS SECTION

Differentiability and Continuity: If a function is differentiable then it must be continuous.

If a function is not continuous then it cannot be differentiable.

A function may be continuous at a point and not differentiable there.

Graphically: CONTINUOUS means **connected.**

DIFFERENTIABLE means continuous, **smooth** and not vertical.

Differentiation Patterns: $D(kf(x))$ $= k \cdot D(f(x))$

$D(f + g)$ $= Df + Dg$

$D(f - g)$ $= Df - Dg$

$D(f \cdot g)$ $= f \cdot Dg + g \cdot Df$

$$D(f/g) = \frac{g \cdot Df - f \cdot Dg}{g^2}$$

The FINAL STEP used to evaluate f indicates the FIRST RULE to use to differentiate f.

To evaluate a derivative at a point, first differentiate and then evaluate.

PROBLEMS

1. The graph of $y = f(x)$ is given in Fig. 6.

(a) At which integers is f continuous?

(b) At which integers is f differentiable?
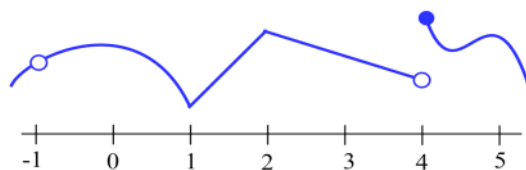
Fig. 6

2. The graph of $y = g(x)$ is given in Fig. 7.

(a) At which integers is g continuous?

(b) At which integers is g differentiable?
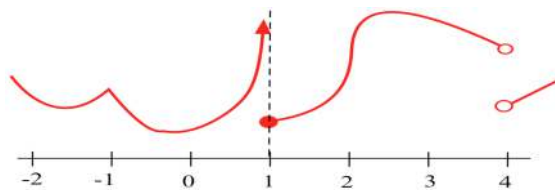
Fig. 7

3. Use the values given in the table to determine the values of $f \cdot g$, $D(f \cdot g)$, f/g and $D(f/g)$.

x	f(x)	f '(x)	g(x)	g '(x)	f(x)·g(x)	D(f(x)·g(x))	f(x)/g(x)	D(f(x)/g(x))
0	2	3	1	5				
1	−3	2	5	−2				
2	0	−3	2	4				
3	1	−1	0	3				

4. Use the values given in the table to determine the values of f•g , **D**(f•g) , f/g and **D**(f/g) .

x	f(x)	f '(x)	g(x)	g '(x)	f(x)•g(x)	**D**(f(x)•g(x))	f(x)/g(x)	**D**(f(x)/g(x))
0	4	2	3	−3				
1	0	3	2	1				
2	−2	5	0	−1				
3	−1	−2	−3	4				
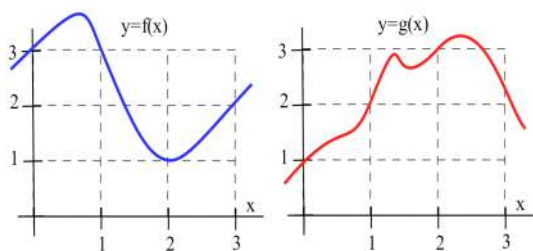
Fig. 8

5. Use the information in Fig. 8 to plot the values of the functions
f + g, f•g and f/g and their derivatives at x = 1, 2 and 3 .

6. Use the information in Fig. 8 to plot the values of the functions
2f, f − g and g/f and their derivatives at x = 1, 2 and 3 .

7. Calculate **D**((x − 5)(3x + 7)) by (a) using the product rule and (b) expanding the product and
then differentiating. Verify that both methods give the same result.

8. Calculate **D**($\sqrt{x}$ •sin(x)) .

9. Calculate $\frac{d}{dx} (\frac{\cos(x)}{x^2})$.

10. Calculate **D**(sin(x) + cos(x)) .

11. Calculate **D**($\sin^2(x)$) and **D**($\cos^2(x)$) **.**

12. Calculate **D**(sin(x)), $\frac{d}{dx}$ (sin(x) + 7) , **D**(sin(x) − 8000) and **D**(sin(x) + k).

13. Find values for the constants a, b and c so that the parabola $f(x) = ax^2 + bx + c$ has f(0) = 0,
f '(0) = 0 and f '(10) = 30.

14. If f is a differentiable function,
(a) how are the graphs of y = f(x) and   y = f(x) + k related?
(b) how are the derivatives of f(x) and f(x) + k related?

15. If f and g are differentiable functions which always differ by a constant (f(x) − g(x) = k for
all x), then what can you conclude about their graphs and their derivatives?

16. If f and g are differentiable functions whose sum is a constant (f(x) + g(x) = k for all x), then what
can you conclude about (a) their graphs? (b) their derivatives?

17. If the product of f and g is a constant (f(x)g(x) = k for all x), then how are $\frac{\textbf{D}(f(x))}{f(x)}$
and $\frac{\textbf{D}(g(x))}{g(x)}$ related?

18. If the quotient of f and g is a constant (f(x)/g(x) = k for all x), then how are g•f ' and f•g ' related?

In problems 19 – 28, (a) calculate f '(1) and (b) determine when f '(x) = 0.

19. $f(x) = x^2 - 5x + 13$

20. $f(x) = 5x^2 - 40x + 73$

21. $f(x) = 3x - 2\cos(x)$

22. $f(x) = |x + 2|$

23. $f(x) = x^3 + 9x^2 + 6$

24. $f(x) = x^3 + 3x^2 + 3x - 1$

25. $f(x) = x^3 + 2x^2 + 2x - 1$

26. $f(x) = \dfrac{7x}{x^2 + 4}$

27. $f(x) = x\cdot\sin(x)$ and $0 \le x \le 5$. (You may need to use the Bisection Algorithm or the "trace" option on a calculator to approximate where f '(x) = 0.)

28. $f(x) = A x^2 + B x + C$ A, B and C are constants and $A \neq 0$.

29. $f(x) = x^3 + A x^2 + B x + C$ with constants A, B and C. Can you find conditions on the constants A, B and C which will guarantee that the graph of y = f(x) has two distinct "vertices"? (Here a "vertex" means a place where the curve changes from increasing to decreasing or from decreasing to increasing.)

Where are the functions in problems 30 – 37 differentiable?

30. $f(x) = |x| \cos(x)$

31. $f(x) = \dfrac{x - 5}{x + 3}$

32. $f(x) = \tan(x)$

33. $f(x) = \dfrac{x^2 + x}{x^2 - 3x}$

34. $f(x) = |x^2 - 4|$

35. $f(x) = |x^3 - 1|$

36. $f(x) = \begin{cases} 0 & \text{if } x < 0 \\ \sin(x) & \text{if } x \ge 0 \end{cases}$

37. $f(x) = \begin{cases} x & \text{if } x < 0 \\ \sin(x) & \text{if } x \ge 0 \end{cases}$

38. For what value(s) of A is $f(x) = \begin{cases} Ax - 4 & \text{if } x < 2 \\ x^2 + x & \text{if } x \ge 2 \end{cases}$ differentiable at x = 2?

39. For what values of A and B is $f(x) = \begin{cases} Ax + B & \text{if } x < 1 \\ x^2 + x & \text{if } x \ge 1 \end{cases}$ differentiable at x = 1?

40. An arrow shot straight up from ground level with an initial velocity of 128 feet per second will be at height $h(x) = -16x^2 + 128x$ feet at x seconds. (Fig.9)

(a) Determine the velocity of the arrow when x = 0, 1 and 2 seconds.

(b) What is the velocity of the arrow, v(x), at any time x?

(c) At what time x will the velocity of the arrow be 0?

(d) What is the greatest height the arrow reaches?

(e) How long will the arrow be aloft?

(f)   Use the answer for the velocity in part (b) to determine the acceleration, a(x) = v '(x), at any time x.

Fig. 9
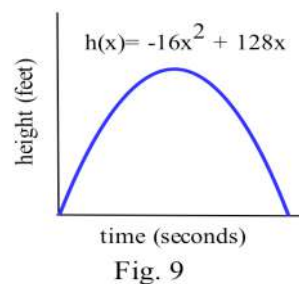
41. If an arrow is shot straight up from ground level on the moon with an initial velocity of 128 feet per second, its height will be $h(x) = -2.65x^2 + 128x$ feet at x seconds. Do parts (a) – (e) of problem 40 using this new equation for h.

42. In general, if an arrow is shot straight upward with an initial velocity of 128 feet per second from ground level on a planet with a constant gravitational acceleration of g feet per second^2 , then its height will be $h(x) = -\frac{g}{2} x^2 + 128x$ feet at x seconds. Answer the questions in problem 40 for arrows shot on Mars and Jupiter (Use the values in Fig. 10).

43. If an object on Earth is propelled upward from ground level with an initial velocity of v_0 feet per second, then its height at x seconds will be $h(x) = -16x^2 + v_0 x$.

Object	g (ft/sec^2)	g (cm/sec^2)
Mercury	11.8	358
Venus	20.1	887
Earth	32.2	981
moon	5.3	162
Mars	12.3	374
Jupiter	85.3	2601
Saturn	36.6	1117
Uranus	34.4	1049
Neptune	43.5	1325
Pluto	7.3	221

Source: CRC Handbook of Chemistry and Physics

Fig. 10: Values of g

  (a) What will be the object's velocity after x seconds.

  (b) What is the greatest height the object will reach?

  (c) How long will the object remain aloft?

44. In order for a 6 foot tall basketball player to dunk the ball, the player must achieve a vertical jump of about 3 feet. Use the information in the previous problems to answer the following questions.

  (a) What is the smallest initial vertical velocity the player can have to dunk the ball?

  (b) With the initial velocity achieved in part (a), how high would the player jump on the moon?

45. The best high jumpers in the world manage to lift their centers of mass approximately 6.5 feet.

  (a) What is the initial vertical velocity these high jumpers attain?

  (b) How long are these high jumpers in the air?

  (c) With the initial velocity in part (a), how high would they lift their centers of mass on the moon?

Fig. 11

46. (a) Find the equation of the line L which is tangent to the curve $y = \frac{1}{x}$ at the point (1,1) .

  (b) Determine where L intersects the x–axis and the y–axis.

  (c) Determine the area of the region in the first quadrant bounded by L, the x–axis and the y–axis. (Fig. 11)

47. (a) Find the equation of the line L which is tangent to the curve $y = \frac{1}{x}$ at the point $(2, \frac{1}{2})$.

  (b) Graph y = 1/x and L and determine where L intersects the x–axis and the y–axis.

  (c) Determine the area of the region in the first quadrant bounded by L, the x–axis and the y–axis.

48. (a) Find the equation of the line L which is tangent to the curve $y = \frac{1}{x}$ at the point $(p, \frac{1}{p})$, $p \geq 0$.

    (b) Determine where L intersects the x–axis and the y–axis.

    (c) Determine the area of the region in the first quadrant bounded by L, the x–axis and the y–axis.

    (d) How does the area of the triangle in part (c) depend on the initial point $(p, \frac{1}{p})$?

49. Find values for the coefficients a, b and c so that the parabola $f(x) = ax^2 + bx + c$ goes

    through the point (1,4) and is tangent to the line $y = 9x - 13$ at the point (3, 14).

50. Find values for the coefficients a, b and c so that the parabola $f(x) = ax^2 + bx + c$ goes

    through the point (0,1) and is tangent to the line $y = 3x - 2$ at the point (2,4).

51. (a) Find a function f so that $\mathbf{D}(f(x)) = 3x^2$.

    (b) Find another function g so that $\mathbf{D}(g(x)) = 3x^2$.

    (c) Can you find more functions whose derivatives are $3x^2$?
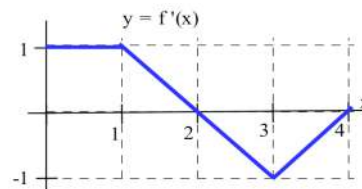
Fig. 12

52. (a) Find a function f so that $\mathbf{f}\,'(x) = 6x + \cos(x)$.

    (b) Find another function g so that $\mathbf{f}\,'(x) = 6x + \cos(x)$.

53. The graph of $y = f\,'(x)$ is given in Fig. 12.

    (a) Assume $f(0) = 0$ and sketch the graph of $y = f(x)$.

    (b) Assume $f(0) = 1$ and graph $y = f(x)$.
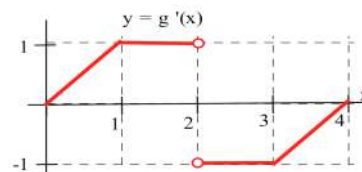
Fig. 13

54. The graph of $y = g\,'(x)$ is given in Fig. 13. Assume that g is continuous.

    (a) Assume $g(0) = 0$ and sketch the graph of $y = g(x)$.

    (b) Assume $g(0) = 1$ and graph $y = g(x)$.

55. Assume that f and g are differentiable functions and that $g(x) \neq 0$. State why

    each step in the following proof of the Quotient Rule is valid.

$$\mathbf{D}(\tfrac{f(x)}{g(x)}) = \lim_{h \to 0} \frac{1}{h} \left\{ \frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)} \right\} = \lim_{h \to 0} \frac{1}{h} \left\{ \frac{f(x+h)g(x) - g(x+h)f(x)}{g(x+h)g(x)} \right\}$$

$$= \lim_{h \to 0} \frac{1}{g(x+h)g(x)} \left\{ \frac{f(x+h)g(x) + \left(-f(x)g(x) + f(x)g(x)\right) - g(x+h)f(x)}{h} \right\}$$

$$= \lim_{h \to 0} \frac{1}{g(x+h)g(x)} \left\{ g(x)\frac{f(x+h) - f(x)}{h} - f(x)\frac{g(x+h) - g(x)}{h} \right\}$$

$$= \frac{1}{g^2(x)} \left\{ g(x) \cdot f'(x) - f(x) \cdot g'(x) \right\} = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{g^2(x)} .$$

Section 3.3   **PRACTICE Answers**

Practice 1:   f is continuous at $x = -1, 0, 2, 4, 6,$ and 7. f is differentiable at $x = -1, 2, 4,$ and 7.

Practice 2: $f(x) = 6x + 8$ and $g(x) = 2$ so $\mathbf{D}(f(x)) = 6$ and $\mathbf{D}(g(x)) = 0$.

$\mathbf{D}(3f(x)) = 3\mathbf{D}(f(x)) = 3(6) = \mathbf{18}, \quad \mathbf{D}(f(x) + g(x)) = \mathbf{D}(f(x)) + \mathbf{D}(g(x)) = 6 + 0 = \mathbf{6}$

$\mathbf{D}(f(x) - g(x)) = \mathbf{D}(f(x)) - \mathbf{D}(g(x)) = 6 - 0 = \mathbf{6}$

$\mathbf{D}(f(x)g(x)) = f(x)\mathbf{D}(g(x)) + g(x)\mathbf{D}(f(x)) = (6x + 8)(0) + (2)(6) = \mathbf{12}$

$\mathbf{D}(f(x)/g(x)) = \dfrac{g(x)\mathbf{D}(f(x)) - f(x)\mathbf{D}(g(x))}{(g(x))^2} = \dfrac{(2)(6) - (6x + 8)(0)}{2^2} = \dfrac{12}{4} = \mathbf{3}$

Practice 3: $\mathbf{D}(x^3 - 5\sin(x)) = \mathbf{D}(x^3) - 5\mathbf{D}(\sin(x)) = \mathbf{3x^2 - 5\cos(x)}$

$\dfrac{\mathbf{d}}{\mathbf{dx}} (\sin(x) - 4x^3) = \dfrac{\mathbf{d}}{\mathbf{dx}} \sin(x) - 4 \dfrac{\mathbf{d}}{\mathbf{dx}} x^3 = \mathbf{\cos(x) - 12x^2}$

Practice 4:

x	f(x)	f '(x)	g(x)	g '(x)	D(3f(x))	D(2f(x) + g(x))	D(3g(x) – f(x))
0	3	−2	−4	3	**−6**	**−1**	**11**
1	2	−1	1	0	**−3**	**−2**	**1**
2	4	2	3	1	**6**	**5**	**1**

Practice 5:

$$\mathbf{D}(f(x) + g(x)) = \lim_{h \to 0} \frac{\{f(x + h) + g(x + h)\} - \{f(x) + g(x)\}}{h}$$

$$= \lim_{h \to 0} \frac{f(x + h) - f(x) + g(x + h) - g(x)}{h}$$

$$= \left(\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}\right) + \left(\lim_{h \to 0} \frac{g(x + h) - g(x)}{h}\right) = \mathbf{D}(f(x)) + \mathbf{D}(g(x)).$$

Practice 6:

x	f(x)	f '(x)	g(x)	g '(x)	D(f(x)·g(x))	D(f(x)/g(x))	D(g(x)/f(x))
0	3	−2	−4	3	$3{\cdot}3+(-4)(-2)=\mathbf{17}$	$\dfrac{-4(-2)-(3)(3)}{(-4)^2} =\dfrac{\mathbf{-1}}{\mathbf{16}}$	$\dfrac{(3)(3)-(-4)(-2)}{3^2} =\dfrac{\mathbf{1}}{\mathbf{9}}$
1	2	−1	1	0	$2{\cdot}0+1(-1) =\mathbf{-1}$	$\dfrac{1(-1)-(2)(0)}{1^2} = \mathbf{-1}$	$\dfrac{2(0)-1(-1)}{2^2} =\dfrac{\mathbf{1}}{\mathbf{4}}$
2	4	2	3	1	$4{\cdot}1 + 3{\cdot}2 = \mathbf{10}$	$\dfrac{3(2)-(4)(1)}{3^2} =\dfrac{\mathbf{2}}{\mathbf{9}}$	$\dfrac{4(1) - 3(2)}{4^2} =\dfrac{\mathbf{-2}}{\mathbf{16}}$

Practice 7:

$$\mathbf{D}((x^2 + 1)(7x - 3)) = (x^2 + 1)\mathbf{D}(7x - 3) + (7x - 3)\mathbf{D}(x^2 + 1) = (x^2 + 1)(7) + (7x - 3)(2x) = \mathbf{21x^2 - 6x + 7}$$

$$\text{or } \mathbf{D}((x^2 + 1)(7x - 3)) = \mathbf{D}(7x^3 - 3x^2 + 7x) = \mathbf{21x^2 - 6x + 7}$$

$$\frac{d}{dt}\left(\frac{3t - 2}{5t + 1}\right) = \frac{(5t + 1)\mathbf{D}(3t - 2) - (3t - 2)\mathbf{D}(5t + 1)}{(5t + 1)^2} = \frac{(5t + 1)(3) - (3t - 2)(5)}{(5t + 1)^2} = \frac{13}{(5t + 1)^2}$$

$$\mathbf{D}\left(\frac{\cos(x)}{x}\right) = \frac{x\mathbf{D}(\cos(x)) - \cos(x)\mathbf{D}(x)}{(x)^2} = \frac{x(-\mathbf{\sin(x)}) - \cos(x)(\mathbf{1})}{x^2} = \frac{-x \cdot \sin(x) - \cos(x)}{x^2}$$

Practice 8: (a) difference rule   (b) product rule   (c) difference rule   (d) quotient rule

Practice 9:

$$\mathbf{D}\left(\frac{x^2 - 5}{\sin(x)}\right) = \frac{\sin(x)\mathbf{D}(x^2 - 5) - (x^2 - 5)\mathbf{D}(\sin(x))}{(\sin(x))^2} = \frac{\sin(x)(\mathbf{2x}) - (x^2 - 5)\mathbf{\cos(x)}}{\sin^2(x)}$$

$$\frac{d}{dt}\left(\frac{t^2 - 5}{t \cdot \sin(t)}\right) = \frac{t \cdot \sin(t)\mathbf{D}(t^2 - 5) - (t^2 - 5)\mathbf{D}(t \cdot \sin(t))}{(t \cdot \sin(t))^2} = \frac{t \cdot \sin(t)(\mathbf{2t}) - (t^2 - 5)\{\mathbf{t \cdot \cos(t) + \sin(t)}\}}{t^2 \cdot \sin^2(t)}$$

Practice 10:   (a) $h(5) = 5 + \dfrac{\sin(5)}{1+5} \approx \mathbf{4.84 \ ft.}$ $v(5) = h'(5) = \dfrac{(1+5)\cos(5) - \sin(5)}{(1+5)^2} \approx \mathbf{0.074 \ ft/sec.}$

"long time": $\displaystyle\lim_{t \to \infty} h(t) = \lim_{t \to \infty} 5 + \frac{\sin(t)}{1 + t} = \mathbf{5 \ feet}.$

$$\lim_{t \to \infty} h'(t) = \lim_{t \to \infty} \frac{(1 + t)\cos(t) - \sin(t)}{(1 + t)^2} = \lim_{t \to \infty}\left\{\frac{\cos(t)}{1 + t} - \frac{\sin(t)}{(1 + t)^2}\right\} = \mathbf{0 \ ft/sec.}$$

Practice 11: $f'(x) = 2x - 10.$ $f'(x) = 0$ when $2x - 10 = 0$ so when $\mathbf{x = 5}$.

$g'(x) = 3x^2 - 12.$ $g'(x) = 0$ when $3x^2 - 12 = 0$ so $x^2 = 4$ and $\mathbf{x = -2, +2}$.

3.4 MORE DIFFERENTIATION PATTERNS

Polynomials are very useful, but they are not the only functions we need. This section uses the ideas of the two previous sections to develop techniques for differentiating **powers of functions**, and to determine the derivatives of some particular functions which occur often in applications, the **trigonometric** and **exponential** functions.

As you focus on learning how to differentiate different types and combinations of functions, it is important to remember what derivatives are and what they measure. Calculators and personal computers are available to calculate derivatives. Part of your job as a professional will be to decide which functions need to be differentiated and how to use the resulting derivatives. You can succeed at that only if you understand what a derivative is and what it measures.

A POWER RULE FOR FUNCTIONS: D(f^n(x))

If we apply the Product Rule to the product of a function with itself, a familiar pattern emerges.

$\mathbf{D}(f^2) = \mathbf{D}(f{\cdot}f) = f{\cdot}\mathbf{D}(f) + f{\cdot}\mathbf{D}(f) = 2f{\cdot}\mathbf{D(f)}.$

$\mathbf{D}(f^3) = \mathbf{D}(f^2{\cdot}f) = f^2{\cdot}\mathbf{D}(f) + f{\cdot}\mathbf{D}(f^2) = f^2{\cdot}\mathbf{D}(f) + f\{ 2f{\cdot}\mathbf{D}(f) \} = f^2{\cdot}\mathbf{D}(f) + 2f^2{\cdot}\mathbf{D}(f) = 3\ f^2{\cdot}\mathbf{D(f)}.$

$\mathbf{D}(f^4) = \mathbf{D}(f^3{\cdot}f) = f^3{\cdot}\mathbf{D}(f) + f{\cdot}\mathbf{D}(f^3) = f^3{\cdot}\mathbf{D}(f) + f\{ 3f^2{\cdot}\mathbf{D}(f)\} = f^3{\cdot}\mathbf{D}(f) + 3f^3{\cdot}\mathbf{D}(f) = 4\ f^3{\cdot}\mathbf{D(f)}.$

Practice 1:   What is the pattern here? What do you think the results will be for **D**(f^5) and **D**(f^{13}) ?

We could keep differentiating higher and higher powers of f(x) by writing them as products of lower powers of f(x) and using the Product Rule, but the Power Rule For Functions guarantees that the pattern we just saw for the small integer powers also works for all constant powers of functions.

> **Power Rule For Functions:**   If   n is any constant,
>
> then $\mathbf{D}(f^n(x)) = n\ f^{n-1}(x) \cdot \mathbf{D}(f(x))$.

The Power Rule for Functions is a special case of a more general theorem, the Chain Rule, which we will examine in Section 2.4. The Power Rule For Functions will be proved after the Chain Rule.

Example 1:   Use the Power Rule for Functions to find

(a) $\mathbf{D}((x^3 - 5)^2)$ (b) $\frac{\mathbf{d}}{\mathbf{dx}}(\sqrt{2x + 3x^5})$ (c) $\mathbf{D}(\sin^2(x)) = \mathbf{D}((\sin(x))^2)$.

Solution: (a) To match the pattern of the Power Rule for $\mathbf{D}((x^3 - 5)^2)$, let $f(x) = x^3 - 5$ and n = 2.

Then $\mathbf{D}((x^3 - 5)^2)\ = \mathbf{D}(f^n(x))$ $= n\ f^{n-1}(x){\cdot}\mathbf{D}(f(x))$

$= 2(x^3 - 5)^1 \mathbf{D}(x^3 - 5) = 2(x^3 - 5)\ (3x^2)$ $= \mathbf{6x^2(x^3 - 5)}.$

(b) To match the pattern for $\dfrac{d}{dx}(\sqrt{2x + 3x^5}) = \dfrac{d}{dx}((2x + 3x^5)^{1/2})$, we can let $f(x) = 2x + 3x^5$

and take $n = 1/2$. Then

$$\frac{d}{dx}((2x + 3x^5)^{1/2}) \quad = \frac{d}{dx}(f^n(x)) \quad = n\, f^{n-1}(x) \frac{d}{dx}(f(x)) \quad = \frac{1}{2}(2x + 3x^5)^{-1/2} \frac{d}{dx}(2x + 3x^5)$$

$$= \frac{1}{2}(2x + 3x^5)^{-1/2}(2 + 15x^4) = \frac{2 + 15x^4}{2\sqrt{2x + 3x^5}} \ .$$

(c) To match the pattern for $D(\sin^2(x))$, Let $f(x) = \sin(x)$ and $n = 2$. Then

$$D(\sin^2(x)) \quad = D(f^n(x)) \quad = n\, f^{n-1}(x) \cdot D(f(x)) = 2\sin^1(x)\, D(\sin(x)) = \mathbf{2\,\sin(x)\,\cos(x)} \ .$$

Practice 2:   Use the Power Rule for Functions to find

(a) $\dfrac{d}{dx}((2x^5 - \pi)^2)$, (b) $D(\sqrt{x + 7x^2})$, (c) $D(\cos^4(x)) = D((\cos(x))^4)$.

Example 2:   Use calculus to show that the line tangent to the circle $x^2 + y^2 = 25$ at the point $(3,4)$

has slope $-3/4$.

Solution: The top half of the circle is the graph of $y = f(x) = \sqrt{25 - x^2}$ so $f'(x) = D((25 - x^2)^{1/2})$

$$= \frac{1}{2}(25 - x^2)^{-1/2} D(25 - x^2) = \frac{-x}{\sqrt{25 - x^2}} \quad \text{and} \quad f'(3) = \frac{-3}{\sqrt{25 - 3^2}} = \frac{-3}{4} \ .$$

As a check, you can verify that the slope of the radial line through the center of the circle $(0,0)$ and the

point $(3,4)$ has slope $4/3$ and is perpendicular to the tangent line which has a slope of $-3/4$.

DERIVATIVES OF TRIGONOMETRIC AND EXPONENTIAL FUNCTIONS

We have some general rules which apply to any elementary combination of differentiable functions, but in

order to use the rules we still need to know the derivatives of each of the particular functions. Here we will

add to the list of functions whose derivatives we know.

Derivatives of the Trigonometric Functions

We know the derivatives of the sine and cosine functions, and each of the other four trigonometric

functions is just a ratio involving sines or cosines. Using the Quotient Rule, we can differentiate the rest of

the trigonometric functions.

Theorem: $D(\tan(x))$ $=$ $\sec^2(x)$	$D(\sec(x))$ $=$ $\sec(x)\tan(x)$	
$D(\cot(x))$ $= -\ \csc^2(x)$	$D(\csc(x))$ $= -\ \csc(x)\cot(x)$.	

Proof: From trigonometry we know $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$, $\cot(x) = \dfrac{\cos(x)}{\sin(x)}$, $\sec(x) = \dfrac{1}{\cos(x)}$, and $\csc(x) = \dfrac{1}{\sin(x)}$,

and we know $\mathbf{D}(\sin(x)) = \cos(x)$ and $\mathbf{D}(\cos(x)) = -\sin(x)$. Using the Quotient Rule,

$$\mathbf{D}(\tan(x)) = \mathbf{D}(\frac{\sin(x)}{\cos(x)}) = \frac{\cos(x){\cdot}\mathbf{D}(\sin(x)) - \sin(x){\cdot}\mathbf{D}(\cos(x))}{(\cos(x))^2}$$

$$= \frac{\cos(x)\,\mathbf{\cos(x)} - \sin(x)\{\,\mathbf{-\sin(x)}\, \}}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \mathbf{sec^2(x)} .$$

$$\mathbf{D}(\sec(x)) = \mathbf{D}(\frac{1}{\cos(x)}) = \frac{\cos(x)\,\mathbf{D}(1) - 1\,\mathbf{D}(\cos(x))}{\cos^2(x)}$$

$$= \frac{\cos(x)\,(\mathbf{0}) - 1\,\{ \mathbf{-\sin(x)} \}}{\cos^2(x)} = \frac{\sin(x)}{\cos^2(x)} = \frac{\sin(x)}{\cos(x)}\,\frac{1}{\cos(x)} = \mathbf{tan(x){\cdot}sec(x)} .$$

Instead of the Quotient Rule, we could have used the Power Rule to calculate $\mathbf{D}(\sec(x)) = \mathbf{D}((\cos(x))^{-1})$.

Practice 3:   Use the Quotient Rule on $f(x) = \cot(x) = \dfrac{\cos(x)}{\sin(x)}$ to prove that $f\,'(x) = -\csc^2(x)$.

Practice 4:   Prove that $\mathbf{D}(\csc(x)) = -\csc(x){\cdot}\cot(x)$. The justification of this result is very similar to

the justification for $\mathbf{D}(\sec(x))$**.**

Practice 5:   Find   (a) $\mathbf{D}(x^5{\cdot}\tan(x))$, (b) $\dfrac{\mathbf{d}}{\mathbf{dt}}(\dfrac{\sec(t)}{t})$ and   (c) $\mathbf{D}(\sqrt{\cot(x) - x}\ )$.

Derivative of e^x

We can use graphs of exponential functions to estimate the slopes of their

tangent lines or we can numerically approximate the slopes.

Example 3:   Estimate the derivative of $f(x) = 2^x$ at the point

$(0, 2^0) = (0, 1)$ by approximating the slope of

the line tangent to $f(x) = 2^x$ at that point.

Solution: We can get estimates from the graph of $f(x) = 2^x$ by carefully

graphing $f(x) = 2^x$ for small values of x, sketching secant lines,

and then measuring the slopes of the secant lines (Fig. 1).
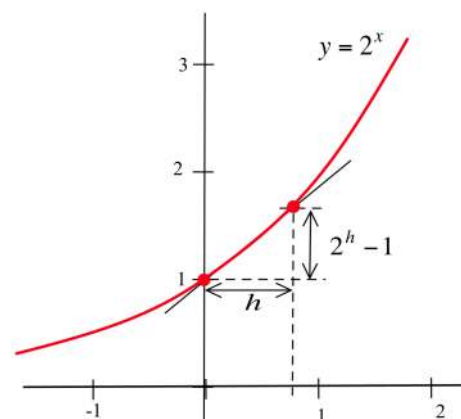
Fig. 1

We can also find the slope numerically by using the definition of the derivative,

$$f\,'(0) \equiv \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{2^{0+h} - 2^0}{h} = \lim_{h \to 0} \frac{2^h - 1}{h} ,\ \text{and evaluating}\ \frac{2^h - 1}{h}$$

for some very small values of h.

h	$\dfrac{2^h - 1}{h}$	$\dfrac{3^h - 1}{h}$	$\dfrac{e^h - 1}{h}$
0.1	0.717734625		
–0.1	0.669670084		
0.01	0.69555		
–0.01	0.690750451		
0.001	0.6933874		
–0.001	0.69290695		
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
0	≈ 0.693	≈ 1.099	1

From the table we can see that $f'(0) \approx .693$.

Practice 6: Fill in the table for $\dfrac{3^h - 1}{h}$, and show that the slope of the line

tangent to $g(x) = 3^X$ at $(0,1)$ is approximately 1.099 . (Fig. 2)

At $(0,1)$, the slope of the tangent to $y = 2^X$ is less than 1 , and the slope of

the tangent to $y = 3^X$ is slightly greater than 1. (Fig. 3) There is a number,

denoted **e**, between 2 and 3 so that the slope of the tangent to $y = e^X$

is exactly 1: $\lim\limits_{h \to 0} \dfrac{e^h - 1}{h} = 1$. The number $e \approx 2.71828182845904$.
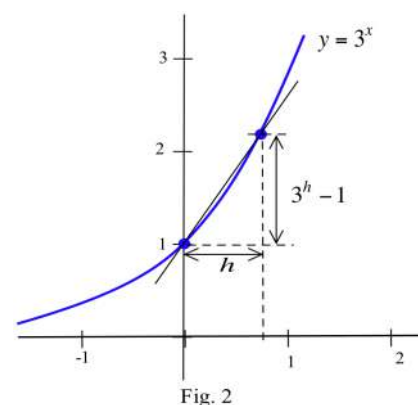
e is irrational and is very important and common in calculus and applications.

Once we grant that $\lim\limits_{h \to 0} \dfrac{e^h - 1}{h} = 1$, it is relatively straightforward to

calculate $D(e^X)$.

Theorem: $D(e^X) = e^X$.

Proof: $D(e^X)$ $\equiv \lim\limits_{h \to 0} \dfrac{e^{x+h} - e^x}{h} = \lim\limits_{h \to 0} \dfrac{e^x \cdot e^h - e^x}{h}$

$= \lim\limits_{h \to 0} \left(e^x \right) \cdot \left(\dfrac{e^h - 1}{h} \right)$

$= \lim\limits_{h \to 0} \left(e^x \right) \cdot \lim\limits_{h \to 0} \left(\dfrac{e^h - 1}{h} \right) = (e^X)(1) = e^X$.

The function $f(x) = e^X$ is its own derivative: $f'(x) = f(x)$. The **height** of $f(x) = e^X$ at any point and the

slope of the tangent to $f(x) = e^X$ at that point are the same: as the graph gets higher, its slope gets steeper.

Example 4:   Find (a) $\dfrac{d}{dt}(t{\cdot}e^{t})$, (b) $\mathbf{D}(e^{x}/\sin(x))$ and (c) $\mathbf{D}(e^{5x}) = \mathbf{D}((e^{x})^{5})$

Solution: (a) Using the Product Rule with $f(t) = t$ and $g(t) = e^{t}$,

$$\dfrac{d}{dt}(t{\cdot}e^{t}) = t{\cdot}\mathbf{D}(e^{t}) + e^{t}{\cdot}\mathbf{D}(t) = t{\cdot}e^{t} + e^{t}{\cdot}(1) = t{\cdot}e^{t} + e^{t} = (t+1)\,e^{t} .$$

(b) Using the Quotient Rule with $f(x) = e^{x}$ and $g(x) = \sin(x)$,

$$\mathbf{D}(\dfrac{e^{x}}{\sin(x)}) = \dfrac{\sin(x)\,\mathbf{D}(e^{x}) - e^{x}\,\mathbf{D}(\sin(x))}{\sin^{2}(x)} = \dfrac{\sin(x)\,e^{x} - e^{x}\cos(x)}{\sin^{2}(x)} .$$

(c) Using the Power Rule for Functions with $f(x) = e^{x}$ and $n = 5$,

$$\mathbf{D}((e^{x})^{5}) = 5(e^{x})^{4}{\cdot}\mathbf{D}(e^{x}) = 5(e^{x})^{4} \cdot e^{x} = 5\,e^{4x}\,e^{x} = 5\,e^{5x} .$$

Practice 7:   Find (a) $\mathbf{D}(x^{3}\,e^{x})$ and (b) $\mathbf{D}((e^{x})^{3})$.

Higher Derivatives: Derivatives of Derivatives

The derivative of a function f is a new function **f '** , and we can calculate the derivative of this new
function to get the derivative of the derivative of f, denoted by f " and called the second derivative of f.
For example, if $f(x) = x^{5}$ then $f'(x) = 5x^{4}$ and $\mathbf{f''(x)} = (f'(x))' = (5x^{4})' = \mathbf{20x^{3}}$.

Definitions: The first derivative of f is $f'(x)$, the rate of change of f.

The second derivative of f is $f''(x) = (f'(x))'$, the rate of change of f ' .

The third derivative of f is $f'''(x) = (f''(x))'$, the rate of change of f " .

For $y = f(x)$, $f'(x) = \dfrac{dy}{dx}$, $f''(x) = \dfrac{d}{dx}\left(\dfrac{dy}{dx} \right) = \dfrac{d^{2}y}{dx^{2}}$, $f'''(x) = \dfrac{d}{dx}\left(\dfrac{d^{2}y}{dx^{2}} \right) = \dfrac{d^{3}y}{dx^{3}}$, and so on.

Practice 8:   Find f ', f ", and f "' for $f(x) = 3x^{7}$, $f(x) = \sin(x)$, and $f(x) = x\cos(x)$.

If f(x) represents the position of a particle at time x , then $v(x) = f'(x)$ will represent the velocity (rate of
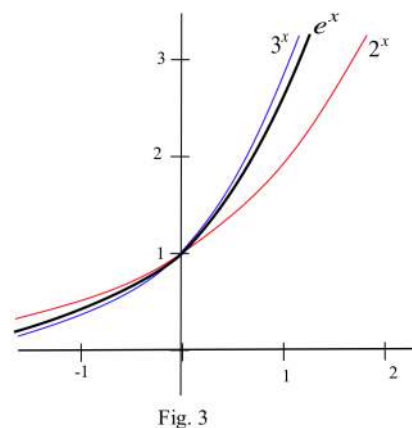change of the position) of the particle and $a(x) = v'(x) = f''(x)$ will represent the acceleration (the rate of
change of the velocity) of the particle.

Example 5:   The height (feet) of a particle at time t seconds is $t^{3} - 4t^{2} + 8t$. Find the height,
velocity and acceleration of the particle when t = 0, 1, and 2 seconds.

Solution: $f(t) = t^{3} - 4t^{2} + 8t$ so $f(0) = 0$ feet, $f(1) = 5$ feet, and $f(2) = 8$ feet.

The velocity is $v(t) = f'(t) = 3t^{2} - 8t + 8$ so $v(0) = 8$ ft/s , $v(1) = 3$ ft/s, and $v(2) = 4$ ft/s. At each of
these times the velocity is positive and the particle is moving upward, increasing in height.

The acceleration is $a(t) = 6t - 8$ so $a(0) = -8$ ft/s^{2} , $a(1) = -2$ ft/s^{2} and $a(2) = 4$ ft/s^{2} .

We will examine the geometric meaning of the second derivative later.

A REALLY "BENT" FUNCTION

In Section 2.3 we saw that the "holey" function $h(x) = \begin{cases} 2 & \text{if } x \text{ is a } \textbf{rational} \text{ number} \\ 1 & \text{if } x \text{ is an } \textbf{irrational} \text{ number} \end{cases}$

is discontinuous at every value of x, so at every x   h(x) is not differentiable. We can create graphs of
continuous functions that are not differentiable at several places just by putting corners at those places, but
how many corners can a continuous function have? How badly can a continuous function fail to be
differentiable?

In the mid–1800s, the German mathematician Karl Weierstrass surprised and even shocked the
mathematical world by creating a function which was **continuous everywhere but differentiable
nowhere** — a function whose graph was everywhere connected and everywhere bent! He used techniques
we have not investigated yet, but we can start to see how such a function could be built.

Start with a function f_1 (Fig. 4) which zigzags between the values +1/2 and –1/2 and has a "corner" at
each integer. This starting function f_1 is continuous everywhere and is differentiable everywhere except at
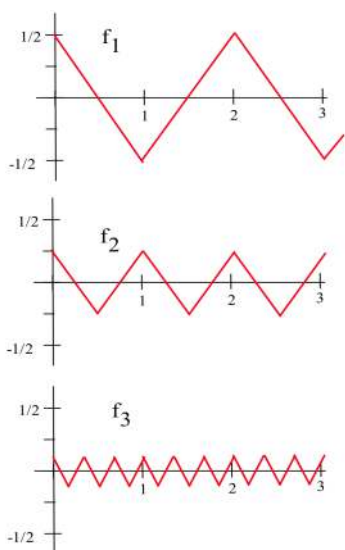
the integers. Next create a list of functions f_2 , f_3 , f_4 , . . . , each of which
is a lot shorter but with many more "corners" than the previous ones.   For
example, we might make f_2 zigzag between the values +1/4 and –1/4
and have "corners" at ± 1/2, ±3/2, ±5/2, etc., and f_3 zigzag between
+1/9 and –1/9 and have "corners" at ±1/3, ±2/3, ±4/3, etc. If we add f_1
and f_2 , we get a continuous function (since the sum of two continuous
functions is continuous) which will have corners at 0, ±1/2, ±1, ±3/2, . . .
If we then add f_3 to the previous sum, we get a new continuous function
with even more corners. If we continue adding the functions in our list
"indefinitely", the final result will be a continuous function which is
differentiable nowhere.

Fig. 4

We haven't developed enough mathematics here to precisely describe what it means to add an infinite
number of functions together or to verify that the resulting function is nowhere differentiable, but we will.
You can at least start to imagine what a strange, totally "bent" function it must be.

Until Weierstrass created his "everywhere continuous, nowhere differentiable" function, most
mathematicians thought a continuous function could only be "bad" in a few places, and Weierstrass' function
was (and is) considered "pathological", a great example of how bad something can be. The mathematician
Hermite expressed a reaction shared by many when they first encounter Weierstrass' function:

"I turn away with fright and horror from this lamentable evil of functions which do not have derivatives."

IMPORTANT RESULTS

Power Rule For Functions: $D(f^n(x)) = n \cdot f^{n-1}(x) \cdot D(f(x))$

Derivatives of the Trigonometric Functions:

$D(\sin(x)) = \cos(x)$ $D(\tan(x)) = \sec^2(x)$ $D(\sec(x)) = \sec(x) \tan(x)$

$D(\cos(x)) = -\sin(x)$ $D(\cot(x)) = -\csc^2(x)$ $D(\csc(x)) = -\csc(x) \cot(x)$

Derivatives of the Exponential Function: $D(e^x) = e^x$

PROBLEMS

1. Let $f(1) = 2$ and $f'(1) = 3$. Find the values of $D(f^2(x))$, $D(f^5(x))$, and $D(\sqrt{f(x)})$ at x=1.

2. Let $f(2) = -2$ and $f'(2) = 5$. Find the values of $D(f^2(x))$, $D(f^{-3}(x))$, and $\frac{d}{dx}(\sqrt{f(x)})$ at x=2.

3. Estimate the values of f(x) and f '(x) in Fig. 5 and determine

y=f(x)
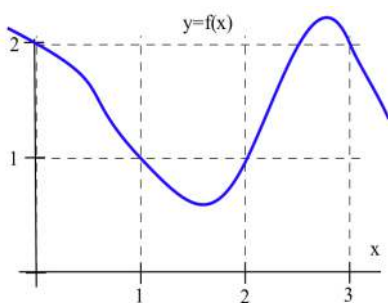
Fig. 5

(a) $\frac{d}{dx}(f^2(x))$ at x = 1 and 3   (b) $D(f^3(x))$ at x = 1 and 3

(c) $D(f^5(x))$ at x = 1 and 3 .

4. Estimate the values of f(x) and f '(x) in Fig. 5 and determine

(a) $D(f^2(x))$ at x = 0 and 2   (b) $\frac{d}{dx}(f^3(x))$ at x = 0 and 2

(c) $\frac{d}{dx}(f^5(x))$ at x = 0 and 2.

In problems 5 – 10 , find the derivative of each function.

5. $f(x) = (2x - 8)^5$ 6. $f(x) = (6x - x^2)^{10}$ 7. $f(x) = x \cdot (3x + 7)^5$

8. $f(x) = (2x + 3)^6 \cdot (x - 2)^4$ 9. $f(x) = \sqrt{x^2 + 6x - 1}$ 10. $f(x) = \dfrac{x - 5}{(x + 3)^4}$

11. A weight attached to a spring is at a height of $h(t) = 3 - 2\sin(t)$ feet above the floor t seconds after it

is released.   (a) Graph h(t)   (b) At what height is the weight when it is released?

(c) How high does the weight ever get above the floor and how close to the floor does it ever get?

(d) Determine the height, velocity and acceleration at time t. (Be sure to include the correct units.)

(e) Why is this an unrealistic model of the motion of a weight on a real spring?

12. A weight attached to a spring is at a height of $h(t) = 3 - \dfrac{2\sin(t)}{1 + 0.1t^2}$ feet above the floor t seconds

after it is released. (a) Graph h(t)   (b) At what height is the weight when it is released?

(c) Determine the height and velocity at time t.

(d) What happens to the height and the velocity of the weight after a "long time?"

13. The kinetic energy K of an object of mass m and velocity v is $\frac{1}{2}$ m·v^2 .

   (a) Find the kinetic energy of an object with mass m and height h(t) = 5t feet at t = 1 and 2 seconds.

   (b) Find the kinetic energy of an object with mass m and height h(t) = t^2 feet at t = 1 and 2 seconds.

14. An object of mass m is attached to a spring and has height h(t) = 3 + sin(t) feet at time t seconds.

   (a) Find the height and kinetic energy of the object when t = 1, 2, and 3 seconds.

   (b) Find the rate of change in the kinetic energy of the object when t = 1, 2, and 3 seconds.

   (c) Can K ever be negative? Can dK/dt ever be negative? Why?

In problems 15 – 20, find the derivatives **df/dx** .

15.   f(x) = x·sin(x)

16.   f(x) = sin^5(x)

17.   f(x) = e^x – sec(x)

18.   f(x) = $\sqrt{\cos(x) + 1}$

19.   f(x) = e^{-x} + sin(x)

20.   f(x) = $\sqrt{x^2 - 4x + 3}$

In problems 21 – 26, find the equation of the line tangent to the graph of the function at the given point.

21. f(x) = (x – 5)^7 at (4, –1)

22. f(x) = e^x at (0,1)

23. f(x) = $\sqrt{25 - x^2}$ at (3,4)

24. f(x) = sin^3(x) at (π,0)

25. f(x) = (x – a)^5 at (a,0)

26. f(x) = x·cos^5(x) at (0, 0)

27. Find the equation of the line tangent to f(x) = e^x at the point (3 , e^3). Where will this tangent line
   intersect the x–axis? Where will the tangent line to f(x) = e^x at the point (p, e^p) intersect the x–axis?

In problems 28 – 33, calculate **f '** and **f ''**.

28. f(x) = 7x^2 + 5x – 3

29. f(x) = cos(x)

30. f(x) = sin(x)

31. f(x) = x^2·sin(x)

32. f(x) = x·sin(x)

33. f(x) = e^x·cos(x)

34. Calculate the first 8 derivatives of f(x) = sin(x). What is the pattern?
   What is the 208th derivative of sin(x)?

35. What will the 2nd derivative of a quadratic polynomial be? The 3rd derivative? The 4th derivative?

36. What will the 3rd derivative of a cubic polynomial be? The 4th derivative?

37. What can you say about the nth and (n+1)st derivatives of a polynomial of degree n?

In problems 38 – 42, you are given f '. Find a function f with the given derivative.

38. f '(x) = 4x + 2

39. f '(x) = 5e^x

40. f '(x) = 3·sin^2(x)·cos(x)

41. f '(x) = 5(1 + e^x)^4·e^x

42. f '(x) = e^x + sin(x)

43. The function $f(x) = \begin{cases} x \cdot \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ in Fig. 6 is continuous at 0 since

$\lim_{h \to 0} f(x) = 0 = f(0)$. Is f differentiable at 0? (Use the definition of f '(0) and consider
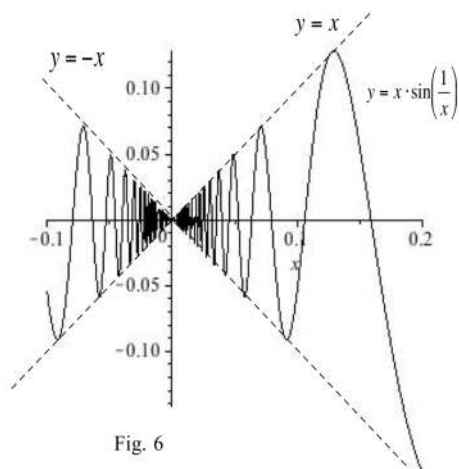
$\lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$.)

Fig. 6
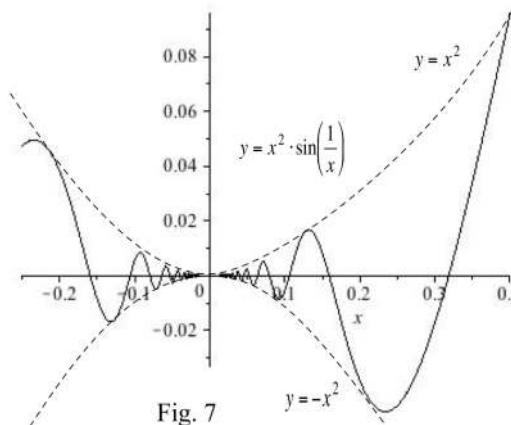
Fig. 7

44. The function $f(x) = \begin{cases} x^2 \cdot \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$ in Fig. 7 is continuous at 0 since

$\lim_{h \to 0} f(x) = 0 = f(0)$. Is f differentiable at 0? (Use the definition of f '(0) and consider

$\lim_{h \to 0} \dfrac{f(0+h) - f(0)}{h}$.)

The number e appears in a variety of unusual situations. Problems 45 – 48 illustrate a few of them.

45. Use your calculator to examine the values of $(1 + \frac{1}{x})^x$ when x is relatively large, for example,

x = 100, 1000, and 10000. Try some other large values for x. If x is large, the value of

$(1 + \frac{1}{x})^x$ is close to what number?

46. If you put \$1 into a bank which pays 1% interest per year and compounds the interest x times a

year, then after one year you will have earned $(1 + \frac{.01}{x})^x$ dollars in the bank.

(a) How much money will you have after 1 year if the bank calculates the interest once a year?

(b) How much money will you have after 1 year if the bank calculates the interest twice a year?

(c) How much money will you have after 1 year if the bank calculates the interest 365 times a year?

(d) How does your answer in part (c) compare with $e^{.01}$.

47. (a) Calculate the value of the sums $s_1 = 1 + \dfrac{1}{1!}$, $s_2 = 1 + \dfrac{1}{1!} + \dfrac{1}{2!}$, $s_3 = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!}$,

$s_4 = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!}$, $s_5 = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + \dfrac{1}{5!}$, and

$s_6 = 1 + \dfrac{1}{1!} + \dfrac{1}{2!} + \dfrac{1}{3!} + \dfrac{1}{4!} + \dfrac{1}{5!} + \dfrac{1}{6!}$.

(b) What value do the sums in part (a) seem to be approaching? Calculate s_7 and s_8 .

(n! = product of all positive integers from 1 to n. For example, 2! = 1·2 = 2, 3! = 1·2·3 = 6, 4! = 24.)

48. If it is late at night and you are tired of studying calculus, try the following experiment with a friend.
Take the 2 through 10 of hearts from a regular deck of cards and shuffle these 9 cards well. Have your
friend do the same with the 2 through 10 of spades. Now compare your cards one at a time. If there is
a match, for example you both play a 5, then the game is over and you win. If you make it through the
entire 9 cards with no match, then your friend wins. If you play the game **many times**, then the ratio
$\dfrac{\text{total number of games played}}{\text{number of times your friend wins}}$ will be approximately equal to e.

Section 3.4   **PRACTICE Answers**

Practice 1:   The pattern is $\mathbf{D}(\, f^n(x)\,) = n\, f^{n-1}(x) \cdot \mathbf{D}(\, f(x)\,)$. $\mathbf{D}(\, f^5\,) = 5f^4\, \mathbf{D(f)}$ and $\mathbf{D}(\, f^{13}\,) = 13f^{12}\mathbf{D(f)}$.

Practice 2: $\dfrac{d}{dx}(2x^5 - \pi)^2 = 2(2x^5 - \pi)^1\, \mathbf{D}(\, 2x^5 - \pi\,) = 2(2x^5 - \pi)^1\, \mathbf{(10x^4)} = 40x^9 - 20\pi x^4$.

$\mathbf{D}(\, (x + 7x^2)^{1/2}\,) = \dfrac{1}{2}(x + 7x^2)^{-1/2}\, \mathbf{D}(\, x + 7x^2\,) = \dfrac{\mathbf{1 + 14x}}{2\sqrt{x + 7x^2}}$.

$\mathbf{D}(\, (\cos(x)\,)^4\,) = 4(\, \cos(x)\,)^3\mathbf{D}(\, \cos(x)\,) = 4(\, \cos(x)\,)^3\, (\, -\mathbf{sin(x)}\,) = -\mathbf{4cos^3(x)sin(x)}$.

Practice 3: $\mathbf{D}(\, \dfrac{\cos(x)}{\sin(x)}\,) = \dfrac{\sin(x)\mathbf{D}(\, \cos(x)\,) - \cos(x)\mathbf{D}(\, \sin(x)\,)}{(\, \sin(x)\,)^2}$

$= \dfrac{\sin(x)(\, -\mathbf{sin(x)}\,) - \cos(x)(\, \mathbf{cos(x)}\,)}{\sin^2(x)} = \dfrac{-(\, \sin^2(x) + \cos^2(x)\,)}{\sin^2(x)} = \dfrac{-1}{\sin^2(x)} = -\mathbf{csc^2(x)}$.

Practice 4: $\mathbf{D}(\, \csc(x)\,) = \mathbf{D}(\, \dfrac{1}{\sin(x)}\,) = \dfrac{\sin(x)\mathbf{D}(\, 1\,) - 1\mathbf{D}(\, \sin(x)\,)}{\sin^2(x)}$

$= \dfrac{\sin(x)\mathbf{(0)} - \mathbf{cos(x)}}{\sin^2(x)} = -\dfrac{\cos(x)}{\sin(x)}\dfrac{1}{\sin(x)} = -\mathbf{cot(x)csc(x)}$.

Practice 5: $D(x^5 \cdot \tan(x)) = x^5 D(\tan(x)) + \tan(x)D(x^5) = x^5 \sec^2(x) + \tan(x)(5x^4)$.

$$\frac{d}{dt} \left(\frac{\sec(t)}{t} \right) = \frac{t D(\sec(t)) - \sec(t)D(t)}{t^2} = \frac{t \cdot \sec(t) \cdot \tan(t) - \sec(t)}{t^2} \ .$$

$$D((\cot(x) - x)^{1/2}) = \frac{1}{2} (\cot(x) - x)^{-1/2} D(\cot(x) - x)$$

$$= \frac{1}{2} (\cot(x) - x)^{-1/2} (-\csc^2(x) - 1) = \frac{-\csc^2(x) - 1}{2 \sqrt{\cot(x) - x}} \ .$$

Practice 6:

h	$\dfrac{2^h - 1}{h}$	$\dfrac{3^h - 1}{h}$	$\dfrac{e^h - 1}{h}$
0.1	0.717734625	**1.16123174**	1.051709181
–0.1	0.669670084	**1.040415402**	0.9516258196
0.01	0.69555	**1.104669194**	1.005016708
–0.01	0.690750451	**1.092599583**	0.9950166251
0.001	0.6933874	**1.099215984**	1.000500167
–0.001	0.69290695	**1.098009035**	0.9995001666
$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
0	≈ 0.693	≈ 1.099	1

Practice 7: $D(x^3 e^x) = x^3 D(e^x) + e^x D(x^3) = x^3 (e^x) + e^x (3x^2) = x^2 \cdot e^x \cdot (x + 3)$.

$$D((e^x)^3) = 3(e^x)^2 D(e^x) = 3(e^x)^2 (e^x) = 3e^{2x} \cdot e^x = 3 e^{3x} \ \text{or}$$

$$D((e^x)^3) = D(e^{3x}) = e^{3x} D(3x) = 3 e^{3x} \ .$$

Practice 8:

$f(x) = 3x^7$	$f(x) = \sin(x)$	$f(x) = x \cdot \cos(x)$
$f'(x) = 21x^6$	$f'(x) = \cos(x)$	$f'(x) = -x \cdot \sin(x) + \cos(x)$
$f''(x) = 126x^5$	$f''(x) = -\sin(x)$	$f''(x) = -x \cdot \cos(x) - 2\sin(x)$
$f'''(x) = 630x^4$	$f'''(x) = -\cos(x)$	$f'''(x) = x \cdot \sin(x) - 3\cos(x)$

3.5   THE CHAIN RULE

The Chain Rule is the most important and most used of the differentiation patterns. It enables us to differentiate **composites** of functions such as $y = \sin(x^2)$. It is a powerful tool for determining the derivatives of some **new functions** such as logarithms and inverse trigonometric functions. And it leads to important **applications** in a variety of fields. You will need the Chain Rule hundreds of times in this course, and practice with it now will save you time and points later. Fortunately, with some practice, the Chain Rule is also easy to use.

We already know how to differentiate the composition of some functions.

Example 1:   For $f(x) = 5x - 4$ and $g(x) = 2x + 1$, find f∘g(x) and **D(** f∘g(x) **)** .

Solution: f∘g(x) = f(g(x)) = 5(2x+1) – 4 = 10x + 1, so **D(** f∘g(x) **)** = **D(** 10x + 1 **)** = 10.

Practice 1:   For $f(x) = 5x - 4$ and $g(x) = x^2$, find f∘g(x), **D(** f∘g(x) **)** , g∘f(x) , and **D(** g∘f(x) **)** .

Some compositions, however, are still very difficult to differentiate. We know the derivatives of $g(x) = x^2$ and $h(x) = \sin(x)$, and we know how to differentiate some combinations of these functions such as $x^2 + \sin(x)$, $x^2 \cdot \sin(x)$, and even $\sin^2(x)$, but the derivative of the simple composition $f(x) = h \circ g(x) = \sin(x^2)$ is hard — until we know the Chain Rule. To see just how hard, try using the definition of derivative on it.

Example 2:

(a) Suppose amplifier Y doubles the strength of the output signal from amplifier U, and U triples the strength of the original signal x. How does the final signal out of Y compare with the signal x?

original signal x → | amplifier U | → | amplifier Y | → final signal

(b) Suppose y changes twice as fast as u, and u changes three times as fast as x. How does the rate of change of y compare with the rate of change of x?

Solution: In each case we are comparing the result of a composition, and the answer to each question is 6, the product of the two amplifications or rates of change.

In (a), we have that $\dfrac{\text{signal out of Y}}{\text{signal x}} = \dfrac{\text{signal out of Y}}{\text{signal out of U}} \cdot \dfrac{\text{signal out of U}}{\text{signal x}} = (2)(3) = 6.$

In (b), $\dfrac{\Delta y}{\Delta x} = \dfrac{\Delta y}{\Delta u} \cdot \dfrac{\Delta u}{\Delta x} = (2)(3) = 6.$

These examples are simple cases of the Chain Rule for differentiating a composition of functions.

THE CHAIN RULE

Chain Rule (Leibniz notation form)

If   y is a differentiable function of u, and u is a differentiable function of x,

then   y is a differentiable function of x and $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$.

Idea for a proof: $\dfrac{dy}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta u} \cdot \dfrac{\Delta u}{\Delta x}$ (*if* $\Delta u \neq 0$)

$$= \left(\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta u} \right) \cdot \left(\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta u} \right)$$ u is continuous, so $\Delta x \to 0$ implies $\Delta u \to 0$

$$= \left(\lim\limits_{\Delta u \to 0} \dfrac{\Delta y}{\Delta u} \right) \cdot \left(\lim\limits_{\Delta x \to 0} \dfrac{\Delta y}{\Delta u} \right) = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$$

Although this nice short argument gets to the heart of why the Chain Rule works, it is not quite valid. If du/dx $\neq 0$, then it is possible to show that $\Delta u \neq 0$ for all very small values of Δx, and the "idea for a proof" is a real proof. There are, however, functions for which $\Delta u = 0$ for lots of small values of Δx, and these create problems for the previous argument. A justification which is true for ALL cases is much more complicated, and one is given in the appendix.

The symbol $\dfrac{dy}{du}$ is a single symbol (as is $\dfrac{du}{dx}$) , and we <u>cannot</u> eliminate du from the product $\dfrac{dy}{du} \cdot$ $\dfrac{du}{dx}$ in the Chain Rule. It is, however, perfectly fine to use the idea of eliminating du to help you remember the statement of the Chain Rule.

Example 3: $y = \cos(x^2 + 3)$ is $y = \cos(u)$ with $u = x^2 + 3$. Find dy/dx.

Solution: $y = \cos(u)$ so $dy/du = -\sin(u)$. $u = x^2 + 3$ so $du/dx = 2x$. Finally, using the Chain Rule,

$$\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = -\sin(u) \cdot 2x = -2x \cdot \sin(x^2 + 3) \ .$$

Practice 2:   Find dy/dx for $y = \sin(4x + e^x)$.

There is also a composition of functions form of the Chain Rule. The notation is different, but it means precisely the same as the Leibniz form.

> **Chain Rule (composition form)**
>
>   If   g is differentiable at x and f is differentiable at g(x),
>
>   then   the composite f∘g is differentiable at x , and $(f \circ g)'(x) = \mathbf{D}(f(g(x))) = \mathbf{f}'(g(x)) \cdot \mathbf{g}'(x)$.

You may find it easier to think of the composition form of the Chain Rule in words:

(f(g(x)))' = "the derivative of the outside function (with respect to the original inside function) times

the derivative of the inside function" where f is the outside function and g is the inside function.

Example 4:   Differentiate $\sin(x^2)$.

Solution: The function $\sin(x^2)$ is the composition $f \circ g$ of two simple functions: $f(x) = \sin(x)$ and

$g(x) = x^2$: $f \circ g(x) = f(g(x)) = f(x^2) = \sin(x^2)$ which is the function we want. Both f and g are

differentiable functions with derivatives $f'(x) = \cos(x)$ and $g'(x) = 2x$, so, by the Chain Rule,

$$\mathbf{D}(\sin(x^2)) = (f \circ g)'(x) = \mathbf{f}'(g(x)) \cdot \mathbf{g}'(x) = \cos(g(x)) \cdot 2x = \cos(x^2) \cdot 2x = 2x\cos(x^2).$$

If you tried using the definition of derivative to calculate the derivative of this function at the beginning of

the section, you can really appreciate the power of the Chain Rule for differentiating compositions.

Example 5:   The table gives values for f, f', g and g' at a number of points. Use these values to
determine $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = -1$ and 0.

x	f(x)	g(x)	f '(x)	g '(x)	(f∘g)(x)	(f∘g) '(x)
−1	2	3	1	0		
0	−1	1	3	2		
1	1	0	−1	3		
2	3	−1	0	1		
3	0	2	2	−1		

Solution: $(f \circ g)(-1) = f(g(-1)) = f(3) = 0$ and $(f \circ g)(0) = f(g(0)) = f(1) = 1$.

$(f \circ g)'(-1) = \mathbf{f}'(g(-1)) \cdot \mathbf{g}'(-1) = \mathbf{f}'(3) \cdot (0) = (2)(0) = 0$ and

$(f \circ g)'(0) = \mathbf{f}'(g(0)) \cdot \mathbf{g}'(0) = \mathbf{f}'(1) \cdot (2) = (-1)(2) = -2$.

Practice 3:   Fill in the table in Example 5 for $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = 1, 2$ and 3.

Neither form of the Chain Rule is inherently superior to the other — use the one you prefer. The Chain

Rule will be used hundreds of times in the rest of this book, and it is important that you master its usage.

The time you spend now mastering and understanding how to use the Chain Rule will be paid back tenfold

in the next several chapters.

Example 6:   Determine $\mathbf{D}(e^{\cos(x)})$ using each form of the Chain Rule.

Solution: Using the Leibniz notation: $y = e^u$ and $u = \cos(x)$. $dy/du = e^u$ and $du/dx = -\sin(x)$ so

$dy/dx = (dy/du) \cdot (du/dx) = (e^u) \cdot (-\sin(x)) = -\sin(x) \cdot e^{\cos(x)}$.

The function $e^{\cos(x)}$ is also the composition of $f(x) = e^x$ with $g(x) = \cos(x)$, so

$\mathbf{D}(e^{\cos(x)})$ $= \mathbf{f}'(g(x)) \cdot \mathbf{g}'(x)$ by the Chain Rule

          $= e^{g(x)} \cdot (-\sin(x))$ since $\mathbf{D}(e^x) = e^x$ and $\mathbf{D}(\cos(x)) = -\sin(x)$

          $= -\sin(x) \cdot e^{\cos(x)}$.

Practice 4:   Calculate $\mathbf{D}(\sin(7x - 1))$, $\dfrac{d}{dx}(\sin(ax + b))$, and $\dfrac{d}{dt}(e^{3t})$.

Practice 5:   Use the graph of g in Fig. 1 and the Chain
Rule to estimate $D(\sin(g(x)))$ and
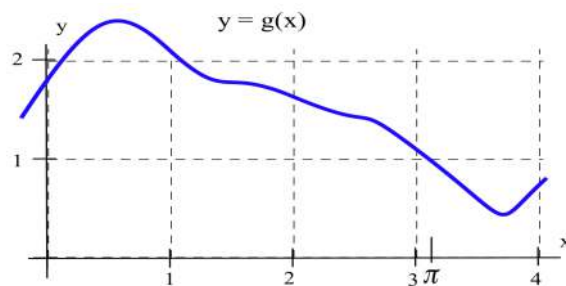$D(g(\sin(x)))$ at $x = \pi$.

Fig. 1

The Chain Rule is a general differentiation pattern, and it can
be used with the other general patterns such as the Product and
Quotient Rules.

Example 7:   Determine $D(e^{3x} \cdot \sin(5x + 7))$ and $\dfrac{d}{dx}(\cos(x \cdot e^{x}))$.

Solution: (a) $e^{3x} \sin(5x + 7)$ is a product of two functions so we need the product rule first:

$D(e^{3x} \cdot \sin(5x + 7))$ $= e^{3x} \cdot D(\sin(5x + 7)) + \sin(5x + 7) \cdot D(e^{3x})$

$= e^{3x} \cos(5x + 7) \cdot 5 + \sin(5x + 7) e^{3x} \cdot 3 = 5 e^{3x} \cos(5x + 7) + 3 e^{3x} \sin(5x + 7)$.

(b) $\cos(x \cdot e^{x})$ is a composition of cosine with a product so we need the Chain Rule first:

$\dfrac{d}{dx}(\cos(x \cdot e^{x}))$ $= - \sin(x \cdot e^{x}) \cdot \dfrac{d}{dx}(x \cdot e^{x})$

$= - \sin(x e^{x}) \cdot \{ x \cdot \dfrac{d}{dx}(e^{x}) + e^{x} \cdot \dfrac{d}{dx}(x) \} = - \sin(x e^{x}) \cdot \{ x e^{x} + e^{x} \}$.

Sometimes we want to differentiate a composition of more than two functions. We can do so if we proceed
in a careful, step–by–step way.

Example 8:   Find $D(\sin(\sqrt{x^3 + 1}))$

Solution: The function $\sin(\sqrt{x^3 + 1})$ can be considered as a composition f∘g of

$f(x) = \sin(x)$ and $g(x) = \sqrt{x^3 + 1}$. Then

$(\sin(\sqrt{x^3 + 1}))'$ $= f '(g(x)) \cdot g '(x) = \cos(g(x)) \cdot g '(x) = \cos(\sqrt{x^3 + 1}) D(\sqrt{x^3 + 1})$

For the derivative of $\sqrt{x^3 + 1}$, we can use the Chain Rule again or its special case, the Power Rule:

$D(\sqrt{x^3 + 1}) = D((x^3 + 1)^{1/2}) = \dfrac{1}{2}(x^3 + 1)^{-1/2} D(x^3 + 1) = \dfrac{1}{2}(x^3 + 1)^{-1/2} 3x^2$.

Finally, $(\sin(\sqrt{x^3 + 1}))'$ $= \cos(\sqrt{x^3 + 1}) D(\sqrt{x^3 + 1})$

$= \cos(\sqrt{x^3 + 1}) \dfrac{1}{2}(x^3 + 1)^{-1/2}(3x^2) = \dfrac{3x^2 \cos(\sqrt{x^3 + 1})}{2\sqrt{x^3 + 1}}$.

This example was more complicated than the earlier ones, but it is just a matter of applying the Chain Rule twice, to a composition of a composition. If you proceed step–by–step and don't get lost in the problem, these multiple applications of the Chain Rule are relatively straightforward.

We can also use the Leibniz form of the Chain Rule for a composition of more than two functions. If

$y = \sin(\sqrt{x^3 + 1} \,)$, then $y = \sin(u)$ with $u = \sqrt{w}$ and $w = x^3 + 1$. The Leibniz form of the Chain

Rule is $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dw} \cdot \dfrac{dw}{dx}$, so $\dfrac{dy}{dx} = \cos(u)\dfrac{1}{2\sqrt{w}} \, 3x^2 = \cos(\sqrt{x^3 + 1}\,) \cdot \dfrac{1}{2\sqrt{x^3+1}} \cdot 3x^2$.

Practice 6:   (a) Find $\mathbf{D}(\sin(\cos(5x)))$. (b) For $y = e^{\cos(3x)}$, find dy/dx .

CHAIN RULE AND TABLES OF DERIVATIVES

With the Chain Rule, the derivatives of all sorts of strange and wonderful functions are available. If we know $\mathbf{f}'$ and $\mathbf{g}'$, then we also know the derivatives of their composition: $(f(g(x)))' = \mathbf{f}'(g(x)) \, \mathbf{g}'(x)$.

Example 9:   Given $\mathbf{D}(\arcsin(x)) = \dfrac{1}{\sqrt{1 - x^2}}$, find $\mathbf{D}(\arcsin(5x))$ and $\dfrac{d(\arcsin(e^x))}{dx}$.

Solution: (a) $\arcsin(5x)$ is the composition of $f(x) = \arcsin(x)$ with $g(x) = 5x$. We know $\mathbf{g}'(x) = 5$,

and $\mathbf{f}'(x) = \dfrac{1}{\sqrt{1 - x^2}}$ so $\mathbf{f}'(g(x)) = \dfrac{1}{\sqrt{1 - (g(x))^2}} = \dfrac{1}{\sqrt{1 - 25x^2}}$.

Then $\mathbf{D}(\arcsin(5x)) = \mathbf{f}'(g(x))\cdot\mathbf{g}'(x) = \dfrac{1}{\sqrt{1 - 25x^2}} \cdot(5) = \dfrac{5}{\sqrt{1 - 25x^2}}$.

(b) $y = \arcsin(e^x)$ is $y = \arcsin(u)$ with $u = e^x$. We know $dy/du = \dfrac{1}{\sqrt{1 - u^2}}$ and $du/dx = e^x$

so $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = \dfrac{1}{\sqrt{1 - u^2}} \cdot e^x = \dfrac{1}{\sqrt{1 - (e^x)^2}} \cdot e^x = \dfrac{e^x}{\sqrt{1 - e^{2x}}}$

In general, $\mathbf{D}(\arcsin(f(x))) = \dfrac{\mathbf{f}'(x)}{\sqrt{1 - (f(x))^2}}$ and $\dfrac{d(\arcsin(u))}{dx} = \dfrac{1}{\sqrt{1 - u^2}} \cdot \dfrac{du}{dx}$.

Practice 7: $\mathbf{D}(\arctan(x)) = \dfrac{1}{1 + x^2}$. Find $\mathbf{D}(\arctan(x^3))$ and $\dfrac{d(\arctan(e^x))}{dx}$.

The Appendix in the back of this book shows the derivative patterns for a variety of functions. You may not know much about some of the functions, but with the differentiation patterns given and the Chain Rule you should be able to calculate derivatives of compositions. It is just a matter of following the pattern.

Practice 8:   Use the patterns $D(\sinh(x)) = \cosh(x)$ and $D(\ln(x)) = 1/x$ to determine

   (a) $D(\sinh(5x - 7))$ (b) $\dfrac{d}{dx} (\ln(3 + e^{2x}))$ (c) $D(\arcsin(1 + 3x))$.

Example 10:   If $D(F(x)) = e^{x} \cdot \sin(x)$, find $D(F(5x))$ and $\dfrac{d(F(t^3))}{dt}$

Solution: (a) $D(F(5x)) = D(F(g(x))$ with $g(x) = 5x$. $F '(x) = e^{x} \cdot \sin(x)$ so

   $D(F(5x)) = F '(g(x)) \cdot g '(x) = e^{g(x)} \cdot \sin(g(x)) \cdot 5 = e^{5x} \cdot \sin(5x) \cdot 5$.

   (b) $y = F(u)$ with $u = t^3$. $\dfrac{dy}{du} = e^{u} \cdot \sin(u)$ and $\dfrac{du}{dt} = 3t^2$ so

   $\dfrac{dy}{dt} = \dfrac{dy}{du} \cdot \dfrac{du}{dt} = e^{u} \cdot \sin(u) \cdot 3t^2 = e^{(t^3)} \cdot \sin(t^3) \cdot 3t^2$

Proof of the Power Rule For Functions

We started using the Power Rule For Functions in section 2.3. Now we can easily prove it.

   Power Rule For Functions: If $y = f^{n}(x)$ and f is differentiable, then $\dfrac{dy}{dx} = n \cdot f^{n-1}(x) \cdot f '(x)$.

Proof: $y = f^{n}(x)$ is $y = u^{n}$ with $u = f(x)$. Then $\dfrac{dy}{du} = n \cdot u^{n-1}$ and $\dfrac{du}{dx} = f '(x)$ so by the Chain

   Rule, $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx} = n \cdot u^{n-1} \cdot f '(x) = n \cdot f^{n-1}(x) \cdot f '(x)$.

PROBLEMS

In problems 1 – 6 , find two functions f and g so that the given function is the composition of f and g.

1. $y = (x^3 - 7x)^5$ 2. $y = \sin^4(3x - 8)$ 3. $y = \sqrt{(2 + \sin(x))^5}$

4. $y = \dfrac{1}{\sqrt{x^2 + 9}}$ 5. $y = | x^2 - 4 |$ 6. $y = \tan(\sqrt{x})$

7.   For each function in problems 1 – 6, write y as a function of u for some u which is a function of x.

Problems 8 and 9 refer to the values given in this table:

x	f(x)	g(x)	f '(x)	g '(x)	(f∘g)(x)	(f∘g)' (x)
−2	2	−1	1	1		
−1	1	2	0	2		
0	−2	1	2	−1		
1	0	−2	−1	2		
2	1	0	1	−1		

8.   Use the table of values to determine (f∘g)(x) and (f∘g)' (x) at x = 1 and 2.

9.   Use the table of values to determine (f∘g)(x) and (f∘g)' (x) at x = −2, −1 and 0.

10. Use Fig. 2 to estimate the values of $g(x)$, $\mathbf{g}$ '$(\mathbf{x})$, $(f{\circ}g)(x)$, $\mathbf{f}$ '$(g(x))$, and $(\mathbf{f}{\bullet}\mathbf{g})$ '(x) at $x = 1$.

11. Use Fig. 2 to estimate the values of $g(x)$, $\mathbf{g}$ '$(\mathbf{x})$, $(f{\circ}g)(x)$, $\mathbf{f}$ '$(g(x))$, and $(\mathbf{f}{\bullet}\mathbf{g})$ '(x) for $x = 2$.
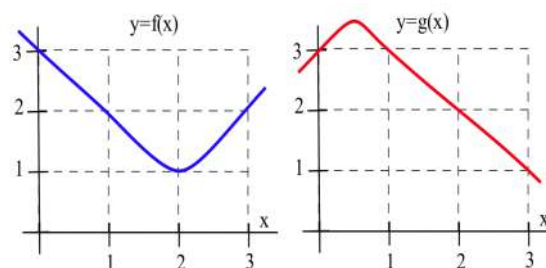
Fig. 2

In problems 12 - 20, differentiate each function.

12. $\mathbf{D}((x^2 + 2x + 3)^{87})$

13. $\mathbf{D}((1 - \frac{3}{x})^4)$

14. $\frac{\mathbf{d}}{\mathbf{dx}} (x + \frac{1}{x})^5$

15. $\mathbf{D}(\frac{5}{\sqrt{2 + \sin(x)}})$

16. $\frac{\mathbf{d}}{\mathbf{dt}} t{\cdot}\sin(3t + 2)$

17. $\mathbf{D}(x^2{\cdot}\sin(x^2 + 3))$

18. $\frac{\mathbf{d}}{\mathbf{dx}} \sin(2x){\cdot}\cos(5x + 1)$

19. $\mathbf{D}(\frac{7}{\cos(x^3 - x)})$

20. $\frac{\mathbf{d}}{\mathbf{dt}} \frac{5}{3 + e^t}$

21. $\mathbf{D}(e^x + e^{-x})$

22. $\frac{\mathbf{d}}{\mathbf{dx}} (e^x - e^{-x})$

23. An object attached to a spring is at a height of $h(t) = 3 - \cos(2t)$ feet above the floor t seconds after it is released.   (a) At what height was it released?

(b) Determine its height, velocity and acceleration at any time t.

(c) If the object has mass m, determine its kinetic energy $K = \frac{1}{2} mv^2$ and dK/dt at any time t.

24.   A manufacturer has determined that an employee with d days of production experience will be able to
produce approximately $P(d) = 3 + 15(1 - e^{-0.2d})$ items per day. Graph $P(d)$.

(a) Approximately how many items will a beginning employee be able to produce each day?

(b) How many items will an experienced employee be able to produce each day?

(c) What is the marginal production rate of an employee with 5 days of experience? (What are the units of your answer, and what does this answer mean?)

25.   The air pressure $P(h)$, in pounds per square inch, at an altitude of h feet above sea level is approximately $P(h) = 14.7 e^{-0.0000385h}$.

(a) What is the air pressure at sea level? What is the air pressure at an altitude of 30,000 feet?

(b) At what altitude is the air pressure 10 pounds per square inch?

(c) If you are in a balloon which is 2000 feet above the Pacific Ocean and is rising at 500 feet per minute, how fast is the air pressure on the balloon changing?

(d) If the temperature of the gas in the balloon remained constant during this ascent, what would happen to the volume of the balloon?

Find the derivatives in problems 26 – 33 .

26. $\mathbf{D}(\dfrac{(2x+3)^2}{(5x-7)^3})$

27. $\dfrac{\mathbf{d}}{\mathbf{dz}}\sqrt{1+\cos^2(z)}$

28. $\mathbf{D}(\sin(3x+5))$

29. $\dfrac{\mathbf{d}}{\mathbf{dx}}\tan(3x+5)$

30. $\dfrac{\mathbf{d}}{\mathbf{dt}}\cos(7t^2)$

31. $\mathbf{D}(\sin(\sqrt{x+1}))$

32. $\mathbf{D}(\sec(\sqrt{x+1}))$

33. $\dfrac{\mathbf{d}}{\mathbf{dx}}(e^{\sin(x)})$

In problems 34 – 37 , calculate $\dfrac{d\,f(x)}{dx}$ and $\dfrac{d\,x(t)}{dt}$ when t = 3 and use these values to determine the value of $\dfrac{d\,f(x(t))}{dt}$ when t = 3.

34. $f(x)=\cos(x)$, $x=t^2-t+5$

35. $f(x)=\sqrt{x}$, $x=2+\dfrac{21}{t}$

36. $f(x)=e^x$, $x=\sin(t)$

37. $f(x)=\tan^3(x)$, $x=8$

In problems 38 – 43, find a function which has the given function as its derivative. (You are given f '(x) in each problem and are asked to find a function f(x).)

38. $f'(x)=(3x+1)^4$

39. $f'(x)=(7x-13)^{10}$

40. $f'(x)=\sqrt{3x-4}$

41. $f'(x)=\sin(2x-3)$

42. $f'(x)=6e^{3x}$

43. $f'(x)=\cos(x)\cdot e^{\sin(x)}$

If two functions are equal, then their derivatives are also equal. In problems 44 – 47 , differentiate each side of the trigonometric identity to find a new identity.

44. $\sin^2(x)=\dfrac{1}{2}-\dfrac{1}{2}\cos(2x)$

45. $\cos(2x)=\cos^2(x)-\sin^2(x)$

46. $\sin(2x)=2\sin(x)\cos(x)$

47. $\sin(3x)=3\sin(x)-4\sin^3(x)$

Derivatives of Families of Functions

So far we have emphasized derivatives of particular functions, but sometimes we want to look at the derivatives of a whole family of functions. In problems 48 –71, the letters A–D represent constants and the given formulas describe families of functions.

For problems 48 – 65, calculate $y'=\dfrac{d\,y}{d\,x}$.

48. $y=Ax^3-B$

49. $y=Ax^3+Bx^2+C$

50. $y=\sin(Ax+B)$

51. $y=\sin(Ax^2+B)$

52. $y=Ax^3+\cos(Bx)$

53. $y=\sqrt{A+Bx^2}$

54. $y=\sqrt{A-Bx^2}$

55. $y=A-\cos(Bx)$

56. $y=\cos(Ax+B)$

57. $y = \cos(Ax^2 + B)$ 58. $y = A \cdot e^{Bx}$ 59. $y = x \cdot e^{Bx}$

60. $y = e^{Ax} + e^{-Ax}$ 61. $y = e^{Ax} - e^{-Ax}$ 62. $y = \dfrac{\sin(Ax)}{x}$

63. $y = \dfrac{Ax}{\sin(Bx)}$ 64. $y = \dfrac{1}{Ax + B}$ 65. $y = \dfrac{Ax + B}{Cx + D}$

In problems 66 – 71, (a) find $y\,'$, (b) find the value(s) of x so that $y\,' = 0$, and (c) find $y\,"$.

Typically your answer in part (b) will contain As, Bs and (sometimes) Cs.

66. For $y = Ax^2 + Bx + C$, (a) find $y\,'$, (b) find the value(s) of x so that $y\,' = 0$, and (c) find $y\,"$.

   (You should recognize the part (b) answer from intermediate algebra. What is it?)

67. $y = Ax(B - x) = ABx - Ax^2$.   68. $y = Ax(B - x^2) = ABx - Ax^3$.

69. $y = Ax^2(B - x) = ABx^2 - Ax^3$.   70. $y = Ax^2 + \dfrac{B}{x}$.   71. $y = Ax^3 + Bx^2 + C$.

Use the given differentiation patterns to differentiate the composite functions in problems 72 – 83 . We have not derived the derivatives for these functions (yet), but if you are handed the derivative pattern for a function then you should be able to take derivatives of a composition involving that function.

Given: $\mathbf{D}(\arctan(x)) = \dfrac{1}{1 + x^2}$, $\mathbf{D}(\arcsin(x)) = \dfrac{1}{\sqrt{1-x^2}}$, $\mathbf{D}(\ln(x)) = \dfrac{1}{x}$.

72. $\mathbf{D}(\arctan(7x))$ 73. $\dfrac{d}{dx}(\arctan(x^2))$ 74. $\dfrac{d}{dt}(\arctan(\ln(t)))$ 75. $\mathbf{D}(\arctan(e^x))$

76. $\dfrac{d}{dw}(\arcsin(4w))$ 77. $\mathbf{D}(\arcsin(x^3))$ 78. $\mathbf{D}(\arcsin(\ln(x)))$ 79. $\dfrac{d}{dt}(\arcsin(e^t))$

80. $\mathbf{D}(\ln(3x + 1))$ 81. $\dfrac{d}{dx}(\ln(\sin(x)))$ 82. $\mathbf{D}(\ln(\arctan(x)))$ 83. $\dfrac{d}{ds}(\ln(e^s))$

Section 3.5   **PRACTICE Answers**

Practice 1: $f(x) = 5x - 4$ and $g(x) = x^2$ so $f'(x) = 5$ and $g'(x) = 2x$. $f \circ g(x) = f(g(x)) = f(x^2) = 5x^2 - 4$.

$D(f \circ g(x)) = f'(g(x)) \cdot g'(x) = 5 \cdot 2x = \mathbf{10x}$ or $D(f \circ g(x)) = D(5x^2 - 4) = \mathbf{10x}$.

$g \circ f(x) = g(f(x)) = g(5x - 4) = (5x - 4)^2 = 25x^2 - 40x + 16$.

$D(g \circ f(x)) = g'(f(x)) \cdot f'(x) = 2(5x - 4) \cdot 5 = \mathbf{50x - 40}$ or

$D(g \circ f(x)) = D(25x^2 - 40x + 16) = \mathbf{50x - 40}$.

Practice 2: $\dfrac{d}{dx}(\sin(4x + e^x)) = \cos(4x + e^x) \cdot D(4x + e^x) = \cos(4x + e^x) \cdot (\mathbf{4 + e^x})$

Practice 3:   Fill in the table in Example 6 for $(f \circ g)(x)$ and $(f \circ g)'(x)$ at $x = 1, 2$ and 3.

x	f(x)	g(x)	f'(x)	g'(x)	(f∘g)(x)	(f∘g)'(x) = f'(g(x))·g'(x)
−1	2	3	1	0	0	0
0	−1	1	3	2	1	−2
1	1	0	−1	3	**−1**	f'(g(1))·g'(1) = f'(0)·(3) = (3)(3) = **9**
2	3	−1	0	1	**2**	f'(g(2))·g'(2) = f'(−1)·(1) = (1)(1) = **1**
3	0	2	2	−1	**3**	f'(g(3)·g'(3) = f'(2)·(−1) = (0)(−1) = **0**

Practice 4: $D(\sin(7x - 1)) = \cos(7x - 1)D(7x - 1) = \mathbf{7} \cdot \cos(7x - 1)$.

$\dfrac{d}{dx}\sin(ax + b) = \cos(ax + b)\, D(ax + b) = \mathbf{a} \cdot \cos(ax + b)$ $\dfrac{d}{dt}(e^{3t}) = e^{3t}\dfrac{d}{dt}(3t) = \mathbf{3} \cdot e^{3t}$

Practice 5: $D(\sin(g(x))) = \cos(g(x)) \cdot g'(x)$. At $x = \pi$, $\cos(g(\pi)) \cdot g'(\pi) \approx \cos(0.86) \cdot (-1) \approx -0.65$.

$D(g(\sin(x))) = g'(\sin(x)) \cdot \cos(x)$. At $x = \pi$, $g'(\sin(\pi)) \cdot \cos(\pi) = g'(0) \cdot (-1) \approx -2$

Practice 6:

$D(\sin(\cos(5x))) = \cos(\cos(5x)) \cdot D(\cos(5x)) = \cos(\cos(5x)) \cdot (-\sin(5x)) \cdot D(5x) = -5 \cdot \sin(5x) \cdot \cos(\cos(5x))$

$\dfrac{d}{dx} e^{\cos(3x)} = e^{\cos(3x)} D(\cos(3x)) = e^{\cos(3x)}(-\sin(3x))D(3x) = -3 \cdot \sin(3x) \cdot e^{\cos(3x)}$.

Practice 7: $D(\arctan(x^3)) = \dfrac{1}{1 + (x^3)^2} D(x^3) = \dfrac{3x^2}{1 + x^6}$

$\dfrac{d}{dx}(\arctan(e^x)) = \dfrac{1}{1 + (e^x)^2} D(e^x) = \dfrac{e^x}{1 + e^{2x}}$

Practice 8: $D(\sinh(5x - 7)) = \cosh(5x - 7) D(5x - 7) = \mathbf{5} \cdot \cosh(5x - 7)$

$\dfrac{d}{dx}\ln(3 + e^{2x}) = \dfrac{1}{3 + e^{2x}} D(3 + e^{2x}) = \dfrac{2e^{2x}}{3 + e^{2x}}$

$D(\arcsin(1 + 3x)) = \dfrac{1}{\sqrt{1 - (1 + 3x)^2}} D(1 + 3x) = \dfrac{3}{\sqrt{1 - (1 + 3x)^2}}$

3.6   SOME APPLICATIONS OF THE CHAIN RULE

The Chain Rule will help us determine the derivatives of logarithms and exponential functions a^x . We will also use it to answer some applied questions and to find slopes of graphs given by parametric equations.

DERIVATIVES OF LOGARITHMS

$$\mathbf{D}(\ \ln(x)\ ) = \frac{1}{x} \quad \text{and} \quad \mathbf{D}(\ \ln(\ g(x)\ )\ ) = \frac{g'(x)}{g(x)}\ .$$

Proof: We know that the natural logarithm $\ln(x)$ is the logarithm with base e, and $e^{\ln(x)} = x$ for $x > 0$.

We also know that $\mathbf{D}(\ e^x\ ) = e^x$, so using the Chain Rule we have $\mathbf{D}(\ e^{f(x)}\ ) = e^{f(x)}\ \mathbf{f}\ '(\mathbf{x}).$

Differentiating each side of the equation $e^{\ln(x)} = x$, we get that

$\mathbf{D}(\ e^{\ln(x)}\ )\ = \mathbf{D}(\ x\ )$ use $\mathbf{D}(\ e^{f(x)}\ ) = e^{f(x)}\ \mathbf{f}\ '(\mathbf{x})$ with $f(x) = \ln(x)$

$e^{\ln(x)}\ \mathbf{D}(\ \ln(x)\ ) = 1$ replace $e^{\ln(x)}$ with x

$x\ \mathbf{D}(\ \ln(x)\ ) = 1$ and solve for $\mathbf{D}(\ \ln(x)\ )$ to get $\mathbf{D}(\ \ln(x)\ ) = \frac{1}{x}$.

The function $\ln(\ g(x)\ )$ is the composition of $f(x) = \ln(x)$ with $g(x)$, so by the Chain Rule,

$$\mathbf{D}(\ \ln(\ g(x)\ ) = \mathbf{D}(\ f(\ g(x)\ )\ ) = \mathbf{f}\ '(\ g(x)\ )\ \mathbf{g}\ '(x) = \frac{1}{g(x)}\ \mathbf{g}\ '(x) = \frac{g'(x)}{g(x)}\ .$$

Example 1: Find $\mathbf{D}(\ \ln(\ \sin(x)\ )\ )$ and $\mathbf{D}(\ \ln(\ x^2 + 3\ )\ )$.

Solution: (a) Using the pattern $\mathbf{D}(\ \ln(\ g(x)\ ) = \frac{g'(x)}{g(x)}$ with $g(x) = \sin(x)$, then

$$\mathbf{D}(\ \ln(\ \sin(x)\ )\ ) = \frac{g'(x)}{g(x)} = \frac{\mathbf{D}(\ \sin(x)\ )}{\sin(x)} = \frac{\cos(x)}{\sin(x)} = \cot(x).$$

(b) Using the pattern with $g(x) = x^2 + 3$, we have $\mathbf{D}(\ \ln(\ x^2 + 3\ )\ ) = \frac{g'(x)}{g(x)} = \frac{2x}{x^2 + 3}$.

We can use the Change of Base Formula from algebra to rewrite any logarithm as a natural logarithm, and then we can differentiate the resulting natural logarithm.

Change of Base Formula for logarithms: $\log_a x = \frac{\log_b x}{\log_b a}$ for all positive a, b and x.

Example 2: Use the Change of Base formula and your calculator to find $\log_\pi 7$ and $\log_2 8$.

Solution: $\log_\pi 7 = \frac{\ln 7}{\ln \pi} \approx \frac{1.946}{1.145} \approx 1.700$. (Check that $\pi^{1.7} \approx 7$) $\log_2 8 = \frac{\ln 8}{\ln 2} = 3$.

Practice 1:   Find the values of $\log_9 20$, $\log_3 20$ and $\log_\pi e$.

Putting $b = e$ in the Change of Base Formula, $\log_a x = \dfrac{\log_e x}{\log_e a} = \dfrac{\ln x}{\ln a}$, so any logarithm can be written as a natural logarithm divided by a constant. Then any logarithm is easy to differentiate.

$$\mathbf{D}(\log_a(x)) = \frac{1}{x \ln(a)} \quad \text{and} \quad \mathbf{D}(\log_a(f(x))) = \frac{f'(x)}{f(x)} \cdot \frac{1}{\ln(a)}$$

Proof: $\mathbf{D}(\log_a(x)) = \mathbf{D}\left(\dfrac{\ln x}{\ln a}\right) = \dfrac{1}{\ln(a)} \cdot \mathbf{D}(\ln x) = \dfrac{1}{\ln(a)} \cdot \dfrac{1}{x} = \dfrac{1}{x \ln(a)}$.

   The second differentiation formula follows from the Chain Rule.

Practice 2:   Calculate $\mathbf{D}(\log_{10}(\sin(x)))$ and $\mathbf{D}(\log_\pi(e^x))$.

The number e might seem like an "unnatural" base for a natural logarithm, but of all the logarithms to different bases, the logarithm with base e has the nicest and easiest derivative. The natural logarithm is even related to the distribution of prime numbers. In 1896, the mathematicians Hadamard and Valle–Poussin proved the following conjecture of Gauss: (The Prime Number Theorem) For large values of x,

{number of primes less than x} $\approx \dfrac{x}{\ln(x)}$.

DERIVATIVE OF a^x

Once we know the derivative of e^x and the Chain Rule, it is relatively easy to determine the derivative of a^x for any $a > 0$.

$$\mathbf{D}(a^x) = a^x \cdot \ln a \quad \text{for } a > 0.$$

Proof: If $a > 0$, then $a^x > 0$ and $a^x = e^{\ln(a^x)} = e^{x \cdot \ln a}$.

   $\mathbf{D}(a^x) = \mathbf{D}(e^{\ln(a^x)}) = \mathbf{D}(e^{x \cdot \ln a}) = e^{x \cdot \ln a} \cdot \mathbf{D}(x \cdot \ln a) = a^x \cdot \mathbf{ln\ a}$.

Example 3:   Calculate $\mathbf{D}(7^x)$ and $\dfrac{d}{dt}(2^{\sin(t)})$

Solution: (a) $\mathbf{D}(7^x) = 7^x \ln 7 \approx (1.95)\, 7^x$.

   (b) We can write $y = 2^{\sin(t)}$ as $y = 2^u$ with $u = \sin(t)$. Using the Chain Rule,

   $\dfrac{dy}{dt} = \dfrac{dy}{du} \cdot \dfrac{du}{dt} = 2^u \cdot \ln(2) \cdot \cos(t) = 2^{\sin(t)} \cdot \ln(2) \cdot \cos(t)$.

Practice 3:   Calculate $\mathbf{D}(\sin(2^x))$ and $\dfrac{d}{dt}(3^{(t^2)})$.

SOME APPLIED PROBLEMS

Now we can examine applications which involve more complicated functions.

Example 4: A ball at the end of a rubber band (Fig. 1) is oscillating up and down, and its height

(in feet) above the floor at time t seconds is $h(t) = 5 + 2 \sin(t/2)$. (t is in radians)

(a) How fast is the ball travelling after 2 seconds? after 4 seconds? after 60 seconds?

(b) Is the ball moving up or down after 2 seconds? after 4 seconds? after 60 seconds?

(c) Is the vertical velocity of the ball ever 0 ?

ball

h(x)
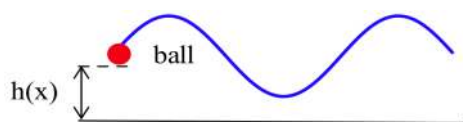
Fig. 1

Solution: (a) $v(t) = \mathbf{D}(h(t)) = \mathbf{D}(5 + 2 \sin(t/2))$

$= 2 \cos(t/2) \mathbf{D}(t/2) = \cos(t/2)$ feet/second so

$v(2) = \cos(2/2) \approx 0.540$ ft/s , $v(4) = \cos(4/2) \approx -0.416$ ft/s , and

$v(60) = \cos(60/2) \approx 0.154$ ft/s.

(b) The ball is moving upward when t = 2 and 60 seconds, downward when t = 4.

(c) $v(t) = \cos(t/2)$ and $\cos(t/2) = 0$ when $t = \pi \pm n{\cdot}2\pi$ $(n = 1, 2, \dots)$.

Example 5: If 2400 people now have a disease, and the number of people with the disease appears to

double every 3 years, then the number of people expected to have the disease in t years is $y = 2400 \cdot 2^{t/3}$.

(a) How many people are expected to have the disease in 2 years?

(b) When are 50,000 people expected to have the disease?

(c) How fast is the number of people with the disease expected to grow now and 2 years from now?

Solution: (a) In 2 years, $y = 2400 \cdot 2^{2/3} \approx 3{,}810$ people.

(b) We know y = 50,000 , and we need to solve $50{,}000 = 2400 \cdot 2^{t/3}$ for t. Taking logarithms of

each side of the equation, $\ln(50{,}000) = \ln(2400{\cdot}2^{t/3}) = \ln(2400) + (t/3) \cdot \ln(2)$ so $10.819 = 7.783 + .231t$

and $t \approx 13.14$ years. We expect 50,000 people to have the disease about 13.14 years from now.

(c) This is asking for dy/dt when t = 0 and 2 years. $\dfrac{dy}{dt} = \dfrac{d(2400{\cdot}2^{t/3})}{dt} = 2400{\cdot}2^{t/3} \cdot \ln(2){\cdot}(1/3)$

$\approx 554.5 \cdot 2^{t/3}$. Now, at t = 0, the rate of growth of the disease is approximately $554.5{\cdot}2^{0} \approx 554.5$

people/year. In 2 years the rate of growth will be approximately $554.5 \cdot 2^{2/3} \approx 880$ people/year.

temperature

$T (h) = \dfrac{72}{1+h}$
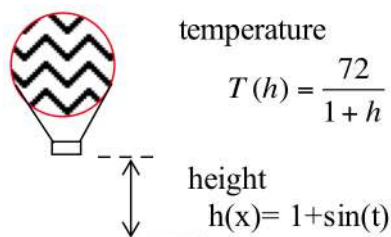
height

h(x)= 1+sin(t)

Fig. 2

Example 6:   You are riding in a balloon, and at time t (in minutes) you

are $h(t) = t + \sin(t)$ feet high. If the temperature at an elevation h   is

$T(h) = \dfrac{72}{1+h}$ degrees Fahrenheit, then how fast is your temperature

changing when t = 5 minutes? (Fig. 2)

Solution: As t changes, your elevation will change, and, as your elevation changes, so will your temperature. It is not difficult to write the temperature as a function of time, and then we could calculate $\dfrac{d\,T(t)}{dt}$ = T '(t) and evaluate T '(5), or we could use the Chain Rule:

$$\frac{d\,T(t)}{dt} \;=\; \frac{d\,T(\,h(t)\,)}{d\,h(t)} \cdot \frac{d\,h(t)}{dt} \;=\; \frac{d\,T(\,h\,)}{d\,h} \cdot \frac{d\,h(t)}{dt} \;=\; \frac{-72}{(1+h)^2} \cdot (\,1+\cos(\,t\,)\,).$$

When t = 5, then h(t) = 5 + sin(5) ≈ 4.04 so T '(5) ≈ $\dfrac{-72}{(1+4.04)^2}$ ·(1 + .284) ≈ –3.64 °/minute.

Practice 4:   Write the temperature T in the previous example as a function of the variable t alone and then differentiate T to determine the value of dT/dt when t = 5 minutes.

Example 7:   A scientist has determined that, under optimum conditions, an initial population of 40 bacteria will grow "exponentially" to $f(t) = 40 \cdot e^{t/5}$ bacteria after t hours.

   (a) Graph y = f(t) for 0 ≤ t ≤ 15 . Calculate f(0), f(5), f(10) .

   (b) How fast is the population increasing at time t? (Find f '(t) .)

   (c) Show that the rate of population increase, f '(t), is proportional to the population, f(t), at any time t.   (Show f '(t) = K·f(t) for some constant K.)

number of bacteria

time (hours)
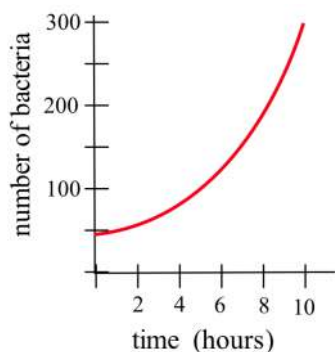
Fig. 3

Solution: (a) The graph of y = f(t) is given in Fig. 3 .

   $f(0) = 40 \cdot e^{0/5} = 40$ bacteria. $f(5) = 40 \cdot e^{5/5} \approx 109$ bacteria,

   and $f(10) = 40 \cdot e^{10/5} \approx 296$ bacteria.

   (b) $f'(t) = \dfrac{d}{dt}(\,f(t)\,) = \dfrac{d}{dt}(\,40 \cdot e^{t/5}\,) = 40 \cdot e^{t/5}\dfrac{d}{dt}(\,t/5\,)$

   $= 40 \cdot e^{t/5}\,(\,1/5\,) = 8 \cdot e^{t/5}$ bacteria/hour.

   (c) $f'(t) = 8 \cdot e^{t/5} = \dfrac{1}{5} \cdot (\,40 \cdot e^{t/5}\,) = \dfrac{1}{5}\,f(t)$ so $f'(t) = K \cdot f(t)$ with K = 1/5 .

PARAMETRIC EQUATIONS

Suppose a robot has been programmed to move in the xy–plane so at time t its x coordinate will be sin(t) and its y coordinate will be t^2 . Both x and y are functions of the independent parameter t, x(t) = sin(t) and $y(t) = t^2$, and the path of the robot (Fig. 4) can be found by plotting (x,y) = (x(t), y(t)) for lots of values of t.

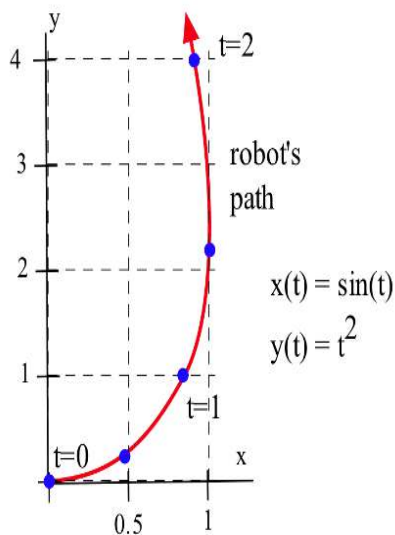| t	x(t) = sin(t)	$y(t) = t^2$	plot point at
0	0	0	(0, 0)
.5	.48	.25	(.48, .25)
1.0	.84	1	(.84, 1)
1.5	1.00	2.25	(1, 2.25)
2.0	.91	4	(.91, 4)

robot's
path

$x(t) = \sin(t)$

$y(t) = t^2$

Fig. 4

Typically we know x(t) and y(t) and need to find dy/dx, the slope of the tangent line to the graph of (x(t), y(t)). The Chain Rule says that

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} \text{ , so , algebraically solving for } \frac{dy}{dx} \text{ , we get } \frac{dy}{dx} = \frac{dy/dt}{dx/dt} \text{ .}$$

If we can calculate dy/dt and dx/dt, the derivatives of y and x with respect to the parameter t, then we can determine dy/dx, the rate of change of y with respect to x.

If   x = x(t) and y = y(t) are differentiable with respect to t , and $\frac{dx}{dt} \neq 0$,

then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$.

Example 8:   Find the slope of the tangent line to the graph of $(x,y) = (\sin(t), t^2)$ when t = 2?

Solution: dx/dt = cos(t) and dy/dt = 2t . When t = 2, the object is at the point (sin(**2**), **2**^{2}) ≈ (.91 , 4)

and the slope of the tangent line to the graph is $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{\cos(t)} = \frac{2\cdot\textbf{2}}{\cos(\textbf{2})} \approx \frac{4}{-.42} \approx -9.61$.

Practice 5:   Graph (x,y) = (3cos(t), 2sin(t)) and find the slope of the tangent line when t = π/2.

When we calculated $\dfrac{dy}{dx}$, the slope of the tangent line to the graph of (x(t), y(t)), we used the derivatives

$\dfrac{dx}{dt}$ and $\dfrac{dy}{dt}$, and each of these derivatives also has a geometric meaning:

$\dfrac{dx}{dt}$ measures the rate of change of x(t) with respect to t -- it tells us whether the x-coordinate is

increasing or decreasing as the t-variable increases.

$\dfrac{dy}{dt}$ measures the rate of change of y(t) with respect to t.

Example 9:   For the parametric graph in Fig. 5, tell whether $\dfrac{dx}{dt}, \dfrac{dy}{dt}$ and $\dfrac{dy}{dx}$

is positive or negative when t=2.

Fig. 5

Solution: As we move through the point B (where t=2) in the direction of increasing

values of t, we are moving to the left so x(t) is decreasing and $\dfrac{dx}{dt}$ is negative.

Similarly, the values of y(t) are increasing so $\dfrac{dy}{dt}$ is positive. Finally, the slope of the tangent

line, $\dfrac{dy}{dx}$, is negative.

(As check on the sign of $\dfrac{dy}{dx}$ we can also use the result $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{positive}{negative} = negative.$)
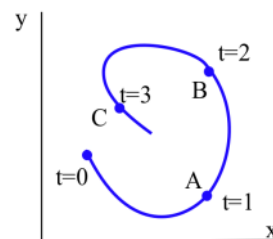
Practice 6: For the parametric graph in the previous example, tell whether $\dfrac{dx}{dt}$, $\dfrac{dy}{dt}$ and $\dfrac{dy}{dx}$

is positive or negative when t=1 and when t=3.

Speed

If we know the position of an object at every time, then we can determine its speed. The formula for speed comes from the distance formula and looks a lot like it, but with derivatives.

If   x = x(t) and y = y(t) give the location of an object at time t
and are differentiable functions of t,

then   the **speed** of the object is $\sqrt{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2}$.

Proof: The speed of an object is the limit, as $\Delta t \to 0$, of $\dfrac{\text{change in position}}{\text{change in time}}$. (Fig. 6)
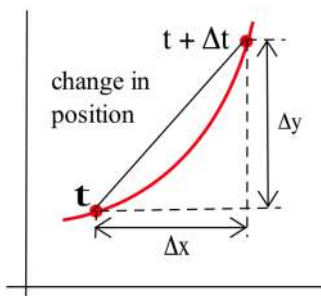
Fig. 6

$$\frac{\text{change in position}}{\text{change in time}} = \frac{\sqrt{(\Delta x)^2 + (\Delta y)^2}}{\Delta t} = \sqrt{\frac{(\Delta x)^2 + (\Delta y)^2}{(\Delta t)^2}}$$

$$= \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}$$

$$\to \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad \text{as } \Delta t \to 0 .$$

Exercise 10: Find the speed of the object whose location at time t is $(x,y) = (\sin(t), t^2)$ when t = 0 and t = 1.

Solution: $dx/dt = \cos(t)$ and $dy/dt = 2t$ so speed $= \sqrt{(\cos(t))^2 + (2t)^2} = \sqrt{\cos^2(t) + 4t^2}$.

When t = 0, speed $= \sqrt{\cos^2(0) + 4(0)^2} = \sqrt{1 + 0} = 1$. When t = 1,

speed $= \sqrt{\cos^2(1) + 4(1)^2} = \sqrt{0.29 + 4} \approx 2.07$.

Practice 7: Show that an object whose location at time t is $(x,y) = (3\sin(t), 3\cos(t))$ has a **constant** speed. (This object is moving on a circular path.)

Practice 8: Is the object whose location at time t is $(x,y) = (3\cos(t), 2\sin(t))$ travelling faster at the top of the ellipse (at $t = \pi/2$) or at the right edge of the ellipse (at t = 0)?

PROBLEMS

Differentiate the functions in problems 1 – 19.

1. $\ln(5x)$
2. $\ln(x^2)$
3. $\ln(x^k)$
4. $\ln(x^x) = x \cdot \ln(x)$
5. $\ln(\cos(x))$

6. $\cos(\ln(x))$
7. $\log_2 5x$
8. $\log_2 kx$
9. $\ln(\sin(x))$
10. $\ln(kx)$

11. $\log_2 (\sin(x))$
12. $\ln(e^x)$
13. $\log_5 5^x$
14. $\ln(e^{f(x)})$
15. $x \cdot \ln(3x)$

16. $e^x \cdot \ln(x)$
17. $\dfrac{\ln(x)}{x}$
18. $\sqrt{x + \ln(3x)}$
19. $\ln(\sqrt{5x - 3} \,)$

20. $\dfrac{d}{dt} \ln(\cos(t))$
21. $\dfrac{d}{dw} \cos(\ln(w))$
22. $\dfrac{d}{dx} \ln(ax + b)$

23. $\dfrac{d}{dt} \ln(\sqrt{t + 1} \,)$
24. $\mathbf{D}(3^x)$
25. $\mathbf{D}(5^{\sin(x)})$

26. $\mathbf{D}(x \cdot \ln(x) - x)$
27. $\dfrac{d}{dx} \ln(\sec(x) + \tan(x))$

28. Find the slope of the line tangent to $f(x) = \ln(x)$ at the point $(e , 1)$. Find the slope of the line
    tangent to $g(x) = e^x$ at the point $(1, e)$. How are the slopes of f and g at these points related?

29. Find a point P on the graph of $f(x) = \ln(x)$ so the tangent line to f at P goes through the origin.

30. You are moving from left to right along the graph of
    $y = \ln(x)$ (Fig. 7) .
    (a) If the x–coordinate of your location at time t
        seconds is $x(t) = 3t + 2$, then how fast is your
        elevation increasing?
    (b) If the x–coordinate of your location at time t
        seconds is $x(t) = e^t$, then how fast is your elevation increasing?
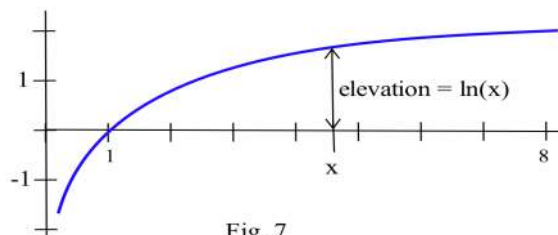
Fig. 7

31. Rumor. The percent of a population, $p(t)$, who have heard a rumor by time t is often modeled
    $$p(t) = \frac{100}{1 + Ae^{-t}} = 100 (1 + Ae^{-t})^{-1}$$ for some positive constant A. Calculate how fast the rumor is
    spreading, $\dfrac{d\,p(t)}{dt}$.

32. Radioactive decay. If we start with A atoms of a radioactive material which has a "half–life" (the
    amount of time for half of the material to decay) of 500 years, then the number of radioactive atoms
    left after t years is $r(t) = A \cdot e^{-Kt}$ where $K = \dfrac{\ln(2)}{500}$. Calculate $r'(t)$ and show that $r'(t)$ is
    proportional to $r(t)$ ($r'(t) = b \cdot r(t)$ for some constant b).

In problems 33 – 41 , find a function with the given derivative.

33. $f'(x) = \dfrac{8}{x}$

34. $h'(x) = \dfrac{3}{3x+5}$

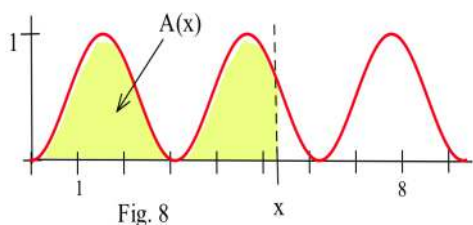35. $f'(x) = \dfrac{\cos(x)}{3+\sin(x)}$

36. $g'(x) = \dfrac{x}{1+x^2}$

37. $g'(x) = 3e^{5x}$

38. $h'(x) = e^2$

39. $f'(x) = 2x \cdot e^{(x^2)}$

40. $g'(x) = \cos(x) \cdot e^{\sin(x)}$

41. $h'(x) = \dfrac{\cos(x)}{\sin(x)}$

Fig. 8

42. Define A(x) to be the **area** bounded between the x–axis, the graph of f(x), and a vertical line at x (Fig. 8). The area under each "hump" of f is 2 square inches.

   (a) Graph A(x) for $0 \le x \le 9$.

   (b) Graph A '(x) for $0 \le x \le 9$.

Problems 43 – 48 involve parametric equations.

43. At time t minutes, robot A is at $(\,t, 2t+1\,)$ and robot B is at $(\,t^2, 2t^2+1\,)$.

   (a) Where is each robot when t=0 and t = 1?

   (b) Sketch the path each robot follows during the first minute.

   (c) Find the slope of the tangent line, dy/dx , to the path of each robot at t = 1 minute.

   (d) Find the speed of each robot at t = 1 minute.

   (e) Discuss the motion of a robot which follows the path $(\,\sin(t), 2\sin(t)+1\,)$ for 20 minutes.

44. $x(t) = t+1$, $y(t) = t^2$. (a) Graph $(\,x(t), y(t)\,)$ for $-1 \le t \le 4$.

   (b) Find dx/dt, dy/dt, the tangent slope dy/dx , and speed when t = 1 and t = 4.

45. For the parametric graph in Fig. 9, determine whether $\dfrac{dx}{dt}, \dfrac{dy}{dt}$ and $\dfrac{dy}{dx}$ are positive, negative or zero when $t = 1$ and $t = 3$.
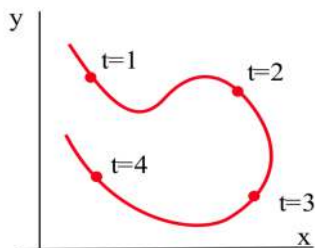
Fig. 9

46. For the parametric graph in Fig. 10, determine whether $\dfrac{dx}{dt}, \dfrac{dy}{dt}$ and $\dfrac{dy}{dx}$ are positive, negative or zero when $t = 1$ and $t = 3$.
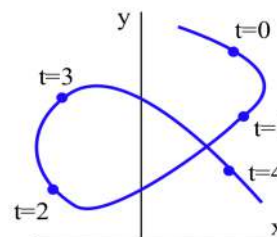

Fig. 10

47. $x(t) = R \cdot (t - \sin(t))$, $y(t) = R \cdot (1 - \cos(t))$. (a) Graph $(x(t), y(t))$ for $0 \le t \le 4\pi$.

(b) Find dx/dt, dy/dt, the tangent slope dy/dx , and speed when $t = \pi/2$ and π.

(The graph of $(x(t), y(t))$ is called a **cycloid** and is the path of a light attached to the edge of a rolling wheel with radius R.)

48. Describe the motion of two particles whose locations at time t are $(\cos(t), \sin(t))$ and $(\cos(t), -\sin(t))$.

49. Describe the path of a robot whose location at time t is

(a) $(3 \cdot \cos(t), 5 \cdot \sin(t))$ (b) $(A \cdot \cos(t), B \cdot \sin(t))$

(c) Give the parametric equations so the robot will move along the same path as in part (a) but in the opposite direction.

50. After t seconds, a projectile hurled with initial velocity v and angle θ will be at $x(t) = v \cdot \cos(\theta) \cdot t$ feet and $y(t) = v \cdot \sin(\theta) \cdot t - 16t^2$ feet. (Fig. 11) (This formula neglects air resistance.)

(a) For an initial velocity of 80 feet/second and an angle of $\pi/4$, find $t > 0$ so $y(t) = 0$. What does this value for t represent physically. Evaluate x(t).

(b) For v and θ in part (a), calculate dy/dx. Find t so dy/dx $= 0$ at t, and evaluate x(t). What does x(t) represent physically.

(c) What initial velocity is needed so a ball hit at an angle of $\pi/4 \approx 0.7854$ will go over a 40 foot high fence 350 feet away?

(d) What initial velocity is needed so a ball hit at an angle of 0.7   will go over a 40 foot high fence 350 feet away?

Fig. 11

Section 3.6   **PRACTICE Answers**

Practice 1: $\log_9 20 = \dfrac{\log(20)}{\log(9)} \approx \mathbf{1.3634165} \approx \dfrac{\ln(\,20\,)}{\ln(\,9\,)}$, $\log_3 20 = \dfrac{\log(20)}{\log(3)} \approx \mathbf{2.726833} \approx \dfrac{\ln(\,20\,)}{\ln(\,3\,)}$

$\log_\pi e = \dfrac{\log(e)}{\log(\pi)} \approx \mathbf{0.8735685} \approx \dfrac{\ln(e)}{\ln(\pi)} = \dfrac{1}{\ln(\pi)}$

Practice 2: $\mathbf{D}(\,\log_{10}(\,\sin(x)\,)\,) = \dfrac{1}{\sin(x)\cdot\ln(10)}$ $\mathbf{D}(\,\sin(x)\,) = \dfrac{\cos(x)}{\sin(x)\cdot\ln(10)}$

$\mathbf{D}(\,\log_\pi(\,e^x\,)\,) = \dfrac{1}{e^x\cdot\ln(\pi)}$ $\mathbf{D}(\,e^x\,) = \dfrac{e^x}{e^x\cdot\ln(\pi)} = \dfrac{1}{\ln(\pi)}$

Practice 3: $\mathbf{D}(\,\sin(\,2^x\,)\,) = \cos(\,2^x\,)\,\mathbf{D}(\,2^x\,) = \cos(\,2^x\,)\cdot\mathbf{2^x\cdot\ln(2)}$

$\dfrac{\mathbf{d}}{\mathbf{dt}}\,3^{(t^2)} = 3^{(t^2)}\,\ln(3)\,\mathbf{D}(\,t^2\,) = 3^{(t^2)}\,\ln(3)\cdot\mathbf{2t}$

Practice 4: $T = \dfrac{72}{1+h} = \dfrac{72}{1+t+\sin(t)}$.

$\dfrac{\mathbf{d}\,T}{\mathbf{dt}} = \dfrac{(\,1+t+\sin(t)\,)\cdot\mathbf{D}(\,72\,) - 72\cdot\mathbf{D}(\,1+t+\sin(t)\,)}{(\,1+t+\sin(t)\,)^2} = \dfrac{-72(\,1+\cos(t))}{(\,1+t+\sin(t)\,)^2}$

When $t=5$, $\dfrac{\mathbf{d}\,T}{\mathbf{dt}} = \dfrac{-72(\,1+\cos(5))}{(\,1+t+\sin(5)\,)^2} \approx \mathbf{-3.63695}$.

Practice 5: $x(t) = 3\cos(t)$ so $dx/dt = -3\sin(t)$. $y(t) = 2\sin(t)$

so $dy/dt = 2\cos(t)$. $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{2\cos(t)}{-3\sin(t)}$.

When $t = \pi/2$, $\dfrac{dy}{dx} = \dfrac{2\cos(\,\pi/2\,)}{-3\sin(\,\pi/2\,)} = \dfrac{2\cdot 0}{-3\cdot 1} = \mathbf{0}$. (See Fig. 12)
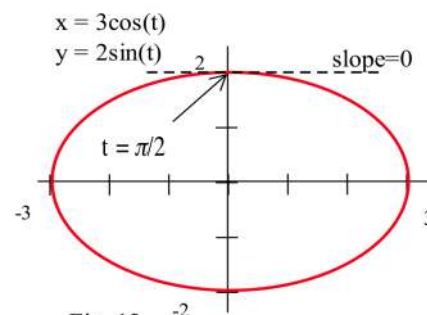
Fig. 12

Practice 6: When x=1: pos., pos., pos.   When x=3: pos., neg., neg.

Practice 7: $x(t) = 3\sin(t)$ and $y(t) = 3\cos(t)$ so $dx/dt = 3\cos(t)$ and $dy/dt = -3\sin(t)$. Then

$\text{speed} = \sqrt{(\,dx/dt\,)^2 + (\,dy/dt\,)^2} = \sqrt{(3\cos(t)\,)^2 + (-3\sin(t)\,)^2}$

$= \sqrt{9\cdot\cos^2(t) + 9\cdot\sin^2(t)} = \sqrt{9} = \mathbf{3}$ **, a constant**.

Practice 8: $x(t) = 3\cos(t)$ and $y(t) = 2\sin(t)$ so $dx/dt = -3\sin(t)$ and $dy/dt = 2\cos(t)$. Then

$\text{speed} = \sqrt{(\,dx/dt\,)^2 + (\,dy/dt\,)^2} = \sqrt{(-3\sin(t)\,)^2 + (2\cos(t)\,)^2} = \sqrt{9\sin^2(t) + 4\cos^2(t)}$.

When $t = 0$, the speed is $\sqrt{9\cdot(0)^2 + 4\cdot(1)^2} = 2$.

When $t = \pi/2$, the speed is $\sqrt{9\cdot(1)^2 + 4\cdot(0)^2} = \mathbf{3}$ **(faster)**.

3.7   RELATED RATES: An Application of Derivatives

In the next several sections we'll look at more uses of derivatives. Probably no single application will be of interest or use to everyone, but at least some of them should be useful to you. Applications also reinforce what you have been practicing; they require that you recall what a derivative means and use the techniques covered in the last several sections. Most people gain a deeper understanding and appreciation of a tool as they use it, and differentiation is both a powerful concept and a useful tool.

The Derivative As A Rate of Change

In Section 3.2, several interpretations were given for the derivative of a function. Here we will examine how the "rate of change of a function" interpretation can be used. If several variables or quantities are related to each other and some of the variables are changing at a known rate, then we can use derivatives to determine how rapidly the other variables must be changing.

Example 1:   Suppose we know that the radius of a circle is increasing at a rate of
    10 feet each second (Fig. 1) , and we want to know how fast the area of
    the circle is increasing when the radius is 5 feet. What can we do?
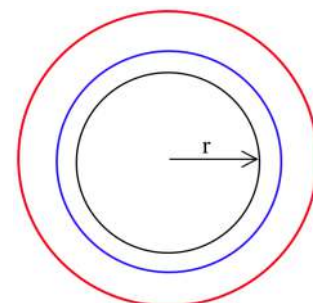
Fig. 1

Solution: We could get an <u>approximate</u> answer by calculating the area of the circle
    when the radius is 5   feet ($A = \pi r^2 = \pi(5 \text{ feet})^2 \approx 78.6 \text{ feet}^2$) and 1 second later
    when the radius is 10 feet larger than before ($A = \pi r^2 = \pi(15 \text{ feet})^2 \approx 706.9 \text{ feet}^2$)
    and then finding $\Delta \text{Area}/\Delta \text{time} = (706.9 \text{ ft}^2 – 78.6 \text{ ft}^2)/(1 \text{ sec}) = 628.3 \text{ ft}^2/\text{sec}$. This
    approximate answer represents the average change in area during the 1 second period when the radius
    increased from 5 feet to 15 feet. It is the slope of the secant line through the points P and Q in
    Fig. 2 , and it is clearly not a very good approximation of the instantaneous rate of change of the area, the
    slope of the tangent line at the point P.
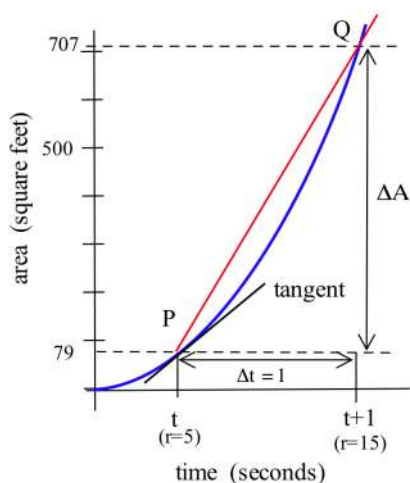
Fig. 2

We could get a <u>better approximation</u> by calculating $\Delta A/\Delta t$ over a shorter time interval, say $\Delta t = .1$ seconds. Then the original area was 78.6 ft^2 , the new area is $A = \pi(6 \text{ feet})^2 \approx 113.1 \text{ feet}^2$ (Why is the new radius 6 feet?) so

$\Delta A/\Delta t = (113.1 \text{ ft}^2 – 78.6 \text{ ft}^2)/(.1 \text{ sec}) = 345 \text{ ft}^2/\text{sec}$. This is the slope of the secant line through the point P and R in Fig. 3, and it is a much better approximation of the slope of the tangent line at P, but it is still only an approximation. Using derivatives, we can get an **exact** answer without doing very much work.

We know that the two variables in this problem, the radius r and the area A, are related to each other by the formula $A = \pi r^2$, and we know that both r and A are changing over time so each of them is a function of an additional variable t = time. We will continue to write the radius and area variables as r and A, but it is important to remember that each of them is really a **function** of t, r = r(t) and A = A(t). The statement that "the radius is increasing at a rate of 10 feet each second" can be translated into a mathematical statement about the rate of change, the derivative of r with respect to time: $\frac{dr}{dt}$ = 10 ft/sec . The question about the rate of change of the area is a question about $\frac{dA}{dt}$. Collecting all of this information, we have

area (square feet)

113
100
79

secant

ΔA

tangent

Δt = 0.1

t
(r=5)

t+0.1
(r=6)

time (seconds)
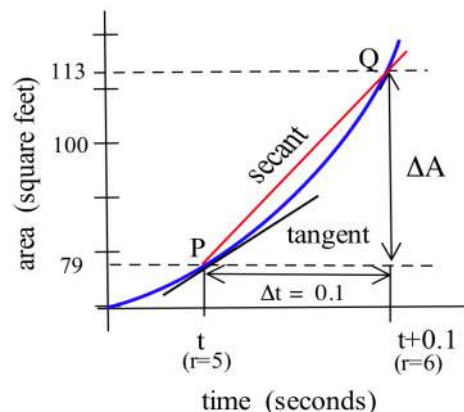
Fig. 3

Variables: r(t) = radius at time t , A(t) = area at time t

We Know: r = 5 feet and $\frac{d\,r(t)}{dt}$ = 10 ft/sec.

We Want: $\frac{d\,A(t)}{dt}$ when r = 5

Connecting Equation: $A = \pi r^2$ or $A(t) = \pi r^2(t)$.

Finally, we are ready to find $\frac{dA}{dt}$ — we just need to differentiate each side of the equation $A = \pi r^2$ with respect to the independent variable t.

$\frac{dA}{dt}$ = $\frac{d\,(\pi r^2)}{dt}$ = $\pi\frac{dr^2}{dt}$ = $\pi\,2r\,\frac{dr}{dt}$. The last piece, $\frac{dr}{dt}$, appears in the derivative because r is a **function** of t and we must use the differentiation rule for a function to a power (or the Chain Rule):

$\frac{d}{dt}\,f^n(t) = nf^{n-1}(t) \cdot \frac{df(t)}{dt}$.

We know from the problem that $\frac{dr}{dt}$ = 10 ft/sec so $\frac{dA}{dt}$ = $\pi\,2r\,\frac{dr}{dt}$ = $\pi 2r(10$ ft/s$)$ = $20\pi r$ ft/s. This answer tells us that the rate of increase of the area of the circle, $\frac{dA}{dt}$, depends on the value of the radius r as well as on the value of $\frac{dr}{dt}$. Since r = 5 feet, the area of the circle will be increasing at a rate of

$\frac{dA}{dt}$ = $20\pi r$ ft/s = $20\pi($**5 feet** $)$ ft/s = 100π ft^2 /s ≈ 314.2 square feet per second.

The key steps in finding the exact rate of change of the area of the circle were to:

- write the known information in a mathematical form, expressing rates of change as derivatives (r = 5 feet and dr/dt = 10 ft/sec)

- write the question in a mathematical form (dA/dt = ?)

- find an equation connecting or relating the variables (A = πr^2)

- differentiate both sides of the equation relating the variables, remembering that the variables are **functions** of t (dA/dt = 2πr dr/dt)

- put all of the known values into the equation in the previous step and solve for the desired part in the resulting equation (dA/dt = 2π(5 ft)(10 ft/sec) = 314.2 ft^2/sec)

Example 2:   Divers lives depend on understanding situations involving related rates. In water, the pressure at a depth of x feet is approximately $P(x) = 15(1 + \frac{x}{33})$ pounds per square inch (compared to approximately 15 pounds per square inch at sea level = P(0)). Volume is inversely proportional to the pressure , v = k/p , so doubling the pressure will result in half the original volume. Remember that volume is a function of the pressure: v = v(p).

(a)   Suppose a diver's lungs, at a depth of 66 feet, contained 1 cubic foot of air , and the diver ascended to the surface without releasing any air, what would happen?

(b)   If a diver started at a depth of 66 feet and ascended at a rate of 2 feet per second, how fast would the pressure be changing?

(Dives deeper than 50 feet also involve a risk of the "bends" or decompression sickness if the ascent is too rapid. Tables are available which show the safe rates of ascent from different depths.)

Solution: (a) The diver would risk getting ruptured lungs. The 1 cubic foot of air at a depth of 66 feet would be at a pressure of $P(66) = 15(1 + \frac{66}{33}) = 45$ pounds per square inch (psi). Since the pressure at sea level , P(0) = 15 psi , is only 1/3 as great, each cubic foot of air would expand to 3 cubic feet, and the diver's lungs would be in danger. Divers are taught to release air as they ascend to avoid this danger.

(b) The diver is ascending at a rate of 2 feet/second so the rate of change of the diver's depth x(t) is $\frac{d}{dt}$

$x(t) = -2$ ft/s. The pressure , $P = 15(1 + \frac{x}{33}) = 15 + \frac{15}{33}$ x , is a function of x (or x(t)) so $\frac{d}{dt}$ P =

$\frac{dP}{dx} \cdot \frac{dx}{dt} = (\frac{15}{33}$ psi/ft)·(-2 ft/sec) $= -\frac{30}{33}$ psi/sec ≈ –0.91 psi/sec.
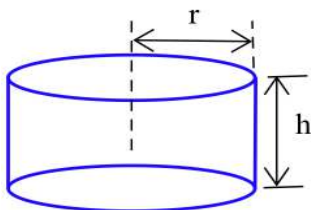
Fig. 4

Example 3:   The height of a cylinder is increasing at 7 meters per second and the radius is increasing at 3 meters per second. How fast is the volume changing when the cylinder is 5 meters high and has a radius of 6 meters? (Fig. 4)

Solution: First we need to translate our known information into a mathematical format. The height and radius are given: h = height = 5 m and r = radius = 6 m. We are also told how fast h and r are changing: dh/dt = 7 m/s and dr/dt = 3 m/s. Finally, we are asked to find dV/dt, and we should expect the units of dV/dt to be the same as $\Delta V/\Delta t$ which are m^3/s.

> **Variables**: h(t) = height at time t , r(t) = radius at t , V(t) = volume at t.
>
> **Know:** h = 5 m , $\dfrac{d\,h(t)}{dt}$ = 7 m/s , r = 6 m , $\dfrac{d\,r(t)}{dt}$ = 3 m/s .
>
> **Want:** $\dfrac{d\,V(t)}{dt}$

We also need an equation which relates the variables h, r and V (all of which are **functions** of time t) to each other:

> **Connecting Equation:** $V = \pi r^2 h$ or $V(t) = \pi r^2(t)\, h(t)$

Then, differentiating each side of this equation with respect to t (remembering that h, r and V are **functions**), we have

$$\frac{dV}{dt} = \frac{d(\pi r^2 h)}{dt} = \pi \frac{d(r^2 h)}{dt} = \pi\left\{ r^2 \frac{dh}{dt} + h \frac{dr^2}{dt} \right\} \qquad \text{by the Product Rule}$$

$$= \pi\left\{ r^2 \frac{dh}{dt} + h(2r)\frac{dr}{dt} \right\} \quad \text{by the Power Rule for functions.}$$

The rest is just substituting values and doing some arithmetic:

$$= \pi\left\{ (6\ m)^2\, (7\ m/s) + (5\ m)2(6\ m)(3\ m/s) \right\}$$
$$= \pi\{\ 252\ m^3/s + 180\ m^3/s\ \}$$
$$= 432\pi\ m^3/s \approx 1357.2\ m^3/s .$$

The volume of the cylinder is increasing at a rate of 1357.2 cubic meters per second. (It is always encouraging when the units of our answer are the ones we expect.)

Practice 1: How fast is the **surface area** of the cylinder changing in the previous example? (Assume that h, r, dh/dt, and dr/dt have the same values as in the example and use Fig. 5 to help you determine an equation relating the surface area of the cylinder to the variables h and r. The cylinder has a top and bottom.)
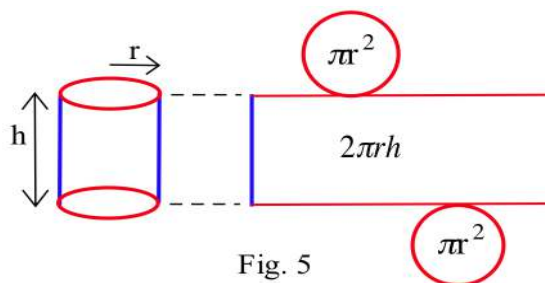
Fig. 5

Practice 2:   How fast is the volume of the cylinder in the previous example changing if the radius is **decreasing** at a rate of 3 meters per second? (The height, radius and rate of change of the height are the same as in the previous example: 5 m, 6 m and 7 m/s respectively.)

Usually, the most difficult part of Related Rate problems is to find an equation which relates or connects all of the variables. In the previous problems, the relating equations required a knowledge of geometry and formulas for areas and volumes (or knowing where to find them). Other Related Rates problems may require other information about similar triangles, the Pythagorean formula, or trigonometry — it depends on the problem.

It is a good idea, a very good idea, to draw a picture of the physical situation whenever possible. It is also a good idea, particularly if the problem is very important (your next raise depends on getting the right answer), to calculate at least one approximate answer as a check of your exact answer.

Example 4:   Water is flowing into a conical tank at a rate of 5 m^3/s. If the radius of the top of the cone is 2 m (see Fig. 6), the height is 7 m, and the depth of the water is 4 m, then how fast is the water level rising?

Fig. 6

Solution: Lets define our variables to be h = height (or depth) of the water in the cone and V = the volume of the water in the cone. Both h and V are changing, and both of them are functions of time t. We are told in the problem that h = 4 m and dV/dt = 5 m^3/s, and we are asked to find dh/dt. We expect that the units of dh/dt will be the same as $\Delta h/\Delta t$ which are meters/second.
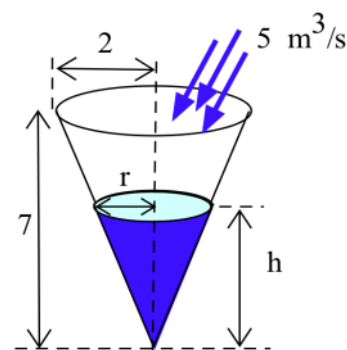
> **Variables:**   h(t) = height at t , r(t) = radius of the top surface of the water at t,
>   V(t) = volume of water at time t
>
> **Know:** h = 4 m , $\dfrac{d\,V(t)}{dt}$ = 5 m^3/s
>
> **Want:** $\dfrac{d\,h(t)}{dt}$

Unfortunately, the equation for the volume of a cone, $V = \frac{1}{3}\pi r^2 h$, also involves an additional variable r, the radius of the cone <u>at the top of the water</u> . This is a situation in which the picture can be a great help by suggesting that we have a pair of similar triangles so r/h = (top radius)/(total height) = (2 m)/(7 m) = 2/7 and $r = \frac{2}{7} h$. Then we can rewrite the volume of the cone of water, $V = \frac{1}{3}\pi r^2 h$, as a function of the single variable h:

> **Connecting Equation:** $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(\frac{2}{7} h)^2 h = \frac{4}{147}\pi h^3$.

The rest of the solution is straightforward.

$$\frac{dV}{dt} = \frac{d(\frac{4}{147}\pi h^3)}{dt} = \frac{4}{147}\pi \frac{d(h^3)}{dt} = \frac{4}{147}\pi\, 3h^2 \frac{dh}{dt} \quad \text{remember, h is a function of t}$$

$$= \frac{4}{147}\pi\, 3(4\text{ m})^2 \frac{dh}{dt} \approx (4.10\text{ m}^2)\frac{dh}{dt} \ .$$

We know that $\frac{dV}{dt} = 5\text{ m}^3/\text{s}$ and $\frac{dV}{dt} = (4.10\text{ m}^2)\frac{dh}{dt}$ so it is easy to solve for

$$\frac{dh}{dt} = \frac{dV/dt}{(4.10\text{ m}^2)} = \frac{5\text{ m}^3/\text{s}}{4.10\text{ m}^2} \approx 1.22\text{ m/s} \ .$$

This example was a little more difficult than the others because we needed to use similar triangles to get

an equation relating V to h and because we eventually needed to do a little arithmetic to solve for dh/dt.

Practice 3:   A rainbow trout has taken the fly at the end of a 60 foot line, and the line is being reeled in
at a rate of 30 feet per minute. If the tip of the rod is 10 feet above the water and the trout is at the
surface of the water, how fast is the trout being pulled toward the angler? (Suggestion: Draw a
picture and use the Pythagorean formula.)

rain

r

R
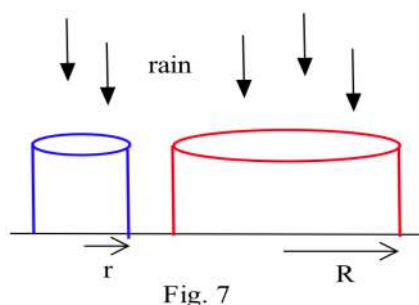
Fig. 7

Example 5:   When rain is falling vertically, the amount (volume) of
rain collected in a cylinder is proportional to the area of the
opening of the cylinder. If you place a narrow cylindrical
glass and a wide cylindrical glass out in the rain (Fig. 7) ,
(a) which glass will collect water faster, and
(b) in which glass will the water level rise faster?

Solution: Let's assume that the smaller glass has a radius of r and the larger has a radius of R, R > r , so
the areas of their openings are πr^2 and $\pi \cdot R^2$ respectively.

(a) The smaller glass will collect water at the rate $\frac{dv}{dt} = K\cdot\pi r^2$, and the larger at the rate

$\frac{dV}{dt} = K\cdot\pi\cdot R^2$ so $\frac{dV}{dt} > \frac{dv}{dt}$, and the larger glass will collect water faster than the smaller glass.

(b) The volume of water in each glass is a function of the radius of the glass and the height of the water in the
glass: $v = \pi r^2 h$ and $V = \pi R^2 H$ where h and H are the heights of the water levels in the smaller and
larger glasses, respectively. The heights h and H vary with t (are functions of t)   so

$$\frac{dv}{dt} = \frac{d(\pi r^2 h)}{dt} = \pi r^2 \frac{dh}{dt} \quad \text{and} \quad \frac{dh}{dt} = \frac{dv/dt}{\pi r^2} = \frac{K\pi r^2}{\pi r^2} = K \quad \text{(we got dv/dt} = K\pi r^2 \text{ in part (a))}.$$

Similarly, $\dfrac{dV}{dt} = \dfrac{d(\pi R^2 H)}{dt} = \pi R^2 \dfrac{dH}{dt}$ so $\dfrac{dH}{dt} = \dfrac{dV/dt}{\pi R^2} = \dfrac{K\pi R^2}{\pi R^2} = K$.

Then $\dfrac{dh}{dt} = K = \dfrac{dH}{dt}$ so the water level in each glass is rising at the same rate. In a one minute period, the larger glass will collect more rain, but the larger glass also requires more rain to raise its water level by each inch. How do you think the volumes and water levels would change if we placed a small glass and a large plastic box side by side in the rain?

PROBLEMS

1. An expandable sphere is being filled with liquid at a constant rate from a tap (imagine a water balloon connected to a faucet). When the radius of the sphere is 3 inches, the radius is increasing at 2 inches per minute. How fast is the liquid coming out of the tap? ($V = \frac{4}{3} \pi r^3$)

16

12
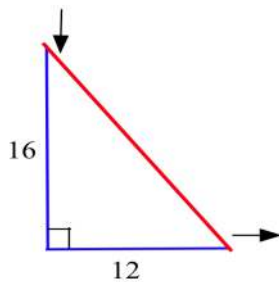
Fig. 8

2. The 12 inch base of a right triangle is growing at 3 inches per hour, and the 16 inch height is shrinking at 3 inches per hour (Fig. 8)

    (a)   Is the area increasing or decreasing?

    (b)   Is the perimeter increasing or decreasing?

    (c)   Is the hypotenuse increasing or decreasing?

13

15
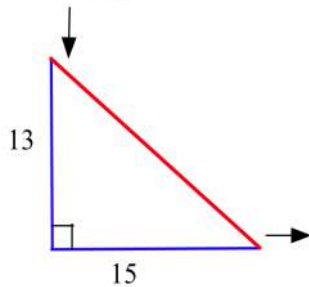
Fig. 9

3. One hour later the right triangle in Problem 2 is 15 inches long and 13 inches high (Fig. 9), and the base and height are changing at the same rate as in Problem 2.

    (a)   Is the area increasing or decreasing now?

    (b)   Is the hypotenuse increasing or decreasing now?

    (c)   Is the perimeter increasing or decreasing now?

4. A young woman and her boyfriend plan to elope, but she must rescue him from his mother who has locked him in his room. The young woman has placed a 20 foot long ladder against his house and is knocking on his window when his mother begins pulling the bottom of the ladder away from the house at a rate of 3 feet per second (Fig. 10). How fast is the top of the ladder (and the young couple) falling when the bottom of the ladder is

(a) 12 feet from the bottom of the wall?

(b) 16 feet from the bottom of the wall?

(c) 19 feet from the bottom of the wall?
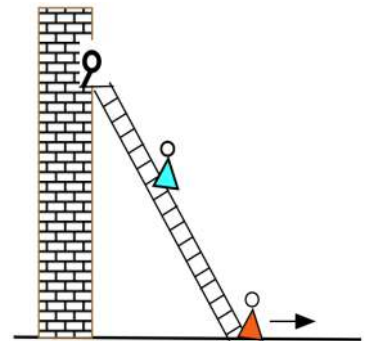
Fig. 10

5.   The length of a 12 foot by 8 foot rectangle is increasing at a rate of 3
      feet per second and the width is decreasing at 2 feet per second (Fig. 11).

2 ft/s

8 ft

12 ft

Fig. 11   3 ft/s ⟶
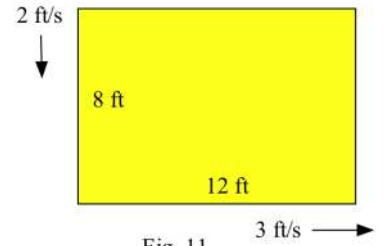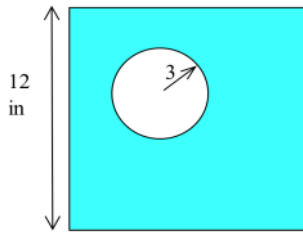
(a) How fast is the perimeter changing?

(b) How fast is the area changing?

12 in

3

Fig. 12

6.   A circle of radius 3 inches is inside a square with
      12 inch sides (Fig. 12). How fast is the area between the circle and square
      changing if the radius is increasing at 4 inches per minute and the sides are
      increasing at 2 inches per minute?

oil spill
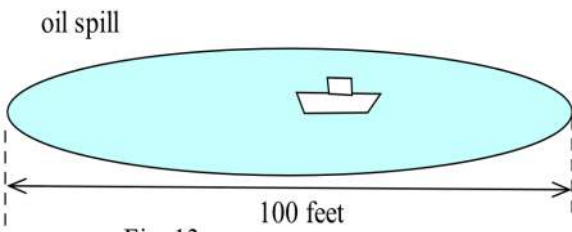
100 feet

Fig. 13

7.   An oil tanker in Puget Sound has sprung a leak, and a
      circular oil slick is forming (Fig. 13). The oil slick is 4
      inches thick everywhere, is 100 feet in diameter, and the
      diameter is increasing at 12 feet per hour. Your job, as
      the Coast Guard commander or the tanker's captain, is to
      determine how fast the oil is leaking from the tanker.

8.   A mathematical species of slug has a semicircular cross section and is always 5 times as long as it is
      high (Fig. 14). When the slug is 5 inches long, it is growing at .2 inches per week.
      (a) How fast is its volume increasing?
      (b) How fast is the area of its "foot" (the part of the slug in
           contact with the ground) increasing?

1 in

slug

5 in
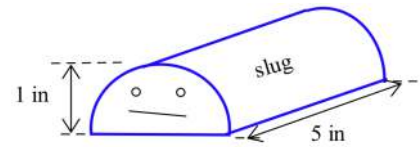
Fig. 14
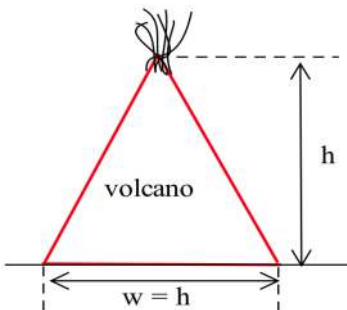
9.   Lava flowing from a hole at the top of a hill is forming a conical mountain whose
      height is always the same as the width of its base (Fig. 15). If the mountain is
      increasing in height at 2 feet per hour when it is 500 feet high, how fast is the lava
      flowing (how fast is the volume of the mountain increasing)? ($V = \frac{1}{3} \pi r^2 h$)

h

volcano

w = h

Fig. 15

10. A six foot tall person is walking away from a 14 foot tall
     lamp post at 3 feet per second (Fig. 16). When the person is
     10 feet from the lamp post,

     (a) How fast is the length of the person's shadow changing?

     (b) How fast is the tip of the shadow moving away from

          the lamp post?

14 ft

6 ft

shadow

Fig. 16
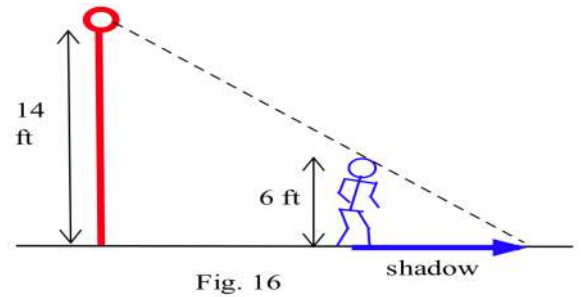
11. Answer parts (a) and (b) in Problem 10 for when the person is 20 feet from
     the lamp post.

12. Water is being poured at a rate of 15 cubic feet per minute into a conical
     reservoir which is 20 feet deep and has a top radius of 10 feet (Fig. 17).

     (a) How long will it take to fill the empty reservoir?

     (b) How fast is the water level rising when the water is 4 feet deep?

     (c) How fast is the water level rising when the water is 16 feet deep?

$15 \text{ ft}^3/\text{min}$

10 ft

r

20 ft

h

Fig. 17

500 feet of string

35°
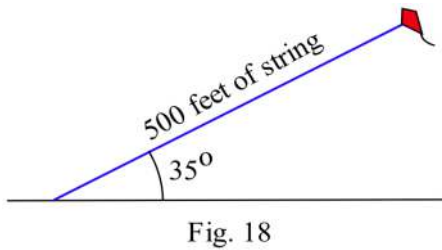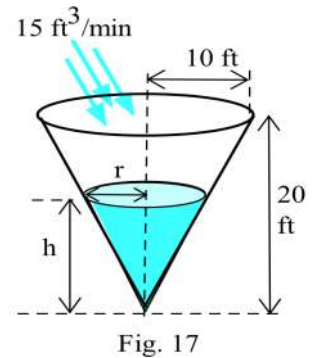
Fig. 18

13. The string of a kite is perfectly taut and always makes

     an angle of 35° above horizontal (Fig. 18).

     (a) If the kite flyer has let out 500 feet of string, how high is the kite?

     (b) If the string is let out at a rate of 10 feet per second, how fast is the

          kite's height increasing?

14. A small tracking telescope is viewing a hot air balloon rise from a
     point 1000 meters away from a point directly under the balloon (Fig. 19).

     (a) When the viewing angle is 20° , it is increasing at a rate of 3° per

          minute. How high is the balloon, and how fast is it rising?

     (b) When the viewing angle is 80° , it is increasing at a rate of 2° per

          minute. How high is the balloon, and how fast is it rising?
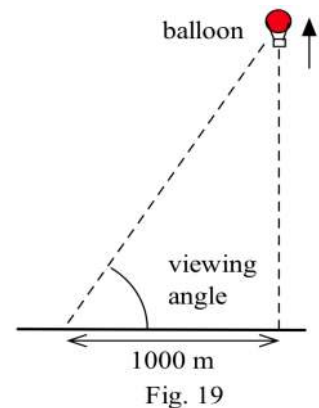
15. The 8 foot diameter of a spherical gas bubble is increasing at 2 feet
     per hour, and the 12 foot long edges of a cube containing the bubble are
     increasing at 3 feet per hour. Is the volume contained between the
     spherical bubble and the cube increasing or decreasing? At what rate?

balloon

viewing
angle

1000 m

Fig. 19

16. In general, the strength S of an animal is proportional to the cross–sectional area of its muscles, and this area is proportional to the square of its height H, so the strength $S = aH^2$. Similarly, the weight W of the animal is proportional to the cube of its height, so $W = bH^3$. Finally, the relative strength R of an animal is the ratio of its strength to its weight. As the animal grows, show that its strength and weight increase, but that the relative strength decreases.

17. The snow in a hemispherical pile melts at a rate proportional to its exposed surface area (the surface area of the hemisphere). Show that the height of the snow pile is decreasing at a constant rate.

18. If the rate at which water vapor condenses onto a spherical raindrop is proportional to the surface area of the raindrop, show that the radius of the raindrop will increase at a constant rate.

19. Define A(x) to be the **area** bounded by the x and y axes, the horizontal line y = 5, and a vertical line at x (Fig. 20).
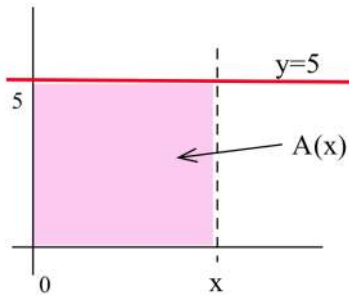
Fig. 20

   (a)   Find a formula for A as a function of x.

   (b)   Determine $\dfrac{d\,A(x)}{dx}$ when x = 1, 2, 4 and 9.

   (c)   Suppose x is a function of time, $x(t) = t^2$, and find a formula for A as a function of t.

   (d)   Determine $\dfrac{d\,A}{dt}$ when t = 1, 2, and 3.

   (e)   Suppose $x(t) = 2 + \sin(t)$. Find a formula for A(t) and determine $\dfrac{d\,A}{dt}$.

20. The point P is going around the circle $x^2 + y^2 = 1$ twice a minute (Fig. 21). How fast is the distance between the point P and the point (4,3) changing

  (a) when P = (1,0)?   (b) when P = (0,1)?   (c) when P = (.8, .6)?

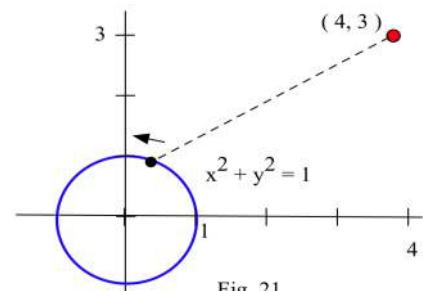(Suggestion: Write x and y as parametric functions of time t.)

Fig. 21

21. You are walking along a sidewalk toward a 40 foot wide sign which is adjacent to the sidewalk and perpendicular to it (Fig. 22).

  (a) If your viewing angle θ is 10°, then how far are you from the nearest corner of the sign?

  (b) If your viewing angle is 10° and you are walking at 25 feet per minute, then how fast is your viewing angle changing?

  (c) If your viewing angle is 10° and is increasing at 2° per minute, then how fast are you walking?

Fig. 21
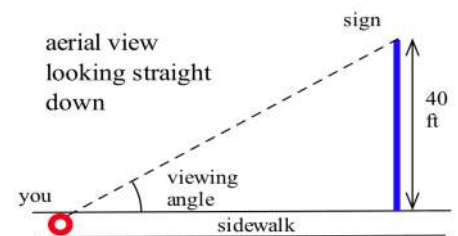
Section 3.7   **PRACTICE Answers**

Practice 1:   The surface area of the cylinder is $SA = 2\pi rh + 2\pi r^2$. From the Example, we know that

   $h' = 7$ m/s and $r' = 3$ m/s, and we want to know how fast the surface area is changing

   when $h = 5$ m and $r = 6$ m.

$$\frac{d\ SA}{dt} = 2\pi r\cdot h' + 2\pi \cdot r'\cdot h + 2\pi \cdot 2r\cdot r'$$

$$= 2\pi(6\text{ m})(\ 7\text{ m/s}) + 2\pi(3\text{ m/s})(5\text{ m}) + 2\pi(2\cdot 6\text{ m})(3\text{ m/s}) = 186\pi \ \text{m}^2/\text{s}$$

   ≈ **584.34 square meters per second**. (Note that the units represent a rate of change of area.)

Practice 2:   The volume of the cylinder is V = (area of the bottom)(height) = $\pi r^2 h$. We are told that

   $r' = -3$ m/s, and that h = 5 m, r = 6 m, and $h' = 7$ m/s.

$$\frac{d\ V}{dt} = \pi r^2\cdot h' + \pi\cdot 2r\cdot r'\cdot h = \pi(6\text{ m})^2(7\text{ m/s}) + \pi(2\cdot 6\text{ m})(-3\text{ m/s})(5\text{ m}) = 72\pi \ \text{m}^3/\text{s}$$

   ≈ **226.19 cubic meters per second**. (Note that the units represent a rate of change of volume.)

Practice 3:   Fig. 23 represents the situation described in this problem. We are told that $L' = -30$ ft/min . The

   variable F represents the distance of the fish from the angler, and we are asked to find F ', the rate of

   change of F when L = 60 ft.
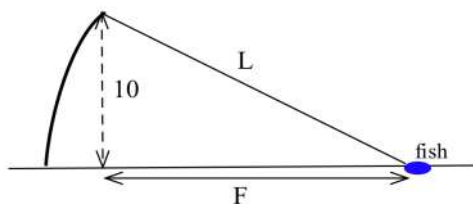
Fig. 23

Fortunately, the problem contains a right triangle so there is a formula (the Pythagorean formula) connecting F and L: $F^2 + 10^2 = L^2$ so

$$F = \sqrt{L^2 - 100} \quad .$$

Then $F' = \frac{1}{2}(L^2 - 100)^{-1/2} \frac{d\ (L^2 - 100)}{dt} = \frac{2L\cdot L'}{2\sqrt{L^2 - 100}} \quad .$

When L = 60 feet, $F' = \dfrac{2(60\text{ ft})(-30\text{ ft/min})}{2\sqrt{(60\text{ ft})^2 - (10\text{ ft})^2}} \approx \dfrac{-3600\text{ ft}^2/\text{min}}{118.32\text{ ft}} \approx$ **–30.43 ft/min.**

We could also find F ' implicitly: $F^2 = L^2 - 100$ so, differentiating each side,

$$2F\cdot F' = 2L\cdot L' \quad \text{and}\quad F' = \frac{L\cdot L'}{F} \quad .$$

Then we could use the given values for L and L ' and value of F (found using the Pythagorean

formula) to evaluate F '.

3.8 NEWTON'S METHOD FOR FINDING ROOTS

Newton's method is a process which can find roots of functions whose graphs cross or just kiss the x–axis. Although this method is a bit harder to apply than the Bisection Algorithm, it often finds roots that the Bisection Algorithm misses, and it usually finds them faster.

Off On A Tangent

The basic idea of Newton's Method is remarkably simple and graphic (Fig. 1):

> at a point $(x, f(x))$ on the graph of f, the tangent line to the
> graph of f "points toward" a root of f, a place where the
> graph touches the x–axis.
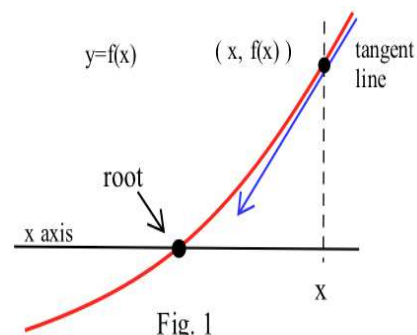
If we want to find a root of f , all we need to do is pick a starting value x_0,
go up or down to the point $(x_0, f(x_0))$ on the graph of f, build a tangent
line there, and follow the tangent line to where it crosses the x–axis, say at x_1.
If x_1 is a root of f , then we are done. If x_1 is not a root of f, then x_1 is usually closer to the root
than x_0 was, and we can repeat the process, using x_1 as our new starting point. Newton's method is an
iterative procedure, that is, the output from one application of the method becomes the starting point for
the next application.
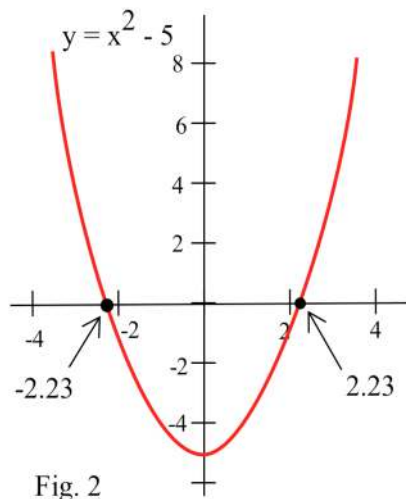
Let's start with a differentiable function $f(x) = x^2 – 5$, (Fig. 2) whose roots we
already know, $x = \pm\sqrt{5} \approx \pm 2.236067977$, and illustrate how Newton's method
works. First we pick some value for x_0 , say $x_0 = 4$ for this example, and move
to the point $(x_0, f(x_0)) = (4, 11)$ on the graph of f.

At $(4, 11)$, the graph of f "points to" a
location on the x–axis which is closer to the root
of f (Fig. 3). We can calculate this location
on the x–axis by finding the equation of the
line tangent to the graph of f at the point
$(4, 11)$ and then finding where this tangent line
intersects the x–axis:
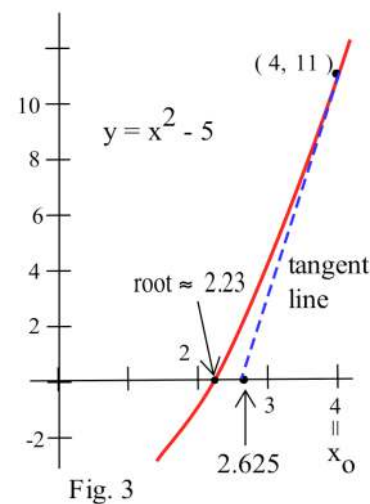
At the point $(4, 11)$, the line tangent to f has slope $m = f'(4) = 2(4) = 8$,
so the equation of the tangent line is $y – 11 = 8(x – 4)$. Setting $y = 0$, we
can find where the tangent line crosses the x–axis:

$$0 - 11 = 8(x - 4) \text{ , so } x = 4 - \frac{11}{8} = \frac{21}{8} = 2.625 .$$

Call this new value x_1 : $x_1 = 2.625$.

The point $x_1 = 2.625$ is closer to the actual root , but it certainly does not

equal the actual root. If Newton's method stopped after one step with the

estimate of 2.625, it would not be very useful. Instead, we can use this new

value for x , $x_1 = 2.625$, to repeat the procedure (Fig. 4):

(i)   move to the point $(x_1, f(x_1)) = (2.625 , 1.890625)$ on the graph,

(ii)   find the equation of the tangent line at the point $(x_1, f(x_1))$:

$$y-1.890625 = 5.25(x- 2.625)$$

(iii)   find the new value where the tangent line intersects the x–axis and

call it x_2. ($x_2 = 2.262880952$)

When we continue repeating this process, (Fig. 5) using each new estimate

for the root of $f(x) = x^2 - 5$ as the beginning point for calculating the next

estimate, we get:

Beginning estimate:	$x_0 = 4$	(0 correct digits)
after 1 iteration:	$x_1 = \underline{2}.625$	(1 correct digit)
after 2 iterations:	$x_2 = \underline{2.2}62880952$	(2 correct digits)
after 3 iterations:	$x_3 = \underline{2.236}251252$	(4 correct digits)
after 4 iterations:	$x_4 = \underline{2.2360679}85$	(8 correct digits)

It only took 4 iterations to get an approximation of $\sqrt{5}$ which is within 0.000000008 of the exact value.
One more iteration gives an approximation x_5 which has 16 correct digits. If we start with $x_0 = -2$ (or
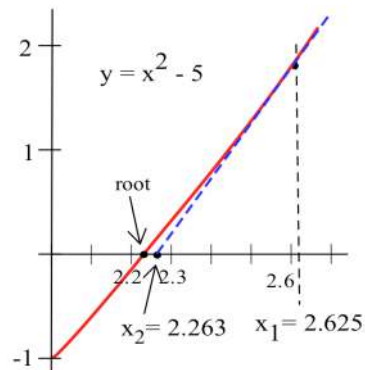any negative number), then the values of x_n approach $-\sqrt{5} \approx -2.23606$.
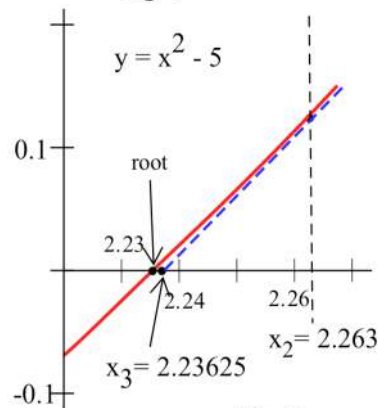
Fig. 6 shows the process for Newton's Method, starting with x_0 and
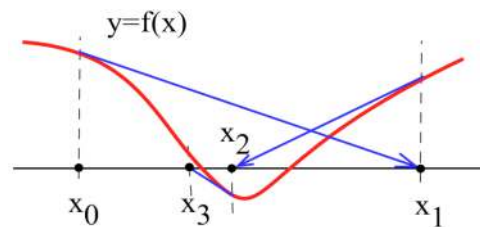graphically finding the locations on the x–axis of $x_1, x_2,$ and x_3 .

Practice 1:   Find where the tangent line to $f(x) = x^3 + 3x - 1$
at $(1, 3)$ intersects the x–axis.

Practice 2:   A starting point and a graph of f are given in Fig. 7.
Label the approximate locations of the next two points
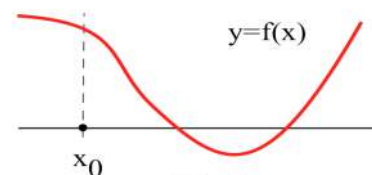on the x–axis which are found by Newton's method.

The Algorithm for Newton's Method

Rather than deal with each particular function and starting point, let's find a pattern for a general function f. For the starting point x_0 , the slope of the tangent line at the point $(x_0, f(x_0))$ is $f'(x_0)$ so the equation of the tangent line is $y - f(x_0) = f'(x_0)(x - x_0)$. This line intersects the x–axis when $y = 0$, so

$0 - f(x_0) = f'(x_0)(x - x_0)$ and $x_1 = x = x_0 - \dfrac{f(x_0)}{f'(x_0)}$. Starting with x_1 and repeating this process we

have $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$; starting with x_2 , we get $x_3 = x_2 - \dfrac{f(x_2)}{f'(x_2)}$; and so on.

In general, if we start with x_n , the line tangent to the graph of f at the point $(x_n, f(x_n))$ intersects the

x–axis at the point $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$, our new estimate for the root of f .

Algorithm for Newton's Method:

(1) Pick a starting value x_0 (preferably close to a root of f).

(2) For each estimate x_n , calculate a new estimate $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.

(3) Repeat step (2) until the estimates are "close enough" to a root or until the method "fails".

When the algorithm for Newton's method is used with $f(x) = x^2 - 5$, the function at the beginning of this section, we have $f'(x) = 2x$ so

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 5}{2x_n} = \frac{2x_n^2 - (x_n^2 - 5)}{2x_n}$$

$$= \frac{x_n^2 + 5}{2x_n} = \frac{1}{2}\left\{ x_n + \frac{5}{x_n} \right\} .$$

The new approximation, x_{n+1} , is the average of the previous approximation, x_n , and 5 divided by the previous approximation, $5/x_n$. Problem 16 asks you to show that this pattern, called Heron's method, approximates the square root of any positive number. Just replace the "5" with the number whose square root you want.

Example 1: Use Newton's method to approximate the root(s) of $f(x) = 2x + x\sin(x+3) - 5$.

Solution: $f'(x) = 2 + x\cos(x+3) + \sin(x+3)$ so

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{2x_n + x_n\sin(x_n+3) - 5}{2 + x_n\cos(x_n+3) + \sin(x_n+3)}$$

The graph of f(x) for $-4 \le x \le 6$ (Fig. 8) indicates only one root of f, and that root is near x = 3 so pick $x_0 = 3$. Then Newton's method yields the values $x_0 = 3$, $x_1 = \underline{2.96}484457$, $x_2 = \underline{2.96446}277$, $x_3 = \underline{2.96446273}$ (the underlined digits agree with the exact root).

If we had picked $x_0 = 4$, Newton's method would have required 4 iterations to get 9 digits of accuracy.
If $x_0 = 5$, then 7 iterations are needed to get 9 digits of accuracy. If we pick $x_0 = 5.1$, then the values

f(x)=2x + xsin(x-3) - 5

root

Fig. 8

of x_n are not close to the actual root after even
100 iterations, $x_{100} \approx -49.183$. Picking a good
value for x_0 can result in values of x_n which

get close to the root quickly. Picking a poor
value for x_0 can result in x_n values which take

longer to get close to the root or which don't
approach the root at all.
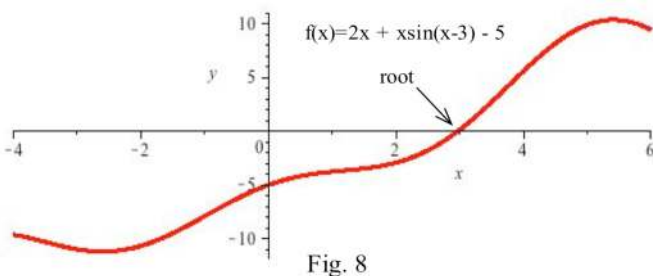
Note: An examination of the graph of the function can help you pick a "good" x_0 .

Practice 3:   Put $x_0 = 3$ and use Newton's method to find
the first two iterates, x_1 and x_2 , for the
function $f(x) = x^3 - 3x^2 + x - 1$.

y=f(x)

x_0
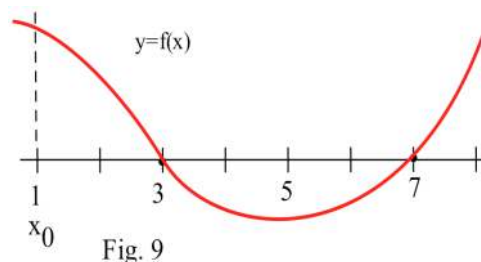
Fig. 9

Example 2:   The function in Fig. 9 has roots at $x = 3$
and $x = 7$. If we pick $x_0 = 1$ and apply

Newton's method, which root do the iterates,
the x_n , approach?

Solution: The iterates of $x_0 = 1$ are labeled in Fig. 10. They are approaching the root at 7.

y=f(x)

x_1

x_2

x_0 Fig. 10

y=f(x)

a   b   c
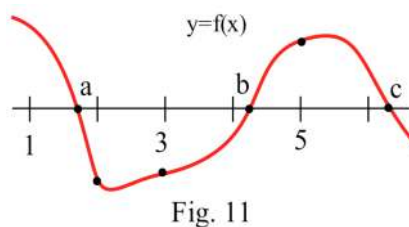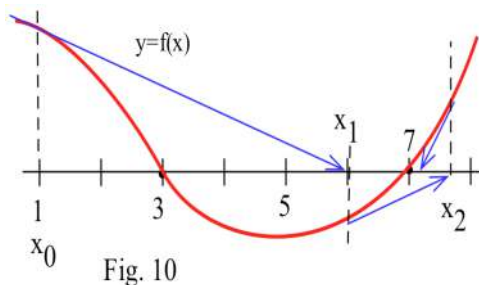
Fig. 11

Practice 4:   For the function in Fig. 11, which root do the iterates of Newton's method approach
if (a) $x_0 = 2$? (b) $x_0 = 3$?   (c) $x_0 = 5$?

Iteration

We have been emphasizing the geometric nature of Newton's method, but Newton's method is also an
example of **iterating a function**. If $N(x) = x - \dfrac{f(x)}{f'(x)}$, the "pattern" in the algorithm, then

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = N(x_0),$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = N(x_1) = N(N(x_0)) = N \circ N(x_0),$$

$$x_3 = N(x_2) = N \circ N \circ N(x_0), \text{ and, in general,}$$

$$x_n = N(x_{n-1}) = n^{th} \text{ iteration of N starting with } x_0.$$

At each step, we are using the output from the function N as the next input into N.

What Can Go Wrong?

When Newton's method works, it usually works very well and the values of the x_n approach a root of f very quickly, often doubling the number of correct digits with each iteration. There are, however, several things which can go wrong.

One obvious problem with Newton's method is that $f'(x_n)$ can be 0. Then we are trying to divide by 0 and x_{n+1} is undefined. Geometrically, if $f'(x_n) = 0$, then the tangent line to the graph of f at x_n is horizontal and does not intersect the x–axis at one point (Fig. 12). If $f'(x_n) = 0$, just pick another starting value x_0 and begin again. In practice, a second or third choice of x_0 usually succeeds.

There are two other less obvious difficulties that are not as easy to overcome — the values of the iterates x_n may become locked into an infinitely repeating loop (Fig. 13), or they may actually move farther away from a root (Fig. 14).
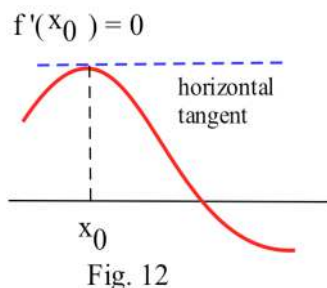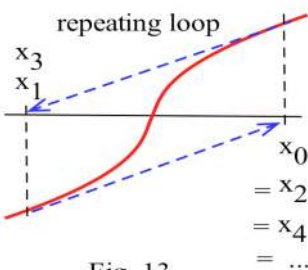

$f'(x_0) = 0$

horizontal tangent

x_0

Fig. 12

repeating loop

x_3
x_1

x_0
$= x_2$
$= x_4$
$= \ldots$

Fig. 13

diverging

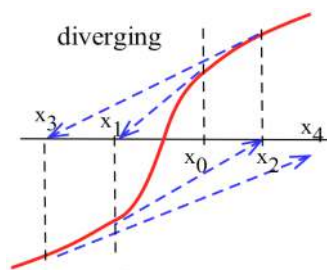x_3 x_1 x_4
x_0 x_2

Fig. 14

Example 3:   Put $x_0 = 1$ and use Newton's method to find the first two iterates, x_1 and x_2, for the function $f(x) = x^3 - 3x^2 + x - 1$.

Solution: This is the same function as in the previous Practice problem, but we are using a different starting value for x_0. $f'(x) = 3x^2 - 6x + 1$ so

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{-2}{-2} = 0 \text{ and } x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 0 - \frac{f(0)}{f'(0)} = 0 - \frac{-1}{1} = 1$$

which is the same as x_0, so $x_3 = x_1 = 0$ and $x_4 = x_2 = 1$. The values of x_n alternate between 1 and 0 and do not approach a root.

Newton's method behaves badly at only a few starting points for this particular function. For most starting points Newton's method converges to the root of this function.

There are some functions which defeat Newton's method for **almost every** starting point.

Practice 5:   For $f(x) = \sqrt[3]{x} = x^{1/3}$ and $x_0 = 1$, verify that $x_1 = -2$, $x_2 = 4$, $x_3 = -8$. Also try $x_0 = -3$, and verify that the same pattern holds: $x_{n+1} = -2x_n$. Graph f and explain why the Newton's method iterates get farther and farther away from the root at 0.

Newton's method is powerful and quick and very easy to program on a calculator or computer. It usually works so well that many people routinely use it as the first method they apply. If Newton's method fails for their particular function, they simply try some other method.

Chaotic Behavior and Newton's Method

An algorithm leads to **chaotic behavior** if two starting points which are close together generate iterates which are sometimes far apart and sometimes close together: $|a_0 - b_0|$ is small but $|a_n - b_n|$ is large for lots (infinitely many) of values of n and $|a_n - b_n|$ is small for lots of values of n.

The iterates of the next simple algorithm exhibit chaotic behavior.

A Simple Chaotic Algorithm: Starting with any number between 0 and 1, double the number and keep the fractional part of the result: x_1 is the fractional part of $2x_0$, x_2 is the fractional part of $2x_1$, and in general, $x_{n+1} = 2x_n - [\, 2x_n \,] = 2x_n - \text{INT}(\, 2x_n \,)$.

If $x_0 = 0.33$, then the iterates of the algorithm are $0.66, 0.32$ (= fractional part of $2 \cdot 0.66$), $0.64, 0.28, 0.56, \ldots$

The iterates for two other starting values close to .33 are given below as well as the iterates of 0.470 and 0.471 :

start = x_0	0.32	0.33	0.34		0.470	0.471
x_1	0.64	0.66	0.68		0.940	0.942
x_2	0.28	0.32	0.36		0.880	0.884
x_3	0.56	0.64	0.72		0.760	0.768
x_4	0.12	0.28	0.44		0.520	0.536
x_5	0.24	0.56	0.88		0.040	0.072
x_6	0.48	0.12	0.76		0.080	0.144
x_7	0.96	0.24	0.56		0.160	0.288
x_8	0.92	0.48	0.12		0.320	0.576
x_9	0.84	0.96	0.24		0.640	0.152

There are starting values as close together as we want whose iterates are far apart infinitely often.

Many physical, biological, and business phenomena exhibit chaotic behavior. Atoms can start out within inches of each other and several weeks later be hundreds of miles apart. The idea that small initial differences can lead to dramatically diverse outcomes is sometimes called the "Butterfly Effect" from the title of a talk ("Predictability: Does the Flap of a Butterfly's Wings in Brazil Set Off a Tornado in Texas?") given by Edward Lorenz, one of the first people to investigate chaos. The "butterfly effect" has important implications about the possibility, or rather the impossibility, of accurate long–range weather forecasting. Chaotic behavior is also an important aspect of studying turbulent air and water flows, the incidence and spread of diseases, and even the fluctuating behavior of the stock market.

Newton's method often exhibits chaotic behavior, and, since it is a relatively easy to study, is often used as a model to study the properties of chaotic behavior. If we use Newton's method to approximate the roots of $f(x) = x^3 - x$ (with roots $0, +1$ and -1) , then starting points which are very close together can have iterates which converge to different roots. The iterates of $.4472$ and $.4473$ converge to the roots 0 and $+1$, respectively. The iterates of the middle point $.44725$ converge to the root -1, and the iterates of another nearby point, $\sqrt{1/5} \approx .44721$, simply cycle between $-\sqrt{1/5}$ and $+\sqrt{1/5}$ and do not converge at all.

Practice 6:   Find the first 4 Newton's method iterates of $x_0 = .997$ and $x_0 = 1.02$ for $f(x) = x^2 + 1$. Try

two other starting values very close to 1 (but not equal to 1) and find their first 4 iterates. Use the graph of $f(x) = x^2 + 1$ to explain how starting points so close together can quickly have iterates so far apart.

PROBLEMS

1. The graph of $y = f(x)$ is given in Fig. 15.   Estimate the locations of x_1
   and x_2 when Newton's method is applied to f with the given starting value x_0

    .

Fig. 15
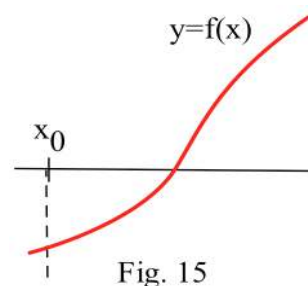
2. The graph of $y = g(x)$ is given in Fig. 16.   Estimate the locations of x_1
    and x_2 when Newton's starting value x_0 .

Fig. 16
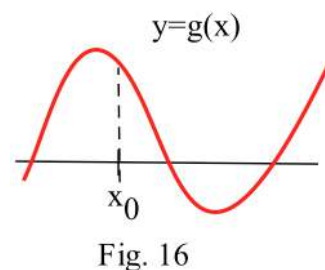
3. The function in Fig. 17 has several roots. Which root do the iterates of
    Newton's method converge to if we start with $x_0 = 1$? $x_0 = 4$?

Fig. 17
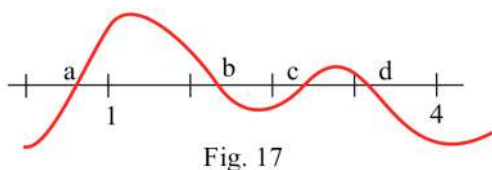
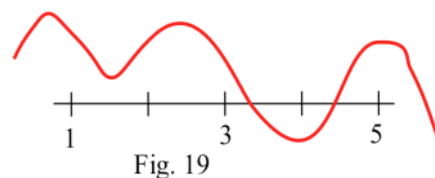4. The function in Fig. 18 has several roots.

    Which root do the iterates of Newton's method
    converge to if we start with $x_0 = 2$? $x_0 = 6$?

Fig. 18
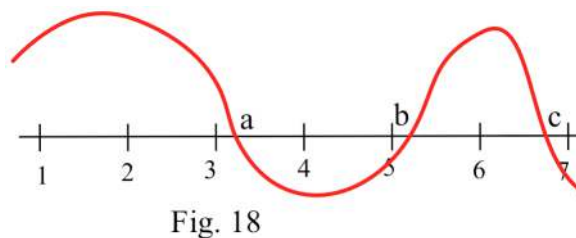
5. What happens to the iterates if we apply Newton's method
    to the function in Fig. 19 and start with $x_0 = 1$? $x_0 = 5$?

Fig. 19

6. What happens if we apply Newton's method to a function f and
    start with $x_0 = a$ root of f?

7. What happens if we apply Newton's method to a function f and start with $x_0 = a$ maximum of f?

In problems 8 and 9, a function and a value for x_0 are given. Apply Newton's method to find x_1 and x_2.

8. $f(x) = x^3 + x - 1$ and $x_0 = 1$ 9. $f(x) = x^4 - x^3 - 5$ and $x_0 = 2$

In problems 10 and 11, use Newton's method to find a root or solution, accurate to 2 decimal places, of the given functions using the given starting points.

10. $f(x) = x^3 - 7$ and $x_0 = 2$ 11. $f(x) = x - \cos(x)$ and $x_0 = 0.7$

In problems 12 – 15, use Newton's method to find **all** roots or solutions, accurate to 2 decimal places. It is helpful to examine a graph of the function to determine a "good" starting value x_0 .

12. $2 + x = e^x$ 13. $\dfrac{x}{x+3} = x^2 - 2$ 14. $x = \sin(x)$ 15. $x = \sqrt[5]{3}$

16.   Show that if Newton's method is applied to $f(x) = x^2 - A$ to approximate the square root of A, then

$x_{n+1} = \dfrac{1}{2}(x_n + \dfrac{A}{x_n})$ so the new estimate of the square root is the average of the previous estimate and

A divided by the previous estimate. This method of approximating square roots is called Heron's method.

17.   Use Newton's method to devise an algorithm for approximating the cube root of a number A.

18.   Use Newton's method to devise an algorithm for approximating the nth root of a number A.

Problems 19 – 22 involve chaotic behavior.

19.   The iterates of numbers using the Simple Chaotic Algorithm have a number of properties.

(a)   Verify that the iterates of $x = 0$ are all equal to 0.

(b)   Verify that if $x = 1/2, 1/4, 1/8$, and , in general, $1/2^n$, then the nth iterate of x is 0 (and so are all of the iterates beyond the nth iterate.)

20.   When Newton's method is applied to $f(x) = x^2 + 1$, most starting values for x_0 lead to chaotic x_n .

Find a value for x_0 so the iterates alternate: $x_1 = - x_0$ and $x_2 = -x_1 = x_0$.

21. $f(x) = \begin{cases} 2x & \text{if } 0 \le x < 1/2 \\ 2 - 2x & \text{if } 1/2 \le x \le 1 \end{cases}$ is called a "stretch and fold" function.

(a) Describe what f does to the points in the interval $[\,0, 1]$.

(b) Examine and describe the behavior of the iterates of $2/3, 2/5, 2/7,$ and $2/9$.

(c) Examine and describe the behavior of the iterates of $.10, .105,$ and $.11$.

(d) Do the iterates of f lead to chaotic behavior?

22. (This problem requires a computer or programmable calculator.)

    For each of the following functions start with x = .5 .

    (a) If f(x) = 2x(1–x), then what happens to the iterates of f after many iterations? ("many" = 50 is fine.)

    (b) If f(x) = 3.3x(1–x), then what happens to the iterates of f after many iterations?

    (c) If f(x) = 3.83x(1–x), then what happens to the iterates of f after many iterations?

    (d) What do you think happens to the iterates of f(x) = 3.7x(1–x)? What actually does happen.

    (e) Repeat parts (a)–(d) with some other starting values for x strictly between 0 and 1 (0<x<1). Does
        the starting value seem to effect the eventual behavior of the iterates?

    (The behavior of the iterates of f depends in a strange way on the numerical value of the leading

    coefficient. The behavior exhibited in part (d) is an example of "chaos".)

Section 3.8   **PRACTICE Answers**

Practice 1: $f(x) = x^3 + 3x + 1$ so $f'(x) = 3x^2 + 3$ and the slope of the tangent line at the point (1,3)

is $f'(1) = 6$. Using the point–slope form for the equation of a line, the equation of the tangent line is

$y - 3 = 6(x - 1)$ or $y = 6x - 3$.

The y–coordinate of a point on the x–axis is 0 so we need to put $y = 0$ and solve the linear equation

for x: $0 = 6x - 3$ so $x = 1/2$.

The line tangent to the graph of $f(x) = x^3 + 3x + 1$ at the point (1,3) intersects the x–axis at

the point **(1/2, 0)**.

Practice 2:   The approximate locations of x_1 and x_2 are shown in Fig. 20.
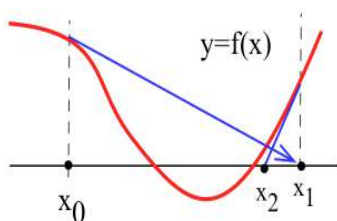

y=f(x)

x_0 x_2 x_1

Fig. 20

Practice 3: $f(x) = x^3 - 3x^2 + x - 1$ so $f'(x) = 3x^2 - 6x + 1$. $x_0 = 3$.

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{2}{10} = \textbf{2.8} .$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = 2.8 - \frac{f(2.8)}{f'(2.8)} = 2.8 - \frac{0.232}{7.72} \approx \textbf{2.769948187}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \approx 2.769292663 .$$

Practice 4:   Fig. 21 shows the first iteration of Newton's Method for $x_0 = 2, 3,$ and 5.

    If $x_0 = 2$, the iterates approach the root at a.

    If $x_0 = 3$, the iterates approach the root at c.

    If $x_0 = 5$, the iterates approach the root at a.
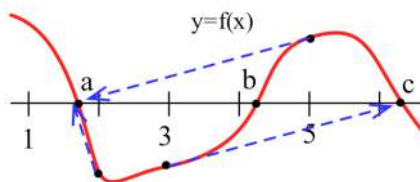

y=f(x)

a   b   c

1   3   5

Fig. 21

Practice 5: $f(x) = x^{1/3}$ so $f'(x) = \frac{1}{3} x^{-2/3}$.

If $x_0 = \mathbf{1}$, then $x_1 = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{1}{1/3} = 1 - 3 = \mathbf{-2}$,

$$x_2 = -2 - \frac{f(-2)}{f'(-2)} = -2 - \frac{(-2)^{1/3}}{\frac{1}{3}(-2)^{-2/3}} = -2 - \frac{-2}{1/3} = \mathbf{4} ,$$

$$x_3 = 4 - \frac{f(4)}{f'(4)} = 4 - \frac{(4)^{1/3}}{\frac{1}{3}(4)^{-2/3}} = 4 - \frac{4}{1/3} = \mathbf{-8} , \text{ and so on.}$$

If $x_0 = \mathbf{-3}$, then $x_1 = -3 - \frac{f(-3)}{f'(-3)} = -3 - \frac{(-3)^{1/3}}{\frac{1}{3}(-3)^{-2/3}} = -3 + 9 = \mathbf{6}$,

$$x_2 = 6 - \frac{f(6)}{f'(6)} = 6 - \frac{6^{1/3}}{\frac{1}{3}6^{-2/3}} = 6 - \frac{6}{1/3} = \mathbf{-12} .$$

The graph of the cube root $f(x) = x^{1/3}$ has a shape similar to Fig. 14, and the behavior of the iterates is similar to the pattern in that figure. Unless $x_0 = 0$ (the only root of f) the iterates alternate in sign and double in magnitude with each iteration: they get progressively farther from the root with each iteration.

Practice 6:   If $x_0 = 0.997$, then $x_1 \approx -0.003$, $x_2 \approx 166.4$, $x_3 \approx 83.2$, $x_4 \approx 41.6$.

If $x_0 = 1.02$, then $x_1 \approx 0.198$, $x_2 \approx -25.2376$, $x_3 \approx -12.6$, $x_4 \approx -6.26$.

3.9 Linear Approximation and Differentials

Newton's method used tangent lines to "point toward" a root of the function. In this section we examine
and use another geometric characteristic of tangent lines:

> If   f is differentiable at a and x is close to a,
>
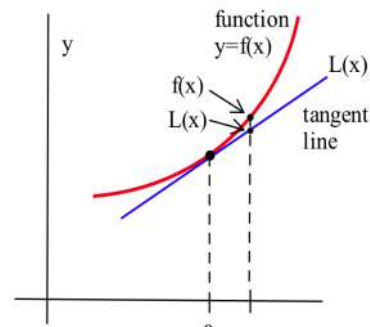> then   the tangent line L(x) is close to f(x). (Fig. 1)

Fig. 1

This idea is used to approximate the values of some commonly used functions
and to predict the "error" or uncertainty in a final calculation if we know the
"error" or uncertainty in our original data. Finally, we define and give some
examples of a related concept called the **differential** of a function.

Linear Approximation

Since this section uses tangent lines frequently, it is worthwhile to recall how we find the equation of the
line tangent to f at a point $x = a$. The line tangent to f at $x = a$ goes through the point $(a, f(a))$ and has
slope $f'(a)$, so, using the point–slope form $y - y_0 = m(x - x_0)$ for linear equations, we have

$$y - f(a) = f'(a) \cdot (x - a) \quad \text{and} \quad y = f(a) + f'(a) \cdot (x - a) .$$

This final result is the equation of the line tangent to f at $x = a$.

> If   f is differentiable at $x = a$,
>
> then   the equation of the line L tangent to f at $x = a$ is
>
> $$\mathbf{L(x) = f(a) + f'(a) \cdot (x - a)} .$$

Example 1:   Find the equation of the line L(x) which is tangent to the graph of $f(x) = \sqrt{x}$ at the point
$(9,3)$. Evaluate L(9.1) and L(8.88) to approximate $\sqrt{9.1}$ and $\sqrt{8.88}$.

Solution: $f(x) = \sqrt{x} = x^{1/2}$ and $f'(x) = \frac{1}{2} x^{-1/2} = \frac{1}{2\sqrt{x}}$ so $f(9) = 3$

and $f'(9) = \frac{1}{2\sqrt{9}} = \frac{1}{6}$. Then

$L(x) = f(9) + f'(9) \cdot (x - 9) = 3 + \frac{1}{6}(x - 9)$. If x is close to 9, then the

value of L(x) is a good approximation of the value of $\sqrt{x}$ (Fig. 2) .

The number 9.1 is close to 9 so

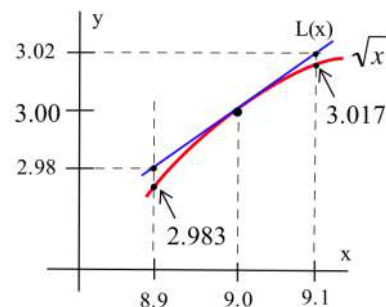$\sqrt{9.1} = f(9.1) \approx L(9.1) = 3 + \frac{1}{6}(9.1 - 9) = 3.016666.$

Fig. 2

Similarly,

$\sqrt{8.88}$ $= f(8.88) \approx L(8.88) = 3 + \dfrac{1}{6}(8.88 - 9)$ $= 2.98$. In fact, $\sqrt{9.1} \approx 3.016621$, so our estimate,

using L(9.1), is within 0.000045 of the exact answer. $\sqrt{8.88} \approx 2.979933$ (accurate to 6 decimal

places) and our estimate is within 0.00007 of the exact answer.

In each example, we got a good estimate of a square root with very little work. The graph indicates
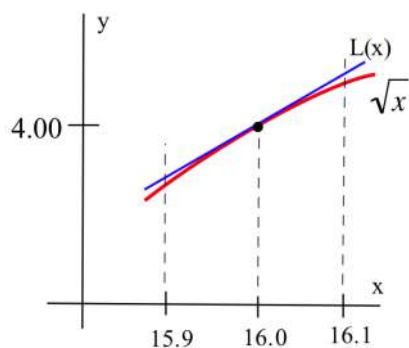the tangent line L(x) is slightly above f(x), and each estimate is slightly larger than the exact value.

Fig. 3

Practice 1:   Find the equation of the line L(x) tangent to the graph of
$f(x) = \sqrt{x}$ at the point (16,4) (Fig. 3). Evaluate L(16.1) and
L(15.92) to approximate $\sqrt{16.1}$ and $\sqrt{15.92}$. Are your
approximations using L larger or smaller than the exact values of
the square roots?

Practice 2:   Find the equation of the line L(x) tangent to the graph of
$f(x) = x^3$ at the point (1,1) and use L(x) to approximate
$(1.02)^3$ and $(0.97)^3$. Do you think your approximations
using L are larger or smaller than the exact values?

The process we have used to approximate square roots and cubics can be used to approximate any
differentiable function, and the main result about the linear approximation follows from the two statements
in the boxes. Putting these two statements together, we have the process for Linear Approximation.

Linear Approximation Process: (Fig. 4)

If   f is differentiable at a and **x is close to a,**

then   (geometrically)   the graph of the tangent line L(x) is close to the

graph of f(x), and

(algebraically)   the values of the tangent line function

L(x) = f(a) + f '(a)·(x – a) approximate the values of f(x):

f(x) ≈ L(x) = f(a) + f '(a)·(x – a) .

Sometimes we replace "x – a" with "Δx" in the last equation, and
the statement becomes f(x) ≈ f(a) + f '(a)·Δx.

Fig. 4

Example 2:   Use the linear approximation process to
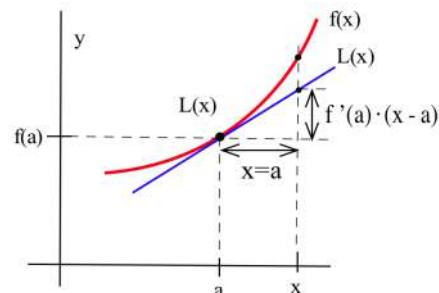approximate the value of $e^{0.1}$.

Solution: $f(x) = e^x$ so $f'(x) = e^x$. We need to pick a value **a** near $x = 0.1$ for which we know the

exact value of $f(\mathbf{a})$ and $f'(\mathbf{a})$, and $\mathbf{a} = 0$ is the obvious choice. Then

$$e^{0.1} = f(0.1) \approx L(0.1) \quad = f(0) + f'(0){\cdot}(0.1 - 0)$$

$$= e^0 \; + \; e^0{\cdot}(0.1) = 1 + 1{\cdot}(0.1) \; = \; 1.1 \; .$$

This approximation is within 0.0052 of the exact value of $e^{0.1}$.

Practice 3:   Approximate the value of $(1.06)^4$, the amount $1 becomes after 4 years in a bank which

pays 6% interest compounded annually. (Take $f(x) = x^4$ and $a = 1$.)

Practice 4:   Use the linear approximation process and the values in the table to estimate the value

of f when $x = 1.1, 1.23$ and 1.38 .

x	f(x)	f'(x)
1	0.7854	0.5
1.208761	0.4098	
1.4	0.9505	0.3378

We can also approximate **functions** as well as numbers.

Example 3:   Find a linear approximation formula $L(x)$ for $\sqrt{1 + x}$ when x is small. Use your result

to approximate $\sqrt{1.1}$ and $\sqrt{.96}$.

Solution: $f(x) = \sqrt{1 + x} = (1 + x)^{1/2}$ so $f'(x) = \frac{1}{2}(1 + x)^{-1/2} = \frac{1}{2\sqrt{1+x}}$. Since "x is small" , we know

that x is close to 0 , and we can pick $a = 0$. Then $f(a) = f(0) = 1$ and $f'(a) = f'(0) = \frac{1}{2}$ so

$$\sqrt{1 + x} \approx L(x) = f(0) + f'(0){\cdot}(x - 0) = 1 + \frac{1}{2}x = 1 + \frac{x}{2} \; .$$

If x is small, then $\sqrt{1 + x} \approx 1 + \frac{x}{2}$. $\sqrt{1.1} = \sqrt{1 + 0.1} \approx 1 + \frac{0.1}{2} = 1.05$ and

$\sqrt{0.96} = \sqrt{1 + (-.04)} \approx 1 + \frac{-.04}{2} = 0.98$. Use your calculator to determine by how much each

estimate differs from the true value.

Applications of Linear Approximation to Measurement "Error"

Most scientific experiments involve using instruments to take measurements, but the instruments are not
perfect, and the measurements we get from them are only accurate up to a certain level. If we know the
level of accuracy of our instruments and measurements, we can use the idea of linear approximation to
estimate the level of accuracy of results we calculate from our measurements.

If we measure the side x of a square to be 8 inches, then, of course, we would calculate its area to be
$A(x) = x^2 = 64$ square inches. Suppose, as is reasonable in a real measurement, that our measuring instrument
could only measure or be read to the nearest 0.05 inches. Then our measurement of 8 inches would really

mean some number between $8-0.05 = 7.95$ inches and $8+0.05 = 8.05$ inches, and the true area of the square

would be between $A(7.95) = 63.2025$ and $A(8.05) = 64.8025$ square inches. Our possible "error" or

"uncertainty", because of the limitations of the instrument, could be as much as $64.8025 - 64 = 0.8025$ square

inches so we could report the area of the square to be 64 ± 0.8025 square inches. We can also use the linear

approximation method to estimate the "error" or uncertainty of the area. (For a function as simple as the area

of a square, this linear approximation method really isn't needed, but it is used to illustrate the idea.)

For a square with side x, the area is $A(x) = x^2$ and $A'(x) = 2x$. If Δx represents the "error" or uncertainty

of our measurement of the side, then, using the linear approximation technique for $A(x)$,

$A(x) \approx A(a) + A'(a) \cdot \Delta x$ so the uncertainty of our calculated area is $A(x) - A(a) \approx A'(x) \cdot \Delta x$. In this example,

$a = 8$ inches and $\Delta x = 0.05$ inches so

$$A(8.05) \approx A(8) + A'(8) \cdot (0.05) = 64 + 2(8) \cdot (0.05) = 64.8 \text{ square inches,}$$

and the uncertainty in our calculated area is approximately

$$A(8 + 0.05) - A(8) \approx A'(8) \cdot \Delta x = 2(8 \text{ inches})(0.05 \text{ inches}) = 0.8 \text{ square inches.}$$

This process can be summarized as:

> **Linear Approximation Error:**
>
> If   the value of the x–variable is measured to be $x = a$ with an "error" of Δx units,
>
> then Δf, the "error" in estimating $f(x)$, is $\mathbf{\Delta f = f(x) - f(a) \approx f'(a) \cdot \Delta x}$.

Practice 5:   If we measure the side of a cube to be 4 cm with an uncertainty of 0.1 cm, what is the

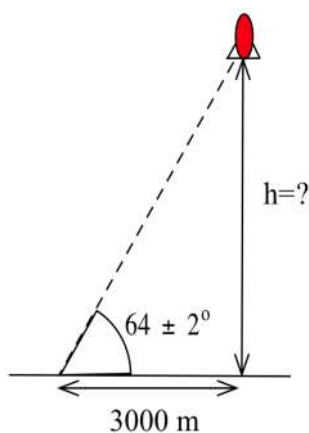volume of the cube and the uncertainty of our calculation of the volume? (Use linear approximation.)

h=?

64 ± 2°

3000 m

Fig. 5

Example 4:   We are using a tracking telescope to follow a small rocket. Suppose

we are 3000 meters from the launch point of the rocket, and, 2 seconds after

the launch, we measure the angle of the inclination of the rocket to be 64°

with a possible "error" of 2° (Fig. 5). How high is the rocket and what is the

possible error in this calculated height?

Solution: Our measured angle is $x = 1.1170$ radians and $\Delta x = 0.0349$ radians (all of our

trigonometric work is in radians), and the height of the rocket at an angle x is

$f(x) = 3000 \cdot \tan(x)$ so $f(1.1170) \approx 6151$ m. Our uncertainty in the height is

$$\Delta f(x) \approx f'(x) \cdot \Delta x \approx 3000 \cdot \sec^2(x) \cdot \Delta x = 3000 \sec^2(1.1170) \cdot (0.0349) = 545 \text{ m.}$$

If our measured angle of 64° can be in error by as much as 2°, then our calculated height of 6151 m

can be in error by as much as 545 m. The height is 6151 ± 545 meters.

Practice 6:   Suppose we measured the angle of inclination of in the previous Example to be $43^{o} \pm 1^{o}$.

Estimate the height of the rocket in the form   "height $\pm$ error ."

In some scientific and engineering applications, the calculated **result** must be within some given specification. You might need to determine how accurate the initial measurements must be in order to guarantee the final calculation is within the specification. Added precision usually costs time and money, so it is important to choose a measuring instrument which is good enough for the job but not too good or too expensive.

Example 5:   Your company produces ball bearings (spheres) with a volume of 10 cm^3, and the volume

must be accurate to within 0.1 cm^3. What radius should the bearings have and what error can you

tolerate in the radius measurement to meet the accuracy specification for the volume? $(V = \frac{4}{3} \pi r^3)$

Solution: Since we want $V = 10$, we can solve $10 = \frac{4}{3} \pi r^3$ for r to get $r = 1.3365$ cm.

$V(r) = \frac{4}{3} \pi r^3$ and $V '(r) = 4\pi r^2$ so $\Delta V \approx V '(r) \cdot \Delta r$. In this case we have been given that

$\Delta V = 0.1 \text{ cm}^3$, and we have calculated $r = 1.3365$ cm so $0.1 \text{ cm}^3 = V '(1.3365$

$\text{cm}) \cdot \Delta r = (22.45 \text{ cm}^2) \cdot \Delta r$.

Solving for Δr, we get $\Delta r \approx 0.0045$ cm. To meet the specifications for the

allowable error in the volume, we must allow no more than 0.0045 cm

variation in the radius. If we measure the diameter of the sphere rather than the

radius, then we want $d = 2r = 2(1.3365 \pm 0.0045) = 2.673 \pm 0.009$ cm .
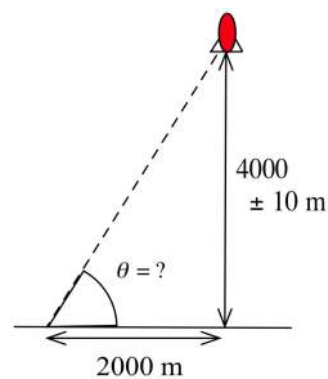
4000
± 10 m

$\theta = ?$

2000 m

Fig. 6

Practice 7:   You want to determine the height of the rocket to within 10 meters

when it is 4000 meters high (Fig. 6). How accurate must your angle

measurement be? (Do your calculations in radians.)

Relative Error and Percentage Error

The "error" we have been examining is called the **absolute error** to distinguish it from two other commonly used terms, the **relative error** and the **percentage error** which compare the absolute error with the magnitude of the number being measured. An "error" of 6 inches in measuring the circumference of the earth would be extremely small, but a 6 inch error in measuring your head for a hat would result in a very bad fit.

Definitions: The **Relative Error** of f is $\dfrac{\text{error of f}}{\text{value of f}} \;\; = \;\; \dfrac{\Delta f}{f}$

The **Percentage Error** of f is $\dfrac{\Delta f}{f} \cdot (100)$.

Example 6:   If the relative error in the calculation of the area of a circle must be less than 0.4 ,   then
what relative error can we tolerate in the measurement of the radius?

Solution: $A(r) = \pi r^2$ so $A'(r) = 2\pi r$ and $\Delta A \approx A'(r)\,\Delta r = 2\pi r\,\Delta r$. The Relative Error of A is

$$\frac{\Delta A}{A} \approx \frac{2\pi r\,\Delta r}{\pi r^2} = 2\frac{\Delta r}{r}\ .$$ We can guarantee that the Relative Error of A , $\frac{\Delta A}{A}$, is less than 0.4

if the Relative Error of r , $\frac{\Delta r}{r} = \frac{1}{2}\frac{\Delta A}{A}$, is less than $\frac{1}{2}(0.4) = 0.2$.

Practice 8:   If you can measure the side of a cube with a percentage error less than 3%, then what will
the percentage error for your calculation of the surface area of the cube be?

The Differential of f

In Fig. 7, the change in value of the function f near the point
$(x, f(x))$ is $\Delta f = f(x + \Delta x) - f(x)$ and the change along the
tangent line is $f'(x)\cdot\Delta x$. If Δx is small, then we have used the
approximation that $\Delta f \approx f'(x)\cdot\Delta x$. This leads to the definition
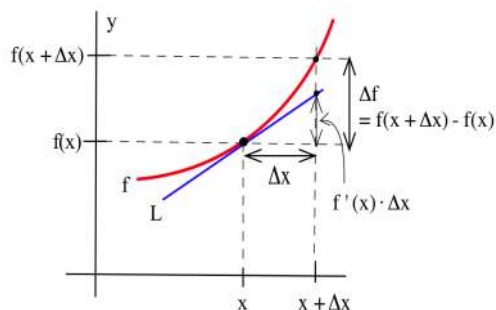of a new quantity, **df**, called the differential of f.

Fig. 7

> **Definition:**   The **differential of f** is   **df ▪ f '(x)·dx** where **dx** is any real number.

The differential of f represents the change in f , as x changes from x to x + dx, along the tangent line to
the graph of f at the point $(x, f(x))$. If we take dx to be the number Δx, then the differential is an
approximation of Δf: $\Delta f \approx f'(x)\cdot\Delta x = f'(x)\cdot dx = df$.

Example 7:   Determine the differential df of each of $f(x) = x^3 - 7x$, $g(x) = \sin(x)$, and $h(r) = \pi r^2$.
Solution: $df = f'(x)\cdot dx = (3x^2 - 7)\,dx$, $dg = g'(x)\cdot dx = \cos(x)\,dx$, and $dh = h'(r)\,dr = 2\pi r\,dr$.

Practice 9:   Determine the differentials of $f(x) = \ln(x)$, $u = \sqrt{1 - 3x}$, and $r = 3\cos(\theta)$.

We will do little with differentials for a while, but are used extensively in integral calculus.

The Linear Approximation "Error" | f(x) – L(x) |

An approximation is most valuable if we also have have some measure of the size of the "error", the distance between the approximate value and the value being approximated,. Typically, we will not know the exact value of the error (why not?), but it is useful to know that the error must be less than some number. For example, if one scale gives the weight of a gold pendant as 10.64 grams with an error less than .3 grams (10.64 ± .3 grams) and another scale gives the weight of the same pendant as 10.53 grams with an error less than .02 grams (10.53 ± .02 grams), then we can have more faith in the second approximate weight because of the smaller "error" guarantee. Before finding a guarantee on the size of the error of the linear approximation process, we will check how well the linear approximation process approximates some functions we can compute exactly. Then we will prove one bound on the possible error and state a somewhat stronger bound.

Example 8:   Let $f(x) = x^2$. Evaluate $f(2+\Delta x), L(2+\Delta x)$ and $| f(2+\Delta x) - L(2+\Delta x) |$ for

$\Delta x = 0.1, 0.05, 0.01, 0.001$ and for a general value of Δx .

Solution: $f(2+\Delta x) = (2+\Delta x)^2 = 2^2 + 4\Delta x + (\Delta x)^2$ and $L(2+\Delta x) = f(2) + f'(2) \cdot \Delta x = 2^2 + 4 \cdot \Delta x$. Then

Δx	$f(2+\Delta x)$	$L(2+\Delta x)$	$| f(2+\Delta x) - L(2+\Delta x) |$
0.1	4.41	4.4	0.01
0.05	4.2025	4.2	0.0025
0.01	4.0401	4.04	0.0001
0.001	4.004001	4.004	0.000001

Cutting the value of Δx in half makes the error 1/4 as large. Cutting Δx to 1/10 as large makes the error 1/100 as large. In general, $| f(2+\Delta x) - L(2+\Delta x) | = | (2^2 + 4 \cdot \Delta x + (\Delta x)^2) - (2^2 + 4 \cdot \Delta x) | = (\Delta x)^2$.
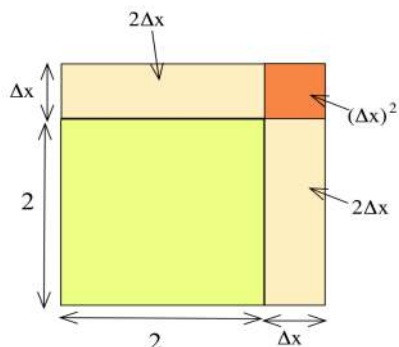

Fig. 8

This function and error also have a nice geometric interpretation (Fig. 8): $f(x) = x^2$ is the area of a square of side x so $f(2+\Delta x)$ is the area of a square of side $2+\Delta x$, and that area is the sum of the pieces with areas 2^2 , $2 \cdot \Delta x, 2 \cdot \Delta x$, and $(\Delta x)^2$. The linear approximation $L(2+\Delta x) = 2^2 + 4 \cdot \Delta x$ to the area of the square includes the 3 largest pieces 2^2 , $2 \cdot \Delta x$ and $2 \cdot \Delta x$, but it omits the small square with area $(\Delta x)^2$ so the approximation is in error by the amount $(\Delta x)^2$.

Practice 10:   Let $f(x) = x^3$. Evaluate $f(4+\Delta x), L(4+\Delta x)$ and

$| f(4+\Delta x) - L(4+\Delta x) |$ for $\Delta x = 0.1, 0.05, 0.01, 0.001$ and for a

general value of Δx. Use Fig. 9 to give a geometric interpretation of

$f(4+\Delta x), L(4+\Delta x)$ and $| f(4+\Delta x) - L(4+\Delta x) |$.

In both the example and practice problem, the error $| f(a+\Delta x) - L(a+\Delta x) |$

turned out to be very small, proportional to $(\Delta x)^2$, when Δx was small.

In general, the error approaches 0 as Δx approaches 0.
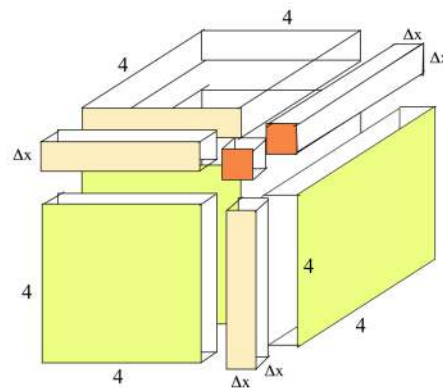
Fig. 9

Theorem :   If $f(x)$ is differentiable at a and $L(a+\Delta x) = f(a) + f'(a) \cdot \Delta x$,

then $\displaystyle\lim_{\Delta x \to 0} | f(a+\Delta x) - L(a+\Delta x) | = 0$ and

$$\lim_{\Delta x \to 0} \frac{| f(a+\Delta x) - L(a+\Delta x) |}{\Delta x} = 0$$

Proof: $| f(a+\Delta x) - L(a+\Delta x) | = | f(a+\Delta x) - f(a) - f'(a) \cdot \Delta x | = \left\{ \dfrac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a) \right\} \cdot \Delta x$. But f is

differentiable at $x=a$ so $\displaystyle\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} = f'(a)$ and $\displaystyle\lim_{\Delta x \to 0} \left\{ \frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a) \right\} = 0$.

Then $\displaystyle\lim_{\Delta x \to 0} | f(a+\Delta x) - L(a+\Delta x) | = \lim_{\Delta x \to 0} \left\{ \frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a) \right\} \cdot \lim_{\Delta x \to 0} \Delta x = 0 \cdot 0 = 0$

Not only does the difference $f(a+\Delta x) - L(a+\Delta x)$ approach 0, but this difference approaches 0 so fast

that we can divide it by Δx, another quantity approaching 0, and the quotient still approaches 0.

In the next chapter we can prove that the error of the linear approximation process is

proportional to $(\Delta x)^2$. For now we just state the result.

Theorem: If f is differentiable at a and $| f''(x) | \leq M$ for all x between a and $a+\Delta x$

then $| \text{"error"} | = | f(a+\Delta x) - L(a+\Delta x) | \leq \dfrac{1}{2} M \cdot (\Delta x)^2$.

Problems

Fig. 10

1.   Fig. 10 shows the tangent line to a function g at the point (2,2) and
     a line segment Δx units long. On the figure, label the locations of
     (a) $2 + \Delta x$ on the x–axis, (b) the point $(2 + \Delta x,\ g(2 + \Delta x))$, and
     (c) the point $(2 + \Delta x, g(2) + g\ '(2){\cdot}\Delta x)$.
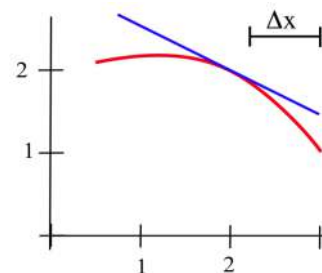   (d) How large is the "error", $(g(2) + g\ '(2){\cdot}\Delta x) - (g(2+\Delta x))$?

y=f(x)
Fig. 11

2.   In Fig. 11 is the linear approximation $L(a + \Delta x)$ larger or smaller than the
     value of $f(a + \Delta x)$ when
     (a) $a = 1$ and $\Delta x = 0.2$? (b) $a = 2$ and $\Delta x = -0.1$? (c) $a = 3$ and $\Delta x = 0.1$?
     (d) $a = 4$ and $\Delta x = 0.2$? and (e) $a = 4$ and $\Delta x = -0.2$?
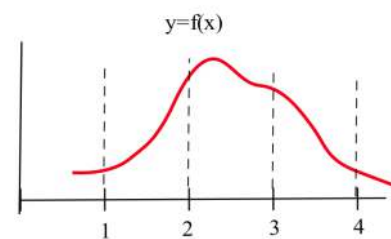
In problems 3 and 4, find the equation of the tangent line L to the given function   f at the given point
$(a, f(a))$. Use the value $L(a + \Delta x)$ to approximate the value of $f(a + \Delta x)$.

3.   (a) $f(x) = \sqrt{x}$, $a = 4, \Delta x = 0.2$ (b) $f(x) = \sqrt{x}$, $a = 81, \Delta x = -1$ (c) $f(x) = \sin(x)$, $a = 0, \Delta x = 0.3$

4.   (a) $f(x) = \ln(x)$, $a = 1,\ \Delta x = 0.3$ (b) $f(x) = e^{x}$, $a = 0, \Delta x = 0.1$ (c) $f(x) = x^{5}$, $a = 1, \Delta x = 0.03$

5.   Show that $(1 + x)^{n} \approx 1 + nx$ if x is "close to" 0. (Suggestion: Put $f(x) = (1+x)^{n}$, $a = 0$, and $\Delta x = x$.)

In problems 6 and 7, use the Linear Approximation Process to derive each approximation formula
for x **"close to" 0**.

6.   (a) $(1 - x)^{n} \approx 1 - nx$ (b) $\sin(x) \approx x$ (c) $e^{x} \approx 1 + x$

7.   (a) $\ln(1 + x) \approx x$ (b) $\cos(x) \approx 1$ (c) $\tan(x) \approx x$ (d) $\sin(\pi/2 + x) \approx 1$
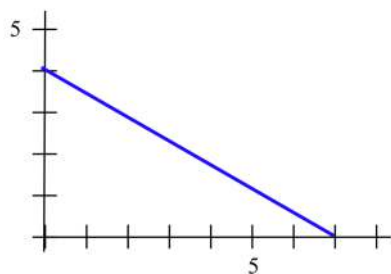

Fig. 12

8.   The height of a triangle is exactly 4 inches, and the base is
     measured to be 7 ± 0.5 inches (Fig. 12). Shade a part of the
     figure which represents the "error" in the calculation of the
     area of the triangle.

9. A rectangle has one side on the x–axis, one side on the y–axis, and a corner
    on the graph of $y = x^2 + 1$ (Fig. 13).
    (a) Use Linear Approximation of the area formula to estimate the increase in
         the area of the rectangle if the base grows from 2 to 2.3 inches.
    (b) Calculate exactly the increase in the area of the rectangle as the
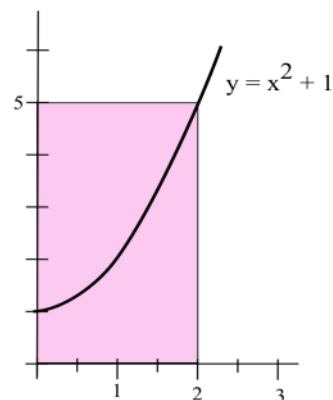   base grows from 2 to 2.3 inches.

10. You can measure the diameter of a circle to within 0.3 cm.
    (a) How large is the "error" in the calculated area of a circle with a measured
         diameter of 7.4 cm?
    (b) How large is the "error" in the calculated area of a circle with a measured diameter of 13.6 cm?
    (c) How large is the percentage error in the calculated area of a circle with a measured diameter of d?

11. You are minting gold coins which must have a volume of 47.3 ± 0.1 cm^3. If you can manufacture
    the coins to be exactly 2 cm high, how much variation can you allow for the radius?

12. If F is the fraction of carbon–14 remaining in a plant sample Y years after it died, then $Y = \frac{5700}{\ln(0.5)} \cdot \ln(F)$.
    (a) Estimate the age of a plant sample in which 83±2 % (0.83 ± 0.02) of the carbon–14 remains.
    (b) Estimate the age of a plant sample in which 13±2 % (0.13 ± 0.02) of the carbon–14 remains.

13. Your company is making dice (cubes) and the specifications require that their volume be 87 ± 2 cm^3.
    How long should each side be and how much variance can a side have in order to meet the
    specifications?

14. If the specifications require a cube with a surface area of 43 ± 0.2 cm^2, how long should each side be
    and how much variance can a side have in order to meet the specifications?

15. The period P, in seconds, for a pendulum to make one complete swing and return to the release point is
    $P = 2\pi \sqrt{L/g}$ where L is the length of the pendulum in feet and g is 32 feet/sec^2.
    (a) If L = 2 feet, what is the period of the pendulum?
    (b) If P = 1 second, how long is the pendulum?
    (c) Estimate the change in P if L increases from 2 feet to 2.1 feet.
    (d) The length of a 24 foot pendulum is increasing 2 inches per hour. Is the period getting longer or
         shorter? How fast is the period changing?

Fig. 13

16. A ball thrown at an angle θ with an initial velocity v will land $\dfrac{v^2}{g}$ $\cdot\sin(2\theta)$ feet from the thrower.

(a) How far away will the ball land if $\theta = \pi/4$ and v = 80 feet/second?

(b) Which will result in a greater change in the distance: a 5% error in the angle θ or a 5% error in the initial velocity v?

17. For the function in Fig. 14, estimate the value of **df** when

(a) x = 2 and dx = 1   (b) x = 4 and dx = –1

(c) x = 3 and dx = 2
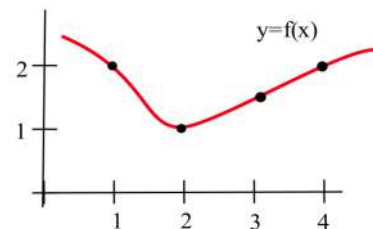
Fig. 14

18. For the function in Fig. 15, estimate the value of **df** when

(a) x = 1 and dx = 2   (b) x = 2 and dx = –1
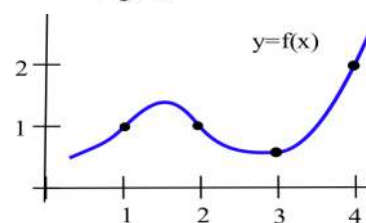
(c) x = 3 and dx = 1

19. Calculate the differentials **df** of the following functions:

(a) $f(x) = x^2 - 3x$ (b) $f(x) = e^x$

(c) $f(x) = \sin(5x)$ (d) $f(x) = x^3 + 2x$ with x = 1 and dx = 0.2

(e) $f(x) = \ln(x)$ with x = e and dx = – 0.1   (f) $f(x) = \sqrt{2x + 5}$ with x = 22 and dx = 3 .

Fig. 15

Section 3.9   **PRACTICE Answers**

Practice 1: $f(x) = x^{1/2}$ so $f'(x) = \dfrac{1}{2\sqrt{x}}$. At the point (16,4) on the graph of f, the slope of the

tangent line is $f'(16) = \dfrac{1}{2\sqrt{16}} = \dfrac{1}{8}$. The equation of the tangent line is

$y - 4 = \dfrac{1}{8}(x - 16)$ or $y = \dfrac{1}{8}x + 2$: $L(x) = \dfrac{1}{8}x + 2$. Then

$\sqrt{16.1} \approx L(16.1) = \dfrac{1}{8}(16.1) + 2 = \mathbf{4.0125}$ and $\sqrt{15.92} \approx L(15.92) = \dfrac{1}{8}(15.92) + 2 = \mathbf{3.99}$

Practice 2: $f(x) = x^3$ so $f'(x) = 3x^2$. At (1,1), the slope of the tangent line is $f'(1) = 3$. The

equation of the tangent line is $y - 1 = 3(x - 1)$ or $y = 3x - 2$: $L(x) = 3x - 2$. Then

$(1.02)^3 \approx L(1.02) = 3(1.02) - 2 = \mathbf{1.06}$ and $(0.97)^3 \approx L(0.97) = 3(0.97) - 2 = \mathbf{0.91}$.

Practice 3: $f(x) = x^4$ so $f'(x) = 4x^3$. a = 1 and $\Delta x = 0.06$.

$(1.06)^4 = f(1.06) \approx L(1.06) = f(1) + f'(1)\cdot(0.06) = 1^4 + 4(1)^3\cdot(0.06) = \mathbf{1.24}$.

Practice 4: $f(1.1) \approx f(1) + f'(1)\cdot(0.1) = 0.7854 + (0.5)\cdot(0.1) = \mathbf{0.8354}$.

$f(1.23) \approx f(1.2) + f'(1.2)\cdot(0.03) = 0.8761 + (0.4098)\cdot(0.03) = \mathbf{0.888394}$.

$f(1.38) \approx f(1.4) + f'(1.4)\cdot(–0.02) = 0.9505 + (0.3378)\cdot(–0.02) = \mathbf{0.943744}$.

Practice 5: $x = 4$ cm and $\Delta x = 0.1$ cm. $f(x) = x^3$ so $f'(x) = 3x^2$ and $f(4) = 4^3 = 64$ cm^3 . Then

"error" $\Delta f \approx f'(x)\Delta x = 3x^2 \cdot \Delta x$. When $x = 4$ and $\Delta x = 0.1$, "error" $\Delta f \approx 3(4)^2(0.1) = $ **4.8 cm^3** .

Practice 6: $43 \pm 1^{\text{o}}$ is 0.75049 ± 0.01745 radians. $f(x) = 3000 \cdot \tan(x)$ so

$f(0.75049) = 3000 \cdot \tan(0.75049) \approx$ **2797.5 m**. $f'(x) = 3000 \sec^2(x)$ so

$\Delta f(x) \approx f'(x) \cdot \Delta x = 3000 \cdot \sec^2(x) \cdot \Delta x = 3000 \cdot \sec^2(0.75049) \cdot (0.01745) = $ **97.9 m** .

The height of the rocket is **2797.5 ± 97.9 m** .

Practice 7: $f(\theta) = 2000 \cdot \tan(\theta)$ so $f'(\theta) = 2000 \cdot \sec^2(\theta)$. We know $4000 = 2000 \cdot \tan(\theta)$ so

$\tan(\theta) = 2$ and $\theta \approx 1.10715$ (radians). $f'(\theta) = 2000 \cdot \sec^2(\theta)$ so $f'(1.10715) = 2000 \cdot \sec^2(1.10715) \approx 10,000$.

Finally, "error" $\Delta f \approx f'(\theta) \cdot \Delta\theta$ so $10 \approx 10,000 \cdot \Delta\theta$ and $\Delta\theta \approx 10/10,000 = $ **0.001** (radians) $\approx 0.057^{\text{o}}$.

Practice 8: $A(r) = 6r^2$ so $A'(r) = 12r$ and $\Delta A \approx A'(r) \cdot \Delta r = 12r \cdot \Delta r$. We are also told that $\Delta r/r < 0.03$.

Percentage error is $\dfrac{\Delta A}{A} \cdot 100 = \dfrac{12r \cdot \Delta r}{6r^2} \cdot 100 = \dfrac{2 \cdot \Delta r}{r} \cdot 100 < 200 \cdot (0.03) = $ **6** .

Practice 9: $f(x) = \ln(x)$ $df = f'(x) \cdot dx = \dfrac{1}{x}\ \mathbf{dx}$

$u = \sqrt{1 - 3x}$ $du = \dfrac{du}{dx} \cdot dx = \dfrac{-3}{2\sqrt{1 - 3x}}\ \cdot \mathbf{dx}$

$r = 3\cos(\theta)\, dr = \dfrac{dr}{d\theta} \cdot d\theta = \mathbf{-3\,\sin(\theta)\ \cdot d\theta}$

Practice 10: $f(x) = x^3$, $f'(x) = 3x^2$, and $L(4 + \Delta x) = f(4) + f'(4)\Delta x = 4^3 + 3(4)^2\Delta x = 64 + 48 \cdot \Delta x$.

Δx	$f(4 + \Delta x)$	$L(4 + \Delta x)$	$\mid f(4+\Delta x) - L(4+\Delta x) \mid$
0.1	68.921	68.8	0.121
0.05	66.430125	66.4	0.030125
0.01	64.481201	64.48	0.001201
0.001	64.048012	64.048	0.000012

$f(4+\Delta x)$ is the actual volume of the cube with side length $4+\Delta x$.

$L(4+\Delta x)$ is the volume of the cube with side length 4 (v = 64) plus the volume of

the 3 "slabs" (v = $3 \cdot 4^2 \cdot \Delta x$)

$\mid f(4+\Delta x) - L(4+\Delta x) \mid$ is the volume of the "leftover" pieces from L: the 3 "rods" (v = $3 \cdot 4 \cdot (\Delta x)^2$)

and the tiny cube (v = $(\Delta x)^3$).

3.10   IMPLICIT and LOGARITHMIC DIFFERENTIATION

This short section presents two final differentiation techniques. These two techniques are more specialized than the ones we have already seen and they are used on a smaller class of functions. For some functions, however, one of these may be the only method that works. The idea of each method is straightforward, but actually using each of them requires that you proceed carefully and practice.

Implicit Differentiation

In our work up until now, the functions we needed to differentiate were either given **explicitly,** such as $y = f(x) = x^2 + \sin(x)$, or it was possible to get an explicit formula for them, such as solving $y^3 - 3x^2 = 5$ to get $y = \sqrt[3]{5 + 3x^2}$. Sometimes, however, we will have an equation relating x and y which is either difficult or impossible to solve explicitly for y , such as $y^2 + 2y = \sin(x) + 4$ or $y + \sin(y) = x^3 - x$. In any case, we can still find $y ' = f '(x)$ by using implicit differentiation.

The key idea behind implicit differentiation is to **assume that y is a function of x** even if we cannot explicitly solve for y. This assumption does not require any work, but we need to be very careful to treat y as a function when we differentiate and to use the Chain Rule or the Power Rule for Functions.

Example 1:   Assume that y is a function of x .

   Calculate (a) $\mathbf{D}(y^3)$, (b) $\dfrac{\mathbf{d}}{\mathbf{dx}}(x^3y^2)$, and   (c) $(\sin(y))'$

Solution: (a) We need the Power Rule for Functions since **y is a function of x**:

   $\mathbf{D}(y^3) = 3\,y^2 \cdot \mathbf{D}(y) = 3\,y^2 \mathbf{y}\,'$.

   (b) We need to use the product rule and the Chain Rule:

   $\dfrac{\mathbf{d}}{\mathbf{dx}}(x^3y^2) = x^3\,\dfrac{\mathbf{d}}{\mathbf{dx}}(y^2) + y^2\,\dfrac{\mathbf{d}}{\mathbf{dx}}(x^3) = x^3\,\mathbf{2y}\dfrac{\mathbf{dy}}{\mathbf{dx}} + y^2 \cdot \mathbf{3x^2} = 2x^3y\dfrac{\mathbf{dy}}{\mathbf{dx}} + 3x^2y^2$.

   (c) We just need to know that $\mathbf{D}(\sin(x)) = \cos(x)$ and then use the Chain Rule:

   $(\sin(y))' = \cos(y) \cdot \mathbf{y}\,'$.

Practice 1: Assume that y is a function of x. Calculate (a) $\mathbf{D}(x^2 + y^2)$ and (b) $\dfrac{\mathbf{d}}{\mathbf{dx}}(\sin(2 + 3y))$.

> **IMPLICIT DIFFERENTIATION:**
>
>   To determine   **y** ' , differentiate each side of the defining equation, **treating y**
>
>   **as a function of x** , and then algebraically solve for   y '.

Example 2:   Find the slope of the tangent line to the circle $x^2 + y^2 = 25$ at the point (3,4) with and without implicit differentiation.

Solution:

Explicitly: We can solve the equation of the circle for $y = +\sqrt{25 - x^2}$ or $y = -\sqrt{25 - x^2}$.

Since the point (3,4) is on the top half of the circle (Fig. 1), $y = +\sqrt{25 - x^2}$ and

$$\mathbf{D}(y) = \mathbf{D}(+\sqrt{25 - x^2}) = \tfrac{1}{2}(25 - x^2)^{-1/2}\,\mathbf{D}(25 - x^2) = \frac{-x}{\sqrt{25 - x^2}}$$

Replacing x with 3, we have $\mathbf{y'} = \dfrac{-3}{\sqrt{25 - 3^2}} = -3/4$.
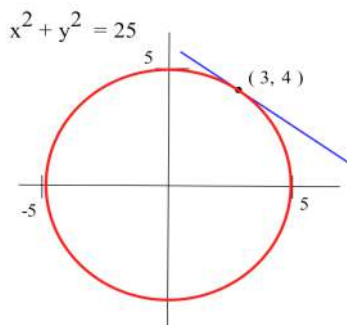
Implicitly: We differentiate each side of the equation $x^2 + y^2 = 25$ and then solve for $\mathbf{y'}$. $\mathbf{D}(x^2 + y^2) = \mathbf{D}(25)$ so $2x + 2y\cdot\mathbf{y'} = 0$.

Solving for $\mathbf{y'}$, we have $\mathbf{y'} = -\dfrac{2x}{2y} = -x/y$, and, at the point (3,4),

$\mathbf{y'} = -3/4$, the same answer we found explicitly.

Fig. 1

Practice 2:   Find the slope of the tangent line to $y^3 - 3x^2 = 15$ at the point (2,3) with and without implicit differentiation.

In the previous example and practice problem, it was easy to explicitly solve for y , and then we could differentiate y to get y '. Because we could explicitly solve for y , we had a choice of methods for calculating y '. Sometimes, however, we can not explicitly solve for y , and the only way of determining y ' is implicit differentiation.

Example 3:   Determine y ' at (0,2) for $y^2 + 2y = \sin(x) + 8$.

Solution: Assuming that y is a function of x and differentiating each side of the equation, we get

$$\mathbf{D}(y^2 + 2y) = \mathbf{D}(\sin(x) + 8) \quad \text{so} \quad 2y\,\mathbf{y'} + 2\,\mathbf{y'} = \cos(x) \quad \text{and} \quad (2y + 2)\,\mathbf{y'} = \cos(x).$$

Then $\mathbf{y'} = \dfrac{\cos(x)}{2y + 2}$ so , at the point (0,2), $\mathbf{y'} = \dfrac{\cos(0)}{2(2) + 2} = 1/6$.

Practice 3:   Determine y ' at (1,0) for $y + \sin(y) = x^3 - x$.

In practice, the equations may be rather complicated, but if you proceed carefully and step–by–step, implicit differentiation is not difficult. Just remember that **y must be treated as a function** so every time you differentiate a term containing a y you should get something which has a **y'** . The algebra needed to solve for y' is always easy -- if you differentiated correctly the resulting equation will be a linear equation in the variable y'.

Example 4:   Find the equation of the tangent line L to the "tilted' parabola in Fig. 1 at the point

$(1, 2)$.

Solution: The line goes through the point $(1, 2)$ so we need to find

the slope there. Differentiating each side of the equation, we get

Fig. 2

$$\mathbf{D}(\, x^2 + 2xy + y^2 + 3x - 7y + 2 \,) = \mathbf{D}(\, 0 \,) \quad \text{so}$$

$$2x + 2x \, \mathbf{y} \, ' + 2y + 2y \, \mathbf{y} \, ' + 3 - 7\mathbf{y} \, ' = 0 \quad \text{and}$$

$$(2x + 2y - 7) \, \mathbf{y} \, ' = -2x - 2y - 3 \, .$$

Solving for $\mathbf{y}$ ', $\mathbf{y} \, ' = \dfrac{-2x - 2y - 3}{2x + 2y - 7}$, so the slope at $(1,2)$ is $m = \mathbf{y} \, ' = \dfrac{-2 - 4 - 3}{2 + 4 - 7} = 9.$

Finally, the equation of the line is $y - 2 = 9(x - 1)$ so $y = 9x - 7.$
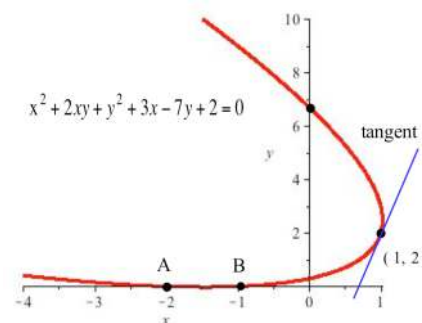
Practice 4:   Find the points where the graph in Fig. 2 crosses the y–axis, and find the slopes of the

tangent lines at those points.

Implicit differentiation is an alternate method for differentiating equations which can be solved explicitly

for the function we want, and it is the only method for finding the derivative of a function which we cannot

describe explicitly.

Logarithmic Differentiation

In section 2.5 we saw that $\mathbf{D}(\, \ln(\, f(x) \,) \,) = \dfrac{f \, '(x)}{f(x)}$. If we simply multiply each side by $f(x)$, we have

$\mathbf{f} \, '(\mathbf{x}) = f(x) \cdot \mathbf{D}(\, \ln(\, f(x) \,) \,)$. When the logarithm of a function is simpler than the function itself, it is often

easier to differentiate the logarithm of f than to differentiate f itself.

> **Logarithmic Differentiation:** $\mathbf{f} \, '(\mathbf{x}) = f(x) \cdot \mathbf{D}(\, \ln(\, f(x) \,) \,)$ **.**

The derivative of f is f times the derivative of the natural logarithm of f. Usually it is easiest to proceed

in three steps: (i)   calculate $\ln(\, f(x) \,)$ and simplify,

(ii)   calculate $\mathbf{D}(\, \ln(\, f(x) \,) \,)$ and simplify, and

(iii)   multiply the result in step (ii) by $f(x)$.

Let's examine what happens when we use this process on an "easy" function, $f(x) = x^2$, and a "hard" one,

$f(x) = 2^x$. Certainly we don't need to use logarithmic differentiation to find the derivative of $f(x) = x^2$, but

sometimes it is instructive to try a new algorithm on a familiar function. Logarithmic differentiation is the

easiest way to find the derivative of $f(x) = 2^x$.

$$f(x) = x^2 \qquad\qquad\qquad f(x) = 2^x$$

(i) $\ln(f(x)) = \ln(x^2) = 2\cdot\ln(x)$

(ii) $\mathbf{D}(\ln(f(x))) = \mathbf{D}(2\cdot\ln(x)) = \dfrac{2}{x}$

(iii) $f\,'(x) = f(x)\cdot \mathbf{D}(\ln(f(x))) = x^2 \cdot \dfrac{2}{x} = \mathbf{2x}$

(i) $\ln(f(x)) = \ln(2^x) = x\cdot\ln(2)$

(ii) $\mathbf{D}(\ln(f(x))) = \mathbf{D}(x\cdot\ln(2)) = \ln(2)$

(iii) $f\,'(x) = f(x)\cdot \mathbf{D}(\ln(f(x))) = \mathbf{2^x\cdot\ln(2)}$

Example 5:   Use the pattern $\mathbf{f}\,'(\mathbf{x}) = f(x)\cdot\mathbf{D}(\ln(f(x)))$ to find the derivative of $f(x) = (3x+7)^5\cdot\sin(2x)$.

Solution: (i) $\ln(f(x)) = \ln((3x+7)^5\cdot\sin(2x)) = 5\cdot\ln(3x+7) + \ln(\sin(2x))$ so

(ii) $\mathbf{D}(\ln(f(x))) = \mathbf{D}(5\cdot\ln(3x+7) + \ln(\sin(2x))) = 5\cdot\dfrac{3}{3x+7} + \dfrac{2\cos(2x)}{\sin(2x)}$.

Then   (iii) $\mathbf{f}\,'(\mathbf{x}) = f(x)\cdot\mathbf{D}(\ln(f(x))) = (3x+7)^5\cdot\sin(2x)\left(\dfrac{15}{3x+7} + \dfrac{2\cos(2x)}{\sin(2x)} \right)$

$$= 15\,(3x+7)^4\,\sin(2x) + 2\,(3x+7)^5\cos(2x),$$

the same result we would obtain using the product rule.

Practice 5:   Use logarithmic differentiation to find the derivative of $f(x) = (2x+1)^3\cdot(3x^2 - 4)^7\cdot(x+7)^4$.

We could have differentiated the functions in the example and practice problem without logarithmic differentiation. There are, however, functions for which logarithmic differentiation is the only method we can use. We know how to differentiate x to a constant power, $\mathbf{D}(x^n) = n\cdot x^{n-1}$, and a constant to the variable power, $\mathbf{D}(c^x) = c^x\cdot\ln(c)$, but the function $f(x) = x^x$ has both a variable base and a variable power so neither differentiation rule applies to x^x. We need to use logarithmic differentiation.

Example 6:   Find $\mathbf{D}(x^x)$ $(x > 0)$.

Solution: (i) $\mathrm{Ln}(f(x) = \ln(x^x) = x\cdot\ln(x)$

(ii) $\mathbf{D}(\ln(f(x))) = \mathbf{D}(x\cdot\ln(x)) = x\cdot\mathbf{D}(\ln(x)) + \ln(x)\cdot\mathbf{D}(x) = x\left(\dfrac{1}{x} \right) + \ln(x)\,(1) = 1 + \ln(x)$.

Then   (iii) $\mathbf{D}(x^x) = \mathbf{f}\,'(\mathbf{x}) = f(x)\cdot\mathbf{D}(\ln(f(x))) = x^x\cdot\left(1 + \ln(x) \right)$.

Practice 6:   Find $\mathbf{D}(x^{\sin(x)})$ $(x > 0)$.

Logarithmic differentiation is an alternate method for differentiating some functions such as products and quotients, and it is the only method we've seen for differentiating some other functions such as variable bases to variable exponents.

PROBLEMS

In problems 1 – 10 find dy/dx in two ways: (a) by differentiating implicitly   and (b) by explicitly solving for y and then differentiating. Then find the value of dy/dx at the given point using your results from both the implicit and the explicit differentiation.

1. $x^2 + y^2 = 100$, point $(6, 8)$

2. $x^2 + 5y^2 = 45$, point $(5, 2)$

3. $x^2 - 3xy + 7y = 5$, point $(2,1)$

4. $\sqrt{x} + \sqrt{y} = 5$, point $(4,9)$

5. $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$, point $(0,4)$

6. $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$, point $(3,0)$

7. $\ln(y) + 3x - 7 = 0$, point $(2,e)$

8. $x^2 - y^2 = 16$, point $(5,3)$

9. $x^2 - y^2 = 16$, point $(5,-3)$

10. $y^2 + 7x^3 - 3x = 8$, point $(1,2)$

11. Find the slopes of the lines tangent to the graph in Fig. 3 at the points $(3,1), (3,3),$ and $(4,2)$.
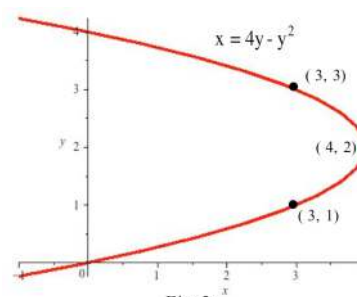
12. Find the slopes of the lines tangent to the graph in Fig. 3 where the graph crosses the y–axis.

Fig. 3

13. Find the slopes of the lines tangent to the graph in Fig. 4 at the points $((5,0),$ $(5,6),$ and $(-4,3)$.
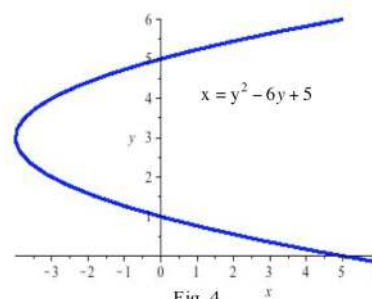
14. Find the slopes of the lines tangent to the graph in Fig. 4 where the graph crosses the y–axis.

In problems 15 – 22 , find dy/dx using **implicit differentiation** and then find the slope of the line tangent to the graph of the equation at the given point.

Fig. 4

15. $y^3 - 5y = 5x^2 + 7$, point $(1,3)$

16. $y^2 - 5xy + x^2 + 21 = 0$, point $(2,5)$

17. $y^2 + \sin(y) = 2x - 6$, point $(3,0)$

18. $y + 2x^2y^3 = 4x + 7$, point $(3,1)$

19. $e^y + \sin(y) = x^2 - 3$, point $(2,0)$

20. $(x^2 + y^2 + 1)^2 - 4x^2 = 81$, point $(0, 2\sqrt{2}\,)$

21. $x^{2/3} + y^{2/3} = 5$, point $(8,1)$

22. $x + \cos(xy) = y + 3$, point $(2,0)$

23. Find the slope of the line tangent to the ellipse in Fig. 5 at the point $(1, 2)$.

24. Find the slopes of the tangent lines at the points where the ellipse in Fig. 5 crosses the y–axis.

25. Find y' for $y = Ax^2 + Bx + C$ and for $x = Ay^2 + By + C$.

26. Find y' for $y = Ax^3 + B$ and for $x = Ay^3 + B$.

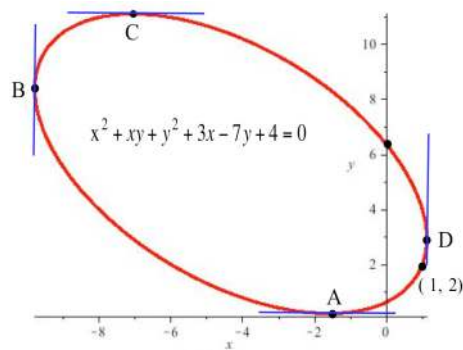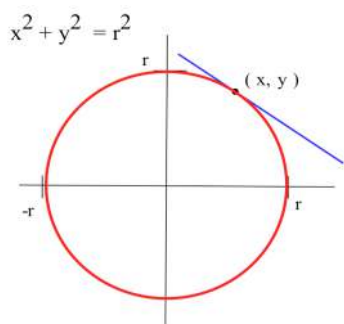27. Find y' for $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

$x^2 + y^2 = r^2$

28. In chapter 1 we assumed that the tangent line to a circle at a point was perpendicular to the radial line through the point and the center of the circle. Use implicit differentiation to prove that the line tangent to the circle $x^2 + y^2 = r^2$ (Fig. 6) at (x,y) is perpendicular to the line through $(0,0)$ and (x,y).

Fig. 6

29. Find the coordinates of point A where the tangent line to the ellipse in Fig. 5 is horizontal.

30. Find the coordinates of point B where the tangent line to the ellipse in Fig. 5 is vertical.

31. Find the coordinates of points C and D on the ellipse in Fig. 5..

In problems 32 – 40 find dy/dx in two ways: (a) by using the "usual" differentiation patterns   and (b) by using logarithmic differentiation.

32. $y = x \cdot \sin(3x)$

33. $y = (x^2 + 5)^7 \cdot (x^3 - 1)^4$

34. $y = \dfrac{\sin(3x - 1)}{x + 7}$

35. $y = x^5 \cdot (3x + 2)^4$

36. $y = 7^x$

37. $y = e^{\sin(x)}$

38. $y = \cos^7(2x + 5)$

39. $y = \sqrt{25 - x^2}$

40. $y = \dfrac{x \cdot \cos(x)}{x^2 + 1}$

In problems 41 – 46 , use logarithmic differentiation to find dy/dx .

41. $y = x^{\cos(x)}$

42. $y = (\cos(x))^x$

43. $y = x^4 \cdot (x - 2)^7 \cdot \sin(3x)$

44. $y = \dfrac{\sqrt{x + 10}}{(2x + 3)^3 \cdot (5x - 1)^7}$

45. $y = (3 + \sin(x))^x$

46. $y = \sqrt{\dfrac{x^2 + 1}{x^2 - 1}}$

In problems 47 – 50, use the values in each table to calculate the values of the derivative in the last column.

47. Use Table 1.   48. Use Table 2   49. Use Table 3.   50. Use Table 4.

Table 1

x	f(x)	ln(f(x))	**D**(ln(f(x)))	f '(x)
1	1	0	1.2	
2	9	2.2	1.8	
3	64	4.2	2.1	

Table 2

x	g(x)	ln(g(x))	**D**(ln(g(x)))	g '(x)
0	5	1.6	0.6	
1	10	2.3	0.7	
2	20	3.0	0.8	

Table 3

x	f(x)	ln(f(x))	**D**(ln(f(x)))	f '(x)
1	5	1.6	−1	
2	2	0.7	0	
3	7	1.9	2	

Table 4

x	g(x)	ln(g(x))	**D**(ln(g(x)))	g '(x)
2	1.4	0.3	1.2	
3	3.3	1.2	0.6	
7	13.6	2.6	0.2	

Problems 51 – 55 illustrate how logarithmic differentiation can be used to verify some differentiation patterns we already know (51 and 52) and to derive some new patterns (53 – 55). Assume that all of the functions are differentiable and that the function combinations are defined.

51. Use logarithmic differentiation on f•g to rederive the product rule: $\mathbf{D}(f•g) = f•g ' + g•f '$.

52. Use logarithmic differentiation on f/g to rederive the quotient rule: $\mathbf{D}(f/g) = \dfrac{g•f ' - f•g '}{g^2}$.

53. Use logarithmic differentiation on f•g•h to derive a product rule for three functions: $\mathbf{D}(f•g•h)$.

54. Use logarithmic differentiation on the exponential function a^x to determine its derivative: $\mathbf{D}(a^x)$.

55. Use logarithmic differentiation to determine a pattern for the derivative of f^g : $\mathbf{D}(f^g)$.

Section 3.10   **PRACTICE Answers**

Practice 1: $\mathbf{D}(x^2 + y^2) = 2x + 2y•\mathbf{y}'$

$\dfrac{d}{dx} (\sin(2 + 3y)) = \cos(2 + 3y)•\mathbf{D}(2 + 3y) = \cos(2 + 3y)•3\mathbf{y}'$

Practice 2:   Explicitly: $y = (3x^2 + 15)^{1/3}$ so $y' = \frac{1}{3} (3x^2 + 15)^{-2/3}•\mathbf{D}(3x^2 + 15) = \frac{1}{3} (3x^2 + 15)^{-2/3}•(6x)$.

When $(x,y) = (2,3)$, $y' = \frac{1}{3}(3(2)^2 + 15)^{-2/3} •(6•2) = 4 (27)^{-2/3} = \frac{4}{9}$.

Implicitly: $\mathbf{D}(y^3 - 3x^2) = \mathbf{D}(15)$ so $3y^2•\mathbf{y}' - 6x = 0$ and $y' = \dfrac{2x}{y^2}$.

When $(x,y) = (2,3)$, $y' = \dfrac{2•(2)}{(3)^2} = \frac{4}{9}$.

Practice 3: $y + \sin(y) = x^3 - x$

$$\mathbf{D}(y + \sin(y)) = \mathbf{D}(x^3 - x) \qquad\qquad \text{differentiating each side}$$

$$\mathbf{y}' + \cos(y)\cdot\mathbf{y}' = 3x^2 - 1$$

$$\mathbf{y}'\cdot(1 + \cos(y)) = 3x^2 - 1$$

$$\mathbf{y}' = \frac{3x^2 - 1}{1 + \cos(y)}$$

Then when $(x,y) = (1,0)$, $\mathbf{y}' = \frac{3(1)^2 - 1}{1 + \cos(0)} = \mathbf{1}$.

Practice 4:   To find where the parabola crosses the y–axis, we can set $x = 0$ and solve for the values of y.

Replacing x with 0 in $x^2 + 2xy + y^2 + 3x - 7y + 2 = 0$, we have $y^2 - 7y + 2 = 0$ so

$$y = \frac{7 \pm \sqrt{(-7)^2 - 4(1)(2)}}{2(1)} = \frac{7 \pm \sqrt{41}}{2} \approx 0.3 \text{ and } 6.7. \text{ The parabola crosses the y–axis}$$

approximately at the points $(0, 0.3)$ and $(0, 6.7)$.

From Example 4, we know that $y' = \frac{-2x - 2y - 3}{2x + 2y - 7}$, so

at the point $(0, 0.3)$, the slope is approximately $\frac{0 - 0.6 - 3}{0 + 0.6 - 7} \approx 0.56$, and

at the point $(0, 6.7)$, the slope is approximately $\frac{0 - 13.4 - 3}{0 + 13.4 - 7} \approx -2.56$.

Practice 5: $f'(x) = f(x)\cdot\mathbf{D}(\ln(f(x)))$ and $f(x) = (2x + 1)^3 (3x^2 - 4)^7 (x + 7)^4$

(i) $\ln(f(x)) = 3\cdot\ln(2x + 1) + 7\cdot\ln(3x^2 - 4) + 4\cdot\ln(x + 7)$.

(ii) $\mathbf{D}(\ln(f(x))) = \dfrac{3}{2x + 1}(2) + \dfrac{7}{3x^2 - 4}(6x) + \dfrac{4}{x + 7}(1)$

(iii) $f'(x) = f(x)\cdot\mathbf{D}(\ln(f(x))) = (2x + 1)^3 (3x^2 - 4)^7 (x + 7)^4\cdot\left\{ \dfrac{2\cdot3}{2x + 1} + \dfrac{7\cdot6x}{3x^2 - 4} + \dfrac{4}{x + 7} \right\}$.

Practice 6: $f'(x) = f(x)\cdot\mathbf{D}(\ln(f(x)))$ and $f(x) = x^{\sin(x)}$ so

(i) $\ln(f(x)) = \ln(x^{\sin(x)}) = \sin(x)\cdot\ln(x)$

(ii) $\mathbf{D}(\ln(f(x))) = \mathbf{D}(\sin(x)\cdot\ln(x)) = \sin(x)\cdot\mathbf{D}(\ln(x)) + \ln(x)\cdot\mathbf{D}(\sin(x)) = \sin(x)\cdot\dfrac{1}{x} + \ln(x)\cdot\cos(x)$

(iii) $f'(x) = f(x)\cdot\mathbf{D}(\ln(f(x))) = x^{\sin(x)}\cdot\left\{ \sin(x)\cdot\dfrac{1}{x} + \ln(x)\cdot\cos(x) \right\}$

Section Three

Section 3.1

1.

x	y = f(x)	m(x) = the estimated slope of the tangent line to y=f(x) at the point (x,y)
0	1	1
0.5	1.4	1/2
1.0	1.6	0
1.5	1.4	−1/2
2.0	1	−2
2.5	0	−2
3.0	−1	−2
3.5	−1.3	0
4.0	−1	1

3.   (a) At $x = 1, 3$, and 4.   (b) f is largest at $x = 4$. f is smallest at $x = 3$.

5.   (a) Graph   (b) Graph   (c) $m(x) = \cos(x)$

7.   The solution is similar to the method used in Example 4. Assume we turn off the engine at the point (p,q) on the curve $y = x^2$, and then find values of p and q so the tangent line to $y = x^2$ at the point (p,q) goes through the given point $(5,16)$. (p,q) is on $y = x^2$ so $q = p^2$. The equation of the tangent line to $y = x^2$ at (p, p^2) is $y = 2px - p^2$ so, substituting $x = 5$ and $y = 16$, we have $16 = 2p(5) - p^2$. Solving $p^2 - 10p + 16 = 0$ we get $p = 2$ or $p = 8$. The solution we want (moving left to right along the curve) is $p = 2, q = p^2 = 4$. ($p = 8, q = 64$ would be the solution if we were moving right to left.)

9.   Impossible. The point $(1, 3)$ is "inside" the parabola.

11.   (a) $m_{sec} = \dfrac{f(x+h) - f(x)}{(x+h) - (x)} = \dfrac{\{ 3(x + h) - 7 \} - \{ 3x - 7 \}}{(x + h) - x} = \dfrac{3h}{h} = 3.$

        (b) $m_{tan} = \lim\limits_{h \to 0} m_{sec} = \lim\limits_{h \to 0} 3 = 3.$ (c) At $x = 2$, $m_{tan} = 3.$

        (d) $f(2) = -1$ so the tangent line is $y - (-1) = 3(x - 2)$ or $y = 3x - 7$.

13.   (a) $m_{sec} = \dfrac{f(x+h) - f(x)}{(x+h) - (x)} = \dfrac{\{ a(x + h) + b \} - \{ ax + b\}}{(x + h) - x} = \dfrac{ah}{h} = a.$

        (b) $m_{tan} = \lim\limits_{h \to 0} m_{sec} = \lim\limits_{h \to 0} a = a$ (c) At $x = 2$, $m_{tan} = a.$

        (d) $f(2) = 2a + b$ so the tangent line is $y - (2a + b) = a(x - 2)$ or $y = ax + b..$

15.   (a) $m_{sec} = \dfrac{f(x+h) - f(x)}{(x+h) - (x)} = \dfrac{\{ 8 - 3(x + h)^2 \} - \{ 8 - 3x^2 \}}{(x + h) - x} = \dfrac{-6xh - 3h^2}{h} = -6x - 3h.$

        (b) $m_{tan} = \lim\limits_{h \to 0} m_{sec} = \lim\limits_{h \to 0} -6x - 3h = -6x$ (c) At $x = 2$, $m_{tan} = -6(2) = -12.$

        (d) $f(2) = -4$ so the tangent line is $y - (-4) = -12(x - 2)$ or $y = -12x + 20...$

17. $a = 1, b = 2, c = 0$, so $m_{tan} = (2)(1)(x) + 2 = 2x + 2$. The problem is to find p for which

               $6 - (p^2 + 2p) = (2p + 2)(3 - p).$

     This reduces to $p^2 - 6p = 0$ so $p = 0$ or 6 and the required points are $(0, 0)$ and $(6, 48)$.

Section 3.2

1.   (a) derivative of g   (b) derivative of h   (c) derivative of f

3.   (a) $m_{sec} = h - 4$, $m_{tan} = \lim\limits_{h \to 0} m_{sec} = -4$.   (b) $m_{sec} = h + 1$, $m_{tan} = \lim\limits_{h \to 0} m_{sec} = 1$.

5.   (a) $m_{sec} = 5 - h$, $m_{tan} = \lim\limits_{h \to 0} m_{sec} = 5$.   (b) $m_{sec} = 7 - 2x - h$, $m_{tan} = \lim\limits_{h \to 0} m_{sec} = 7 - 2x$.

7.   (a) -1 (b) -1 (c) 0   (d) $+1$ (e) DNE   (f) DNE

9. $f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{\{(x + h)^2 + 8\} - \{x^2 + 8\}}{h} = \lim\limits_{h \to 0} \dfrac{2xh + h^2}{h} = \lim\limits_{h \to 0} 2x + h = 2x$. $f'(3) = 6$.

11. $f'(x) = \lim\limits_{h \to 0} \dfrac{\{2(x + h)^3 - 5(x + h)\} - \{2x^3 - 5x\}}{h} = \lim\limits_{h \to 0} \dfrac{6x^2 h + 6xh^2 + 2h^3 - 5h}{h} = 6x^2 - 5$. $f'(3) = 49$.

13.   For any constant C, if $f(x) = x^2 + C$, then

$f'(x) = \lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{\{(x + h)^2 + C\} - \{x^2 + C\}}{h} = \lim\limits_{h \to 0} \dfrac{2xh + h^2}{h} = \lim\limits_{h \to 0} 2x + h = 2x$.

The graphs of $f(x) = x^2$, $g(x) = x^2 + 3$ and $h(x) = x^2 - 5$ are "parallel" parabolas: g is f shifted up 3 units, and h is f shifted down 5 units.

15. $f'(x) = 2x$. Then $f'(1) = 2$ and the equation of the tangent line at $(1,9)$ is $y - 9 = 2(x - 1)$
or $y = 2x + 7$.
$f'(-2) = -4$ and the equation of the tangent line at $(-2,12)$ is $y - 12 = -4(x + 2)$ or $y = -4x + 4$.

17. $f'(x) = \cos(x)$. Then $f'(\pi) = \cos(\pi) = -1$ and the equation of the tangent line at $(\pi,0)$ is
$y - 0 = -1(x - \pi)$ or $y = -x + \pi$.
$f'(\pi/2) = \cos(\pi/2) = 0$ and the equation of the tangent line at $(\pi/2,1)$ is $y - 1 = 0(x - \pi/2)$ or $y = 1$.

19.   (a) $y - 5 = 4(x - 2)$ or $y = 4x - 3$ (b) $x + 4y = 22$ or $y = -0.25x + 5.5$
(c) $f'(x) = 2x$ so the tangent line is horizontal when $x = 0$: at the point $(0,1)$.
(d) $f'(p) = 2p$ (the slope of the tangent line) so $y - q = 2p(x - p)$ or $y = 2px + (q - 2p^2)$.
    Since $q = p^2 + 1$, the equation of the tangent line becomes $y = 2px + (p^2 + 1 - 2p^2) = 2px - p^2 + 1$.
(e) We need p such that $-7 = 2p(1) - p^2 + 1$ or $p^2 - 2p - 8 = 0$. Then $p = -2, 4$. There are two
    points with the property we want: $(-2, 5)$ and $(4, 17)$.

21.   (a) $y' = 2x$, so when $x = 1, y' = 2$. Angle $= \arctan(2) \approx 1.107$ radians $\approx 63^o$.
(b) $y' = 3x^2$, so when $x = 1, y' = 3$. Angle $= \arctan(3) \approx 1.249$ radians $\approx 72^o$.
(c) Angle $\approx 1.249 - 1.107$ radians $= 0.142$ radians (or angle $= 72^o - 63^o = 9^o$)

23.   Graph. On the graph of upward velocity, the units on the horizontal axis are "seconds" and the units on the vertical axis are "feet per second."

25.   (a) $d(4) = 256$ ft. $d(5) = 400$ ft.   (b) $d'(x) = 32x$ $d'(4) = 128$ ft/sec $d'(5) = 160$ ft/sec.

27. $C(x) = \sqrt{x}$ dollars to produce x golf balls.

Marginal production cost is $C'(x) = \dfrac{1}{2\sqrt{x}}$ dollars per golf ball.

$C'(25) = \dfrac{1}{2\sqrt{25}} = \dfrac{1}{10}$ dollars per golf ball. $C'(100) = \dfrac{1}{2\sqrt{100}} = \dfrac{1}{20}$ dollars per golf ball.

29.   (a) $A(0) = 0$, $A(1) = 1/2$, $A(2) = 2$ and $A(3) = 9/2$.   (b) $A(x) = x^2/2$ $(x \geq 0)$.   (c) $\dfrac{d\,A(x)}{dx} = x$.

(d) $\dfrac{d\,A(x)}{dx}$ represents the rate at which $A(x)$ is increasing, the rate at which area is accumulating.

31.   (a) $9x^8$ (b) $\dfrac{2}{3x^{1/3}}$ (c) $\dfrac{-4}{x^5}$ (d) $\pi x^{\pi-1}$ (e) 1 if $x > -5$ and -1 if $x < -5$

33. $f(x) = x^3 + 4x^2$ (plus any constant)   35. $f(x) = 5 \cdot \sin(t)$ 37. $f(x) = \dfrac{1}{2} x^2 + \dfrac{1}{3} x^3$

Section 3.3

1. (a) Cont. at $0, 1, 2, 3, 5$ (b) Diff. at $0, 3, 5$

3.

x	f(x)	f '(x)	g(x)	g '(x)	f(x)•g(x)	D(f(x)•g(x))	f(x)/g(x)	D(f(x)/g(x))
0	2	3	1	5	**2**	**13**	**2**	**−7**
1	−3	2	5	−2	**−15**	**16**	**−3/5**	**4/25**
2	0	−3	2	4	**0**	**−6**	**0**	**−3/2**
3	1	−1	0	3	**0**	**3**	**undef**	**undef**

5.

x	f(x)	f '(x)	g(x)	g '(x)	f(x)+g(x)	f(x)•g(x)	f(x)/g(x)	D(f(x)+g(x))	D(f(x)•g(x))	D(f(x)/g(x))
1	3	−2	2	2	5	6	3/2	0	2	−10/4
2	1	0	3	1/2	4	3	1/3	1/2	1/2	−1/18
3	2	1	2	−1	4	4	1	0	0	1

7.   (a) $\mathbf{D}((x{-}5)(3x{+}7)) = (x{-}5)3 + (3x{+}7)1 = 6x - 8$

     (b) $\mathbf{D}(3x^2 - 8x - 35) = 6x - 8$, the same result as in (a)

9. $\dfrac{d}{dx} \dfrac{\cos(x)}{x^2} = \dfrac{x^2(-\sin(x)) - (\cos(x))(2x)}{(x^2)^2} = -\dfrac{x\sin(x) + 2\cos(x)}{x^3}$

11. $\mathbf{D}(\sin^2(x)) = \sin(x)\cos(x) + \sin(x)\cos(x) = 2\sin(x)\cos(x)$,

      $\mathbf{D}(\cos^2(x)) = \cos(x)(-\sin(x)) + \cos(x)(-\sin(x)) = -2\sin(x)\cos(x)$

13. $f(x) = ax^2 + bx + c$ so $f(0) = c$. Then $f(0) = 0$ implies that $c = 0$.
      $f '(x) = 2ax + b$ so $f '(0) = b$ and $f '(0) = 0$ implies that $b = 0$.
      Finally, $f '(10) = 20a + b = 20a$ so $f '(10) = 30$ implies that $20a = 30$ and $a = 3/2$.

      $f(x) = \dfrac{3}{2} x^2 + 0x + 0$ has $f(0) = 0, f '(0) = 0$, and $f '(10) = 30$.

15.   Their graphs are vertical shifts of each other, and their derivatives are equal.

17. $f(x)g(x) = k$ so $\mathbf{D}(f(x)g(x)) = \mathbf{D}(k) = 0$ and $f(x)g '(x) + g(x)f '(x) = 0$.

      If $f(x) \neq 0$ and $g(x) \neq 0$, then $\dfrac{f '(x)}{f(x)} = -\dfrac{g '(x)}{g(x)}$.

19. $f '(x) = 2x - 5$ so $f '(1) = -3$. $f '(x) = 0$ if $x = 5/2$.

21. $f '(x) = 3 + 2\sin(x)$ so $f '(1) = 3 + 2\sin(1) \approx 4.69$. $f '(x)$ never equals 0 since $\sin(x)$ never equals $-3/2$.

23. $f '(x) = 3x^2 + 18x = 3x(x + 6)$ so $f '(1) = 21$. $f '(x) = 0$ if $x = 0$ or -6.

25. $f '(x) = 3x^2 + 4x + 2$ so $f '(1) = 9$. $f '(x) = 0$ for no values of x (the discriminant $4^2 - 4(3)(2) < 0$).

27. $f '(x) = x{\cdot}\cos(x) + \sin(x)$ so $f '(1) = 1{\cdot}\cos(1) + \sin(1) \approx 1.38$. The graph of f '(x) crosses the x–axis infinitely often. The root of f ' at $x = 0$ is easy to see (and verify). Other roots of f ', such as near $x = 2.03$ and 4.91 and -2.03, can be found numerically using the Bisection algorithm or graphically using the "zoom" or "trace" features on some calculators.

29. $f '(x) = 3x^2 + 2Ax + B$. The graph of $y = f(x)$ has two distinct "vertices" if $f '(x) = 0$ for two distinct values of x. This occurs if the discriminant of $3x^2 + 2Ax + B$ is greater than 0: $(2A)^2 - 4(3)(B) > 0$.

31.   Everywhere except at $x = -3$.   33.   Everywhere except at $x = 0$ and 3.

35.   Everywhere except at $x = 1$.

37.   Everywhere. The only possible difficulty is at $x = 0$, and the definition of the derivative gives $f'(0) = 1$.
       The derivatives of the "two pieces" of f match at $x = 0$ to give a differentiable function there.

39.   Continuity of f at $x = 1$ requires $A + B = 2$. The "left derivative" of f at $x = 1$ is $\mathbf{D}(Ax + B) = A$
       and the "right derivative" of f at $x = 1$ is 3 (if $x > 1$ then $\mathbf{D}(x^2 + x) = 2x + 1$) so to achieve
       differentiability $A = 3$ and $B = 2 - A = -1$.

41. $h(x) = 128x - 2.65x^2$ ft.
       (a) $h'(x) = 128 - 5.3x$ so $h'(0) = 128$ ft/sec, $h'(1) = 122.7$ ft/sec, and $h'(2) = 117.4$ ft/sec.
       (b) $v(x) = h'(x) = 128 - 5.3 x$ ft/sec.   (c) $v(x) = 0$ when $x = 128/5.3 \approx 24.15$ sec.
       (d) $h(24.15) \approx 1{,}545.66$ ft.   (e) about 48.3 seconds: 24.15 up and 24.15 down

43. $h(x) = v_0 x - 16x^2$ ft.
       (a) $h'(x) = v_0 - 32x$ ft/sec
       (b) Max height when $x = v_0/32$: max height $= h(v_0/32) = v_0(v_0/32) - 16(v_0/32)^2 = (v_0)^2/64$ feet.
       (c) Time aloft $= 2(v_0/32) = v_0/16$ seconds.

45.   (a) $(v_0)^2/64 = 6.5$, so $v_0 = 8\sqrt{6.5} \approx 20.396$ ft/sec.

       (b) $2(v_0/32) = \dfrac{8\sqrt{6.5}}{16} \approx 1.27$ seconds.

       (c) Max height $= \dfrac{(v_0)^2}{2g} = \dfrac{(8\sqrt{6.5})^2}{2(5.3)} = \dfrac{416}{10.6} \approx 39.25$ feet.

47.   (a) $y' = -\dfrac{1}{x^2}$; $y'(2) = -1/4$ so $y - 1/2 = (-1/4)(x - 2)$ or $y = (-1/4)x + 1$.

       (b) x–intercept at $x = 4$, y–intercept at $y = 1$ (c) $A = \frac{1}{2}$(base)(height) $= \frac{1}{2}(4)(1) = 2$.

49.   Since $(1,4)$ and $(3,14)$ are on the parabola, we need $a + b + c = 4$ and $9a + 3b + c = 14$.
       Then, subtracting the first equation from the second, $8a + 2b = 10$.
       $f'(x) = 2ax + b$ so $f'(3) = 6a + b = 9$, the slope of $y = 9x - 13$. Now solve the system $8a + 2b = 10$
       and $6a + b = 9$ to get $a = 2$ and $b = -3$. Then use $a + b + c = 4$ to get $c = 5$. $a = 2, b = -3, c = 5$.

51.   (a) $f(x) = x^3$ (b) $g(x) = x^3 + 1$
       (c)   If $h(x) = x^3 + C$ for any constant C, then $\mathbf{D}(h(x)) = 3x^2$.

53.   (a)   For $0 \le x \le 1$, $f'(x) = 1$ so $f(x) = x + C$. Since $f(0) = 0$, we know $C = 0$ and $f(x) = x$.

              For $1 \le x \le 3$, $f'(x) = 2 - x$ so $f(x) = 2x - \frac{1}{2}x^2 + K$. Since $f(1) = 1$, we know

                     $K = -1/2$ and $f(x) = 2x - \frac{1}{2}x^2 - \frac{1}{2}$.

              For $3 \le x \le 4$, $f'(x) = x - 4$ so $f(x) = \frac{1}{2}x^2 - 4x + L$. Since $f(3) = 1$, we know $L = 17/2$

                     and $f(x) = \frac{1}{2}x^2 - 4x + \frac{17}{2}$.
       (b)   This graph is a vertical shift, up 1 unit, of the graph in part (a).

Section 3.4

1. $\mathbf{D}(\,f^2(x)\,) = 2\cdot f^1(x)\cdot f\,'(x)$. At $x = 1$, $\mathbf{D}(\,f^2(x)\,) =\ 2(2)(3) = 12$.

$\mathbf{D}(\,f^5(x)\,) = 5\cdot f^4(x)\cdot f\,'(x)$. At $x = 1$, $\mathbf{D}(\,f^5(x)\,) =\ 5(2)^4(3) = 240$.

$\mathbf{D}(\,f^{1/2}(x)\,) = (1/2)\cdot f^{-1/2}(x)\cdot f\,'(x)$. At $x = 1$, $\mathbf{D}(\,f^{1/2}(x)\,) =\ (1/2)(2)^{-1/2}(3) = \dfrac{3}{2\sqrt{2}}\ = \dfrac{3\sqrt{2}}{4}$.

3.

x	f(x)	f '(x)	$\mathbf{D}(\,f^2(x)\,)$	$\mathbf{D}(\,f^3(x)\,)$	$\mathbf{D}(\,f^5(x)\,)$
1	1	–1	–2	–3	–5
3	2	–3	–12	–36	–240

5. $f\,'(x) = 5\cdot(2x - 8)^4\cdot(2)$

7. $f\,'(x) = x\cdot5\cdot(3x + 7)^4\cdot3 + 1\cdot(3x + 7)^5\ = (3x + 7)^4\{\ 15x + (3x + 7)\ \}\ =\ (3x + 7)^4\cdot(\ 18x + 7)$

9. $f\,'(x) = \dfrac{1}{2}(x^2 + 6x - 1)^{-1/2}\cdot(2x + 6)\ =\ \dfrac{x + 3}{\sqrt{x^2 + 6x - 1}}$

11.   (a)   graph $h(t) = 3 - 2\sin(t)$ (b) When $t = 0$, $h(0) = 3$ feet.
          (c)   Highest = 5 feet above the floor. Lowest = 1 foot above the floor.
          (d) $h(t) = 3 - 2\sin(t)$ feet, $v(t) = h\,'(t) = -2\cos(t)$ ft/sec, and $a(t) = v\,'(t) = 2\sin(t)$ ft/sec^2 .
          (e)   This spring oscillates forever. The motion of a real spring would "damp out" due to friction.

13. $K = \dfrac{1}{2}\,mv^2$ (a) If $h(t) = 5t$, then $v(t) = h\,'(t) = 5$. Then $K(1) = K(2) = \dfrac{1}{2}\,m(5)^2\ =\ 12.5m$.

          (b) If $h(t) = t^2$, then $v(t) = h\,'(t) = 2t$ so $v(1) = 2$ and $v(2) = 4$. Then $K(1) = \dfrac{1}{2}\,m(2)^2 = 2m$

          and $K(2) = \dfrac{1}{2}\,m(4)^2 = 8m$.

15. $\dfrac{df}{dx}\ =\ x\cdot\mathbf{D}(\,\sin(x)\,) + \sin(x)\cdot\mathbf{D}(\,x\,) = x\cdot\cos(x) + \sin(x)$

17. $f\,'(x) = e^x - \sec(x)\cdot\tan(x)$ 19. $f\,'(x) = -e^{-x} + \cos(x)$

21. $f\,'(x) = 7(x - 5)^6(1)$ so $f\,'(4) = 7(-1)^6(1)\ = 7$. Then $y - (-1) = 7(x - 4)$ and $y = 7x - 29$.

23. $f\,'(x) = \dfrac{1}{2}(25 - x^2)^{-1/2}(-2x)\ =\ \dfrac{-x}{\sqrt{25 - x^2}}$ so $f\,'(3) = \dfrac{-3}{4}$. Then $y - 4 = -\dfrac{3}{4}(x - 3)$ or $3x + 4y = 25$.

25. $f\,'(x) = 5(x - a)^4(1)$ so $f\,'(a) = 5(a - a)^4(1) = 0$. Then $y - 0 = 0(x - a)$ or $y = 0$.

27. $f\,'(x) = e^x$ so $f\,'(3) = e^3$. Then $y - e^3 = e^3(x - 3)$.
          x–intercept (y=0): $0 - e^3 = e^3(x - 3)$ so $-1 = x - 3$ and $x = 2$.
          (y–intercept (x=0): $y - e^3 = e^3(0 - 3)$ so $y = -3e^3 + e^3 = -2e^3$.)

          At (p, e^p), $f\,'(p) = e^p$ so $y - e^p = e^p(x - p)$.
          x–intercept (y=0): $0 - e^p = e^p(x - p)$ so $-1 = x - p$ and $x = p - 1$.

29. $f\,'(x) = -\sin(x)$, $f\,''(x) = -\cos(x)$

31. $f\,'(x) = x^2\cos(x) + 2x\sin(x)$,
          $f\,''(x) = -x^2\sin(x) + 2x\cos(x) + 2x\cos(x) + 2\sin(x) = -x^2\sin(x) + 4x\cos(x) + 2\sin(x)$

33. $f'(x) = e^x \cdot \cos(x) - e^x \cdot \sin(x)$, $f''(x) = -2 e^x \cdot \sin(x)$

35. $q' = $ linear, $q'' = $ constant, $q''' = q^{(4)} = q^{(5)} = ... = 0$

37. $p^{(n)} = $ constant, $p^{(n+1)} = 0$ 39. $f(x) = 5e^x$ 41. $f(x) = (1 + e^x)^5$

43.   No. $\displaystyle\lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{(0+h) \cdot \sin\left(\dfrac{1}{0+h}\right) - 0}{h} = \lim_{h \to 0} \sin\left(\dfrac{1}{h}\right)$

       which does not exist.

       (To see that this last limit does not exist, graph sin(1/h) for $-1 \le h \le 1$, or evalaute sin(1/h) for lots of small

       values of h, e.g., h 0.1, 0.01, 0.001, ...)

       €   €   €

45. $\left(1 + \dfrac{1}{x}\right)^x \approx 2.718 ... = $ e when x is large.

47.   (a) $s_2 = 2.5$, $s_3 \approx 2.67$, $s_4 \approx 2.708$, $s_5 \approx 2.716$, $s_6 \approx 2.718$, $s_7 \approx 2.71825$, $s_8 \approx 2.718178$

       (b) They are approaching e .

Section 3.5

1.   If $f(x) = x^5$ and $g(x) = x^3 - 7x$, then $f{\circ}g(x) = (x^3 - 7x)^5$

3.   If $f(x) = x^{5/2}$ and $g(x) = 2 + \sin(x)$, then $f{\circ}g(x) = $

      $\sqrt{(2 + \sin(x))^5}$ (The pair $f(x) = \sqrt{x}$ and $g(x) = (2 + \sin(x))^5$ also work.)

5.   If $f(x) = |x|$ and $g(x) = x^2 - 4$, then $f{\circ}g(x) = | x^2 - 4 |$

7.   (1) $y = u^5$, $u = x^3 - 7x$ (2) $y = u^4$, $u = \sin(3x - 8)$
      (3) $y = u^{5/2}$, $u = 2 + \sin(x)$ (4) $y = 1/\sqrt{u}$, $u = x^2 + 9$
      (5) $y = |u|$, $u = x^2 - 4$ (6) $y = \tan(u)$, $u = \sqrt{x}$

8. & 9.

x	f(x)	g(x)	f'(x)	g'(x)	(f∘g)(x)	(f∘g)' (x)
−2	2	−1	1	1	1	0
−1	1	2	0	2	1	2
0	−2	1	2	−1	0	1
1	0	−2	−1	2	2	2
2	1	0	1	−1	−2	−2

11. $g(2) \approx 2$, $g'(2) \approx -1$, $(f{\circ}g)(2) = f(g(2)) \approx f(2) \approx 1$

       $f'(g(2)) \approx f'(2) \approx 0$, $(f{\circ}g)'(2) = f'(g(2)) \cdot g'(2) \approx 0$

13. $\mathbf{D}\left(\left(1 - \dfrac{3}{x} \right)^4 \right) = 4\left(1 - \dfrac{3}{x} \right)^3 \left(\dfrac{3}{x^2} \right)$

15. $\mathbf{D}\left(\dfrac{5}{\sqrt{2 + \sin(x)}} \right) = 5\left(\dfrac{-1}{2} \right)(2 + \sin(x))^{-3/2} \cos(x) = \dfrac{-5 \cos(x)}{2(2 + \sin(x))^{3/2}}$

17. $\mathbf{D}(x^2 \cdot \sin(x^2 + 3)) = x^2\{ \cos(x^2 + 3) \}(2x) + \{ \sin(x^2 + 3) \}(2x) = 2x\{ x^2 \cdot \cos(x^2 + 3) + \sin(x^2 + 3) \}$

19. $\mathbf{D}(\dfrac{7}{\cos(x^3 - x)}) = \mathbf{D}(7\sec(x^3 - x)) = 7(3x^2 - 1)\cdot\sec(x^3 - x)\cdot\tan(x^3 - x)$

21. $\mathbf{D}(e^x + e^{-x}) = e^x - e^{-x}$

23. $h(t) = 3 - \cos(2t)$ feet.   (a) $h(0) = 2$ feet above the floor.

(b) $h(t) = 3 - \cos(2t)$ feet, $v(t) = h'(t) = 2\sin(2t)$ ft/sec , $a(t) = v'(t) = 4\cos(2t)$ ft/sec^2

(c) $K = \dfrac{1}{2}mv^2 = \dfrac{1}{2}m(2\sin(2t))^2 = 2m\cdot\sin^2(2t)$, $dK/dt = 8m\cdot\sin(2t)\cdot\cos(2t)$

25. $P(h) = 14.7\, e^{-0.0000385h}$. (a) $P(0) = 14.7$ psi (pounds per square inch), $P(30,000) \approx 4.63$ psi

(b) $10 = 14.7\, e^{-0.0000385h}$ so $\dfrac{10}{14.7} = e^{-0.0000385h}$ and $h = \dfrac{1}{-0.0000385}\ln(\dfrac{10}{14.7}) \approx 10,007$ ft.

(c) $dP/dh = 14.7(-0.0000385)\, e^{-0.0000385h}$ psi/ft

At $h = 2,000$ feet, $dP/dh = 14.7(-0.0000385)\, e^{-0.0000385(2,000)}$ psi/ft ≈ -0.000524 psi/ft.
Finally, $dP/dt = 500(-0.000524) \approx -0.262$ psi/minute

(d) If the temperature is constant, then (pressure)(volume) is a constant (from physics!) so a decrease in pressure results in an increase in volume.

27. $\dfrac{d}{dz}\sqrt{1 + \cos^2(z)}\quad = \dfrac{2\cos(z)\{-\sin(z)\}}{2\sqrt{1 + \cos^2(z)}}\quad = \dfrac{-\sin(2z)}{2\sqrt{1 + \cos^2(z)}}$

29. $\dfrac{d}{dx}\tan(3x + 5) = 3\cdot\sec^2(3x + 5)$

31. $\mathbf{D}(\sin(\sqrt{x + 1})) = \{\cos(\sqrt{x + 1})\}\dfrac{1}{2\sqrt{x + 1}}$

33. $\dfrac{d}{dx}(e^{\sin(x)}) = e^{\sin(x)}\cdot\cos(x)$

35. $f(x) = \sqrt{x}$ so $\dfrac{d\,f(x)}{dx} = \dfrac{1}{2\sqrt{x}}$. $x(t) = 2 + \dfrac{21}{t}$ so $\dfrac{d\,x(t)}{dt} = -\dfrac{21}{t^2}$.

At $t = 3$, $x = 9$ and $\dfrac{d\,x(t)}{dt} = -\dfrac{21}{9} = -\dfrac{7}{3}$ so $\dfrac{d\,(f_\infty x)}{dt} = (\dfrac{1}{2\sqrt{9}})(-\dfrac{7}{3}) = -\dfrac{7}{18}$.

37. $f(x) = \tan^3(x)$ so $\dfrac{d\,f(x)}{dx} = 3\cdot\tan^2(x)\cdot\sec^2(x)$. $x(t) = 8$ so $\dfrac{d\,x(t)}{dt} = 0$.

At $t = 3$, $x = 8$ and $\dfrac{d\,x(t)}{dt} = 0$ so $\dfrac{d\,(f_\infty x)}{dt} = 0$.

39. $f(x) = \dfrac{1}{77}(7x - 13)^{11}$ 41. $f(x) = -\dfrac{1}{2}\cos(2x - 3)$ 43. $f(x) = e^{\sin(x)}$

45. Then $-2\sin(2x) = 2\cos(x)\{-\sin(x)\} - 2\sin(x)\cdot\cos(x)$ or $\sin(2x) = 2\sin(x)\cdot\cos(x)$.

47. $3\cos(3x) = 3\cos(x) - 12\sin^2(x)\cdot\cos(x)$
so $\cos(3x) = \cos(x)\{1 - 4\sin^2(x)\} = \cos(x)\{1 - 4 + 4\cos^2(x)\} = 4\cos^3(x) - 3\cos(x)$

49. $y' = 3Ax^2 + 2Bx$ 51. $y' = 2Ax\cdot\cos(Ax^2 + B)$ 53. $y' = \dfrac{Bx}{\sqrt{A + Bx^2}}$

55. $y' = B\cdot\sin(Bx)$ 57. $y' = -2Ax\cdot\sin(Ax^2 + B)$ 59. $y' = x(B\cdot e^{Bx}) + e^{Bx} = (Bx+1)\cdot e^{Bx}$

61. $y' = A \cdot e^{Ax} + A \cdot e^{-Ax}$ 63. $y' = \dfrac{A \cdot \sin(Bx) - Ax \cdot B \cdot \cos(Bx)}{\sin^2(Bx)}$

65. $y' = \dfrac{(Cx+D)A - (Ax+B)C}{(Cx+D)^2} = \dfrac{AD - BC}{(Cx+D)^2}$

67.   (a) $y' = AB - 2Ax$, (b) $x = \dfrac{AB}{2A} = \dfrac{B}{2}$, (c) $y'' = -2A$.

69.   (a) $y' = 2ABx - 3Ax^2 = Ax \cdot (2B - 3x)$, (b) $x = 0, 2B/3$, (c) $y'' = 2AB - 6Ax$.

71.   (a) $y' = 3Ax^2 + 2Bx = x \cdot (3Ax + 2B)$, (b) $x = 0, \dfrac{-2B}{3A}$, (c) $y'' = 6Ax + 2B$.

73. $\dfrac{d}{dx}(\arctan(x^2)) = \dfrac{2x}{1 + x^4}$ 75. $\mathbf{D}(\arctan(e^x)) = \dfrac{1}{1 + (e^x)^2} \cdot e^x = \dfrac{e^x}{1 + e^{2x}}$

77. $\mathbf{D}(\arcsin(x^3)) = \dfrac{3x^2}{\sqrt{1 - x^6}}$ 79. $\dfrac{d}{dt}(\arcsin(e^t)) = \dfrac{1}{\sqrt{1 - (e^t)^2}} \cdot e^t = \dfrac{e^t}{\sqrt{1 - e^{2t}}}$

81. $\dfrac{d}{dx}(\ln(\sin(x))) = \dfrac{1}{\sin(x)} \cos(x) = \cot(x)$ 83. $\dfrac{d}{ds}(\ln(e^s)) = \dfrac{1}{e^s} \cdot e^s = 1$, or $\dfrac{d}{ds}(\ln(e^s)) = \dfrac{d}{ds}(s) = 1$

Section 3.6

1. $\mathbf{D}(\ln(5x)) = \dfrac{1}{5x} 5 = \dfrac{1}{x}$ 3. $\mathbf{D}(\ln(x^k)) = \dfrac{1}{x^k} k x^{k-1} = \dfrac{k}{x}$

5. $\mathbf{D}(\ln(\cos(x))) = \dfrac{1}{\cos(x)} (- \sin(x)) = - \tan(x)$ 7. $\mathbf{D}(\log_2(5x)) = \dfrac{1}{5x \ln(2)} (5) = \dfrac{1}{x \ln(2)}$

9. $\mathbf{D}(\ln(\sin(x))) = \dfrac{1}{\sin(x)} (\cos(x)) = \cot(x)$

11. $\mathbf{D}(\log_2(\sin(x))) = \dfrac{1}{\sin(x)} \dfrac{1}{\ln(2)} (\cos(x)) = \dfrac{\cot(x)}{\ln(2)}$ 13. $\mathbf{D}(\log_5(5^x)) = \mathbf{D}(x) = 1$

15. $\mathbf{D}(x \ln(3x)) = x \cdot \dfrac{1}{3x} \cdot 3 + \ln(3x) = 1 + \ln(3x)$ 17. $\mathbf{D}(\dfrac{\ln(x)}{x}) = \dfrac{x \cdot \frac{1}{x} - \ln(x) \cdot 1}{x^2} = \dfrac{1 - \ln(x)}{x^2}$

19. $\mathbf{D}(\ln((5x-3)^{1/2})) = \dfrac{1}{(5x-3)^{1/2}} \cdot \mathbf{D}((5x-3)^{1/2}) = \dfrac{1}{(5x-3)^{1/2}} \cdot \dfrac{1}{2} \cdot (5x-3)^{-1/2} \cdot \mathbf{D}(5x-3) = \dfrac{5}{2} \cdot \dfrac{1}{5x-3}$.

21. $\dfrac{d}{dw}(\cos(\ln(w))) = \{ - \sin(\ln(w)) \} \dfrac{1}{w} = \dfrac{-\sin(\ln(w))}{w}$

23. $\dfrac{d}{dt}(\ln(\sqrt{t+1})) = \dfrac{1}{2(t + 1)}$ 25. $\mathbf{D}(5^{\sin(x)}) = 5^{\sin(x)} \ln(5) \cos(x)$

27. $\dfrac{d}{dx} \ln(\sec(x) + \tan(x)) = \dfrac{1}{\sec(x) + \tan(x)} (\sec(x)\tan(x) + \sec^2(x)) = \sec(x)$

29. $f(x) = \ln(x)$, $f'(x) = \frac{1}{x}$. Let $P = (p, \ln(p))$. Then we must satisfy $y - \ln(p) = \frac{1}{p}(x - p)$ with $x = 0$ and

$y = 0$: $-\ln(p) = -1$ so $p = e$ and $P = (e, 1)$.

31. $p(t) = 100(1 + Ae^{-t})^{-1}$. $\frac{d}{dt}\,p(t) = 100\,(-1)\,(1 + Ae^{-t})^{-2}\,(Ae^{-t}(-1)) = \dfrac{100\,Ae^{-t}}{(1 + Ae^{-t})^2}$.

33. $f(x) = 8\ln(x) +$ any constant   35. $f(x) = \ln(3 + \sin(x)) +$ any constant

37. $g(x) = \frac{3}{5}\,e^{5x} +$ any constant   39. $f(x) = e^{x^2} +$ any constant

41. $h(x) = \ln(\sin(x)) +$ any constant

43.   A: $(t, 2t + 1)$, B: $(t^2, 2t^2 + 1)$
        (a)   When $t = 0$, A is at $(0,1)$ and B is at $(0,1)$.   When $t = 1$, A is at $(1,3)$, B is at $(1,3)$
        (b)   graph
        (c) $dy/dx = 2$ for each, since $y = 2x + 1$.

        (d)   A: $dx/dt = 1, dy/dt = 2$ so speed $= \sqrt{1^2 + 2^2} = \sqrt{5}$

        B: $dx/dt = 2t$, $dy/dt = 4t$ so speed $= \sqrt{(2t)^2 + (4t)^2} = 2\sqrt{5}$ t. At t=1, B's speed is $2\sqrt{5}$.
        (e)   This robot moves on the same path $y = 2x + 1$, but it moves to the right and up for about
                1.57 minutes, reverses its direction and returns to its starting point, then continues left and
                down for another 1.57 minutes, reverses, and continues to oscillate.

45.   When t=1, dx/dt = + , dy/dt= − , dy/dx= − . WHen t=3, dx/dt = − , dy/dt= − , dy/dx= + .

47. $x(t) = R(t - \sin(t))$ (a) graph
        $y(t) = R(1 - \cos(t))$

        (b) $dx/dt = R(1 - \cos(t))$, $dy/dt = R\sin(t)$, so $\dfrac{dy}{dx} = \dfrac{\sin(t)}{1 - \cos(t)}$.

        When $t = \pi/2$, then $dx/dt = R, dy/dt = R$ so $dy/dx = 1$ and speed $= \sqrt{R^2 + R^2} = R\sqrt{2}$

        When $t = \pi$, $dx/dt = 2R$, $dy/dt = 0$ so $dy/dx = 0$ and speed $= \sqrt{(2R)^2 + 0} = 2R$.

49.   (a) The ellipse $(\frac{x}{3})^2 + (\frac{y}{5})^2 = 1$.

        (b) The ellipse $(\frac{x}{A})^2 + (\frac{y}{B})^2 = 1$ if $A \neq 0$ and $B \neq 0$.   (c) $(3\cdot\cos(t), -5\cdot\sin(t))$ works.
            If $A = 0$, the motion is oscillatory along the x–axis.
            If $B = 0$, the motion is oscillatory along the y–axis.

Section 3.7

1. $V = \frac{4}{3}\pi r^3$ $(r = r(t))$ so $\dfrac{dV}{dt} = 4\pi r^2\dfrac{dr}{dt}$.

    When $r = 3$ in. , $\dfrac{dr}{dt} = 2$ in/min, so $\dfrac{dV}{dtt} = 4\pi(3\text{ in})^2(2\text{ in/min}) = 72\pi$ in^3/min ≈ 226.19 in^3/min.

3. $b = 15$ in., $h = 13$ in., $\dfrac{db}{dt} = 3$ in/hr, $\dfrac{dh}{dt} = -3$ in/hr.

    (a) $A = \frac{1}{2}bh$ so $\dfrac{dA}{dt} = \frac{1}{2}\{b\dfrac{dh}{dt} + h\dfrac{db}{dt}\} = \frac{1}{2}\{(15\text{ in})(-3\text{ in/hr}) + (13\text{ in})(3\text{ in/hr})\} < 0$ so A is decreasing.

    (b) Hypotenuse $C = \sqrt{b^2 + h^2}$ so $\dfrac{dC}{dt} = \dfrac{b\dfrac{db}{dt} + h\dfrac{dh}{dt}}{\sqrt{b^2 + h^2}} = \dfrac{15(3) + 13(-3)}{\sqrt{15^2 + 13^2}} > 0$ so C is increasing.

    (c) Perimeter $P = b + h + C$ so $\dfrac{dP}{dt} = \dfrac{db}{dt} + \dfrac{dh}{dt} + \dfrac{dC}{dt} = (3) + (-3) + \dfrac{6}{\sqrt{394}} > 0$ so P is increasing.

5. (a) $P = 2x + 2y$ so $\dfrac{dP}{dt} = 2\dfrac{dx}{dt} + 2\dfrac{dy}{dt} = 2(3 \text{ ft/sec}) + 2(-2 \text{ ft/sec}) = 2 \text{ ft/sec}.$

(b) $A = xy$ so $\dfrac{dA}{dt} = x\dfrac{dy}{dt} + y\dfrac{dx}{dt} = (12 \text{ ft})(-2 \text{ ft/sec}) + (8 \text{ ft})(3 \text{ ft/sec}) = 0 \text{ ft}^2/\text{sec}.$

7. $V = \pi r^2 h = \pi r^2 (1/3)$ so $\dfrac{dV}{dt} = \dfrac{2\pi}{3} r \dfrac{dr}{dt}.$

When $r = 50$ ft. and $\dfrac{dr}{dt} = 6$ ft/hr, then $\dfrac{dV}{dt} = \dfrac{2\pi}{3}(50 \text{ ft})(6 \text{ ft/hr}) = 200\pi \text{ ft}^3/\text{hr} \approx 628.32 \text{ ft}^3/\text{hr}.$

9. $w(t) = h(t)$ for all t so $\dfrac{dw}{dt} = \dfrac{dh}{dt}$. $V = \dfrac{1}{3} \pi r^2 h$ and $r = w/2 = h/2$ so $V = \dfrac{1}{3} \pi (h/2)^2 h = \dfrac{1}{12} \pi h^3$

and $\dfrac{dV}{dt} = \dfrac{1}{4} \pi h^2 \dfrac{dh}{dt}$. When $h = 500$ ft and $\dfrac{dh}{dt} = 2$ ft/hr, then $\dfrac{dV}{dt} = \dfrac{1}{4} \pi(500)^2(2) = 125{,}000\pi \text{ ft}^3/\text{hr}.$

11. Let x be the distance from the lamp post to the person, and L be the length of the shadow, both in feet. By similar

triangles, $\dfrac{L}{6} = \dfrac{x}{8}$ so $L = \dfrac{3}{4} x$. $\dfrac{dx}{dt} = 3$ ft/sec.

(a) $\dfrac{dL}{dt} = \dfrac{3}{4}\dfrac{dx}{dt} = \dfrac{3}{4}(3 \text{ ft/sec}) = 2.25$ ft/sec.   (b) $\dfrac{d}{dt}(x + L) = \dfrac{dx}{dt} + \dfrac{dL}{dt} = 5.25$ ft/sec.

(The value of x does not enter into the calculations.)

13. (a) $\sin(35^{\text{o}}) = \dfrac{h}{500}$ so $h = 500 \cdot \sin(35^{\text{o}}) \approx 287$ ft.

(b) $L = $ length of the string so $h = L \cdot \sin(35^{\text{o}})$ and $\dfrac{dh}{dt} = \sin(35^{\text{o}}) \dfrac{dL}{dt} = \sin(35^{\text{o}}) (10 \text{ ft/sec}) \approx 5.7$ ft/sec.

15. $V = s^3 - \dfrac{4}{3} \pi r^3$. $r = \dfrac{1}{2}(\text{diameter}) = 4$ ft, $\dfrac{dr}{dt} = 1$ ft/hr , $s = 12$ ft, $\dfrac{ds}{dt} = 3$ ft/hr.

$\dfrac{dV}{dt} = 3s^2 \dfrac{ds}{dt} - 4\pi r^2 \dfrac{dr}{dt} = 3(12 \text{ ft})^2 (3 \text{ ft/hr}) - 4\pi(4 \text{ ft})^2(1 \text{ ft/hr}) \approx 1094.94 \text{ ft}^3/\text{hr}.$ The volume is **increasing** at

about $1094.94 \text{ ft}^3/\text{hr}.$

17. Given: $\dfrac{dV}{dt} = k \cdot 2\pi r^2$ with k constant. We also have $V = \dfrac{2}{3} \pi r^3$ so $\dfrac{dV}{dt} = 2\pi r^2 \dfrac{dr}{dt}$.

Therefore, $k \cdot 2\pi r^2 = 2\pi r^2 \dfrac{dr}{dt}$ so $\dfrac{dr}{dt} = k$. The radius r   is changing at a constant rate.

19. (a) $A = 5x$ (b) $\dfrac{dA}{dx} = 5$ for all $x > 0$.   (c) $A = 5t^2$

(d) $\dfrac{dA}{dt} = 10t$. When $t = 1$, $\dfrac{dA}{dt} = 10$; when $t = 2$, $\dfrac{dA}{dt} = 20$; when $t = 3$, $\dfrac{dA}{dt} = 30$.

(e) $A = 10 + 5 \cdot \sin(t)$. $\dfrac{dA}{dt} = 5 \cdot \cos(t)$.

21. (a) $\tan(10^{\text{o}}) = \dfrac{40}{x}$ so $x = \dfrac{40}{\tan(10^{\text{o}})} \approx 226.9$ ft.

For parts (b) and (c) we need to work in radians since our formulas for the derivatives of the trigonometric

functions assume that the angles are measured in radians: $360^{\text{o}} \approx 2\pi$ radians so $10^{\text{o}} \approx 0.1745$ radians and

$2^{\text{o}} \approx 0.0349$ radians,

(b) $x = \dfrac{40}{\tan(\theta)} = 40 \cot(\theta)$ so $\dfrac{dx}{dt} = -40 \csc^2(\theta) \dfrac{d\theta}{dt}$ and

$\dfrac{d\theta}{dt} = \dfrac{\sin^2(\theta) \dfrac{dx}{dt}}{-40} = \dfrac{\sin^2(0.1745) (-25)}{-40} \approx \dfrac{(0.1736)^2(-25)}{-40} \approx 0.0188$ radians/min $\approx 1.079^{\text{o}}$ /min.

(c) $\dfrac{dx}{dt} -40 \csc^2(\theta) \dfrac{d\theta}{dt} = -40 \dfrac{1}{\sin^2(\theta)} \dfrac{d\theta}{dt} \approx -40 \dfrac{1}{(0.1736)^2} (0.0349) \approx -46.3$ ft/min. (The " $-$ "

indicates the distance to the sign is decreasing: you are approaching the sign.) Your speed is 46.3 ft/min.

Section 3.8

1. See Fig. 15

3. $x_0 = 1$: a. $x_0 = 4$: b.

Fig. 1

5. $x_0 = 1$: $1, 2, 1, 2, 1, ...$
    $x_0 = 5$: x_1 is undefined since $f\,'(5) = 0$.
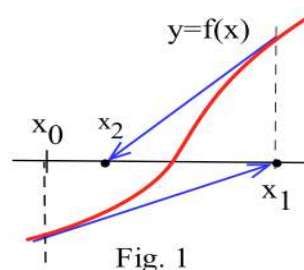
7. If f is differentiable, then $f\,'(x_0) = 0$ and $x_1 = x_0 - \dfrac{f(x_0)}{f\,'(x_0)}$ is undefined.

9. $f(x) = x^4 - x^3 - 5$, $x_0 = 2$. $f\,'(x) = 4x^3 - 3x^2$.

    Then $x_1 = 2 - \dfrac{3}{20} = \dfrac{37}{20} = 1.85$ and $x_2 = 1.85 - \dfrac{1.85^4 - 1.85^3 - 5}{4(1.85)^3 - 3(1.85)^2} \approx 1.824641$.

11. $f(x) = x - \cos(x)$, $f\,'(x) = 1 + \sin(x)$, $x_0 = 0.7$.   Then $x_1 = 0.7394364978$, $x_2 = 0.7390851605$, and   root
    ≈ 0.74 .

13. $\dfrac{x}{x+3} = x^2 - 2$ so we can use $f(x) = x^2 - 2 - \dfrac{x}{x+3}$. If $x_0 = -4$, then the iterates $x_n \to -3.3615$. If x_0
    $= -2$, then $x_n \to -1.1674$. If $x_0 = 2$, then the iterates $x_n \to 1.5289$.

15. $x^5 - 3 = 0$ and $x_0 = 1$. Then $x_n \to 1.2457$.

17. $f(x) = x^3 - A$ so $f\,'(x) = 3x^2$.

    Then $x_{n+1} = x_n - \dfrac{(x_n)^3 - A}{3(x_n)^2} = x_n - \dfrac{x_n}{3} + \dfrac{A}{3(x_n)^2} = \dfrac{1}{3}\left\{ 2x_n + \dfrac{A}{(x_n)^2} \right\}$.

19. (a) $2(0) - \text{INT}(\,2(0)\,) = 0 - 0 = 0$.

    (b) $2(\,1/2\,) - \text{INT}(\,2(\,1/2\,)\,) = 1 - 1 = 0$.
         $2(\,1/4\,) - \text{INT}(\,2(\,1/4\,)\,) = 1/2 - 0 = 1/2 \to 0$.
         $2(\,1/8\,) - \text{INT}(\,2(\,1/8\,)\,) = 1/4 - 0 = 1/4 \to 1/2 \to 0$.
         $2(\,1/2^n\,) - \text{INT}(\,2(\,1/2^n\,)\,) = 1/2^{n-1} - 0 = 1/2^{n-1} \to 1/2^{n-2} \to ... \to 1/4 \to 1/2 \to 0$.

21. (a) If $0 \le x \le 1/2$, then f stretches x to twice its value, $2x$.
         If $1/2 < x \le 1$, then f stretches x to twice its value ($2x$) and "folds" the part above the
             value 1 ($2x - 1$) to below 1: $1 - (\,2x - 1\,) = 2 - 2x$.

    (b) $f(\,2/3\,) = 2/3$.
         $f(\,2/5\,) = 4/5$, $f(\,4/5\,) = 2/5$, and the values continues to cycle.
         $f(\,2/7\,) = 4/7$, $f(\,4/7\,) = 6/7$, $f(\,6/7\,) = 2/7$, and the values continues to cycle.
         $f(\,2/9\,) = 4/9$, $f(\,4/9\,) = 8/9$, $f(\,8/9\,) = 2/9$, and the values continues to cycle.

    (c) $0.1, 0.2, \mathbf{0.4}, 0.8, \mathbf{0.4}, 0.8,$ and the pair of values 0.4 and 0.8 continue to cycle.
         $0.105, 0.210, 0.42, 0.84, \mathbf{0.32}, 0.64, 0.72, 0.56, 0.88, 0.24, 0.48, 0.96, 0.08, 0.16, \mathbf{0.32}$, ...
         $0.11, 0.22, 0.44, \mathbf{0.88}, 0.24, 0.48, 0.96, 0.08, 0.16, 0.32, 0.64, 0.72, 0.56, \mathbf{0.88}$, ...

    (d) Probably so.

Section 3.9

1. See Fig. 1.

$(2 + \Delta x,\ g(2) + g\ '(2) \cdot \Delta x)$

"error"

$(2 + \Delta x,\ g(2 + \Delta x))$

Fig. 1

3. (a) $f(4) = 2,\ f\ '(4) = \dfrac{1}{2\sqrt{4}}\ = \dfrac{1}{4}$.

   Then $y - 2 = \dfrac{1}{4}(x - 4)$ so $y = \dfrac{1}{4}\ x + 1$.

   $\sqrt{4.2}\ = f(4.2) \approx \dfrac{1}{4}(4.2)\ + 1 = 2.05$.

   (b) $f(81) = 9,\ f\ '(81) = \dfrac{1}{18}$ so

   $y - 9 = \dfrac{1}{18}(x - 81)$ so $y = \dfrac{1}{18}(x - 81)\ + 9$.

   $\sqrt{80}\ = f(80) \approx \dfrac{1}{18}(80 - 81)\ + 9 = 9 - \dfrac{1}{18}\ \approx 8.944$.

   (c) $f(0) = 0,\ f\ '(0) = 1$ so $y - 0 = 1(x - 0)$ and $y = x$. Then $\sin(0.3) = f(0.3) \approx 0.3$.

4. (a) $f(1) = 0,\ f\ '(1) = 1$ so $y - 0 = 1(x - 1)$. Then $\ln(1.3) = f(1.3) \approx 1.3 - 1 = 0.3$.

   (b) $f(0) = 1,\ f\ '(0) = 1$ so $y - 1 = 1(x - 0)$ and $y = 1 + x$. Then $e^{0.1} = f(0.1) \approx 1.1$.

   (c) $f(1) = 1,\ f\ '(1) = 5$ so $y - 1 = 5(x - 1)$ or $y = 1 + 5(x{-}1)$. Then $(1.3)^5 \approx 1 + 5(0.03) = 1.15$.

5. $f(x) = (1 + x)^n,\ f\ '(x) = n(1 + x)^{n-1}$, $f(0) = 1$ and $f\ '(0) = n$. Then $y - 1 = n(x - 0)$ or $y = 1 + nx$.
   Therefore, $(1 + x)^n \approx 1 + nx$ (when x is close to 0).

6. (a) $f(x) = (1 - x)^n,\ f\ '(x) = -n(1 - x)^{n-1}$, and $f\ '(0) = -n$. Then $y - 1 = -n(x - 0)$ or $y = 1 - nx$.
   Therefore, $(1 - x)^n \approx 1 - nx$ (when x is close to 0).

   (b) $f(x) = \sin(x),\ f\ '(x) = \cos(x)$, and $f\ '(0) = 1$. Then $y - 0 = 1(x - 0)$ or $y = x$.
   Therefore, $\sin(x) \approx x$ for x close to 0.

   (c) $f(x) = e^x,\ f\ '(x) = e^x$, and $f\ '(0) = 1$. Then $y - 1 = 1(x - 0)$ or $y = x + 1$. $e^x \approx x + 1$.

7. (a) $f(x) = \ln(1 + x),\ f\ '(x) = \dfrac{1}{1+x}$, and $f\ '(0) = 1$. Then $y - 0 = 1(x - 0)$ so $y = x$ and $\ln(1 + x) \approx x$.

   (b) $f(x) = \cos(x),\ f\ '(x) = -\sin(x)$, and $f\ '(0) = 0$. Then $y - 1 = 0(x - 0)$ so $y = 1$ and $\cos(x) \approx 1$.

   (c) $f(x) = \tan(x),\ f\ '(x) = \sec^2(x)$, and $f\ '(0) = 1$. Then $y - 0 = 1(x - 0)$ so $y = x$ and $\tan(x) \approx x$.

   (d) $f(x) = \sin(\dfrac{\pi}{2} + x)$, $f\ '(x) = \cos(\dfrac{\pi}{2} + x)$, and $f\ '(0) = \cos(\dfrac{\pi}{2} + 0) = 0$. Then $y - 1 = 0(x - 0)$ so $y = 1$

   and $\sin(\dfrac{\pi}{2} + x) \approx 1$.

9. (a) Area $A(x) = (base)(height) = x(x^2 + 1) = x^3 + x$. Then $A'(x) = 3x^2 + 1$ so $A'(2) = 13$

   Then $\Delta A \approx A'(2) \cdot \Delta x = (13)(2.3 - 2) = 3.9$

   (b) (base)(height) $= (2.3)((2.3)^2 + 1) = 14.467$. Then the actual difference $= 14.467 - (2)(2^2 + 1) = 4.467$.
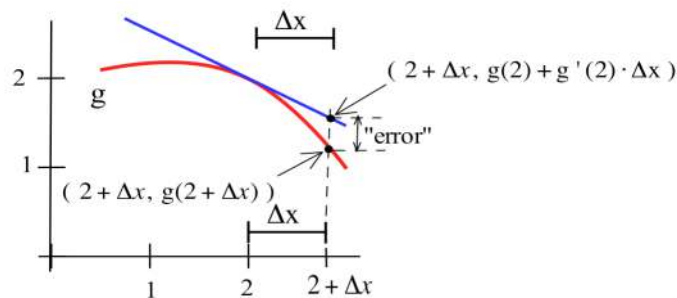
11. $V = \pi r^2 h = 2\pi r^2$ and $\Delta V = 2\pi 2r \, \Delta r = 4\pi r \, \Delta r$. Since $V = 2\pi r^2 = 47.3$, we have $r = \sqrt{47.3/(2\pi)} \approx 2.7437$ cm.

We know $\Delta V = \pm 0.1$ so, using $\Delta V = 4\pi r \, \Delta r$, we have $\pm 0.1 = 4\pi(2.7437)\Delta r$ and

$\Delta r = \dfrac{\pm 0.1}{4\pi(2.7437)} \approx \pm 0.0029$ cm.   The required tolerance is ± 0.0029 cm. (C.W. comment: "A coin 2 cm

high is one hell of a coin! My eyeball estimate is that $47.3 \ \text{cm}^3$ of gold weighs around 2 pounds.")

13. $V = x^3$. When $V = 87$, $x \approx 4.431$ cm. $\Delta V = 3x^2 \, \Delta x$ so $\Delta x = \dfrac{\Delta V}{3x^2} \approx \dfrac{2}{3(4.431)^2} = 0.034$ cm.

15. $P = 2\pi \sqrt{\dfrac{L}{g}}$ with $g = 32 \ \text{ft/sec}^2$.   (a) $P = 2\pi \sqrt{\dfrac{2}{32}} = \dfrac{\pi}{2} \approx 1.57$ seconds.

(b) $1 = 2\pi \sqrt{\dfrac{L}{32}}$ so $L = \dfrac{8}{\pi^2} \approx 0.81$ feet.   (c) $dP = \dfrac{2\pi}{\sqrt{32}} \dfrac{1}{2\sqrt{2}} \, dL = \dfrac{2\pi}{4\sqrt{2}} \dfrac{1}{2\sqrt{2}} (0.1) \approx 0.039$ seconds.

(d) 2 in/hr = 1/6 ft/hr $= \dfrac{1}{21600}$ ft/sec. $\dfrac{dP}{dt} = \dfrac{2\pi}{4\sqrt{2}} \dfrac{1}{2\sqrt{24}} \dfrac{1}{21600} \approx 5.25 \ \text{x} \ 10^{-6}$.

17. (a) $df = f\,'(2) \, dx \approx (0)(1) = 0$ (b) $df = f\,'(4) \, dx \approx (0.3)(-1) = -0.3$ (c) $df = f\,'(3) \, dx \approx (0.5)(2) = 1$

19. (a) $f(x) = x^2 - 3x$. $f\,'(x) = 2x - 3$. $df = (2x - 3) \, dx$.   (b) $f(x) = e^x$. $f\,'(x) = e^x$. $df = e^x \, dx$.

(c) $f(x) = \sin(5x)$. $f\,'(x) = 5 \cos(5x)$. $df = 5 \cos(5x) \, dx$.

(d) $f(x) = x^3 + 2x$. $f\,'(x) = 3x^2 + 2$. $df = (3x^2 + 2) \, dx$. When $x = 1$ and $dx = 0.2$, $df = (3 \cdot 1^2 + 2) \, (0.2) = 1$.

(e) $f\,'(x) = 1/x$. $df = \dfrac{1}{x} \, dx$. When $x = e$ and $dx = -0.1$, $df = = \dfrac{1}{e} (-0.1) = -\dfrac{1}{10e}$.

(f) $f(x) = \sqrt{2x + 5}$. $f\,'(x) = \dfrac{1}{\sqrt{2x + 5}}$. $df = \dfrac{1}{\sqrt{2x + 5}} \, dx$. When $x = 22$ and $dx = 3$, $df = \dfrac{1}{\sqrt{49}} (3) = \dfrac{3}{7}$.

Section 3.10

1.   (a) $x^2 + y^2 = 100$ so $2x + 2yy' = 0$ and $y' = -x/y$. At $(6,8)$, $y' = -6/8 = -3/4$.

(b) $y = \sqrt{100 - x^2}$ so $y' = \dfrac{-x}{\sqrt{100 - x^2}}$. At $(6,8)$, $y' = \dfrac{-6}{\sqrt{100 - 36}} = \dfrac{-6}{8} = -\dfrac{3}{4}$.

3.   (a) $x^2 - 3xy + 7y = 5$ so $2x - 3(y + xy') + 7y' = 0$ and $y' = \dfrac{3y - 2x}{7 - 3x}$. At $(2,1)$, $y' = \dfrac{3-4}{7-6} = -1$.

(b) $y = \dfrac{5 - x^2}{7 - 3x}$ so $y' = \dfrac{(7 - 3x)(-2x) - (5 - x^2)(-3)}{(7 - 3x)^2}$. At $(2,1)$, $y' = \dfrac{(1)(-4) - (1)(-3)}{(1)^2} = -1$.

5.   (a) $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$ so $\dfrac{2x}{9} + \dfrac{2y}{16} \, y' = 0$ and $y' = -\dfrac{16}{9} \dfrac{x}{y}$. At $(0,4)$, $y' = 0$.

(b) $y = 4 \sqrt{1 - \dfrac{x^2}{9}} = \dfrac{4}{3}\sqrt{9 - x^2}$ so $y' = \dfrac{4}{3} \dfrac{-x}{\sqrt{9 - x^2}}$. At $(0,4)$, $y' = 0$.

7.   (a) $\ln(y) + 3x - 7 = 0$ so $\frac{1}{y}\,y' + 3 = 0$ and $y' = -3y$. At (2,e), $y' = -3e$.

(b) $y = e^{7-3x}$ so $y' = -3\,e^{7-3x}$. At (2,e), $y' = -3\,e^{7-6} = -3e$.

9.   (a) $x^2 - y^2 = 16$ so $2x - 2yy' = 0$ and $y' = x/y$. At (5,–3), $y' = -5/3$.

(b) The point $(5, -3)$ is on the bottom half of the circle so

$$y = -\sqrt{x^2 - 16} \quad . \text{ Then } y' = -\frac{x}{\sqrt{x^2 - 16}} \quad . \text{ At } (5,\text{–}3),\ y' = -\frac{5}{\sqrt{25 - 16}} = -\frac{5}{3} \ .$$

11. $x = 4y - y^2$ so, differentiating each side, $1 = 4y' - 2y\,y'$ and $y' = \frac{1}{4 - 2y}$.

At (3,1), $y' = \frac{1}{4 - 2(1)} = \frac{1}{2}$.   At (3,3), $y' = \frac{1}{4 - 2(3)} = -\frac{1}{2}$.

At (4,2), $y' = \frac{1}{4 - 2(2)}$ is undefined (the tangent line is vertical).

13. $x = y^2 - 6y + 5$. Differentiating each side, $1 = 2y\,y' - 6\,y'$ and $y' = \frac{1}{2y - 6}$.

At (5,0), $y' = \frac{1}{2(0) - 6} = -\frac{1}{6}$. At (5,6), $y' = \frac{1}{2(6) - 6} = \frac{1}{6}$.

At (–4,3), $y' = \frac{1}{2(3) - 6}$ is undefined (vertical tangent line).

15. $3y^2 y' - 5y' = 10x$ so $y' = \frac{10x}{3y^2 - 5}$. At (1,3), $m = 10/22 = 5/11$.

17. $y^2 + \sin(y) = 2x - 6$ so $2yy' + \cos(y)y' = 2$ and $y' = \frac{2}{2y + \cos(y)}$. At (3,0), $m = \frac{2}{0+1} = 2$.

19. $e^y + \sin(y) = x^2 - 3$ so $e^y y' + \cos(y)y' = 2x$ and $y' = \frac{2x}{e^y + \cos(y)}$. At (2,0), $m = \frac{4}{1+1} = 2$.

21. $x^{2/3} + y^{2/3} = 5$ so $\frac{2}{3}\,x^{-1/3} + \frac{2}{3}\,y^{-1/3}y' = 0$ and $y' = -(x/y)^{-1/3}$. At (8,1), $m = -(8/1)^{-1/3} = -\frac{1}{2}$.

23.   Using implicit differentiation, $2x + xy' + y + 2yy' + 3 - 7y' = 0$ so $y' = \frac{-2x - y - 3}{x + 2y - 7}$.

At (1,2), $y' = \frac{-2(1) - (2) - 3}{(1) + 2(2) - 7} = \frac{7}{2}$.

25. $y = Ax^2 + Bx + C$ so $y' = 2Ax + B$. (explicitly)

$x = Ay^2 + By + C$ so (implicitly) $1 = 2Ayy' + By'$ and $y' = \frac{1}{2Ay + B}$.

27. $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

Then $2Ax + Bxy' + By + 2Cyy' + D + Ey' = 0$ so $y' = \frac{-2Ax - By - D}{Bx + 2Cy + E}$.

28. $x^2 + y^2 = r^2$ so $2x + 2yy' = 0$ and $y' = -\frac{x}{y}$. The slope of the tangent line is –x/y. The slope of the line

through the points (0,0) and (x,y) is y/x, so the slopes of the lines are negative reciprocals of each other

and the lines are perpendicular.

29.   From problem 23, $y' = \dfrac{-2x - y - 3}{x + 2y - 7}$ so $y' = 0$ when $-2x - y - 3 = 0$ and $y = -2x - 3$.

Substituting $y = -2x - 3$ into the original equation, we have

$x^2 + x(-2x - 3) + (-2x - 3)^2 + 3x - 7(-2x - 3) + 4 = 0$ so

$3x^2 + 26x + 34 = 0$ and $x = \dfrac{-26 \pm \sqrt{26^2 - 4(3)(34)}}{2(3)} = \dfrac{-26 \pm \sqrt{268}}{6} \approx -1.605$ and -7.062.

If $x \approx -1.605$ (point A), then $y = -2x - 3 \approx -2(-1.605) - 3 = 0.21$. Point A is **(−1.605, 0.21)**.

If $x \approx -7.062$ (point C), then $y = -2x - 3 \approx -2(-7.062) - 3 = 11.124$. Point C is **(−7.062, 11.124)**.

31.   From the solution to problem 29, point C is $(-7.062, 11.124)$.

At D, $y' = \dfrac{-2x - y - 3}{x + 2y - 7}$ is undefined so $x + 2y - 7 = 0$ and $x = 7 - 2y$.

Substituting $x = 7 - 2y$ into the original equation, we have

$(7-2y)^2 + (7-2y)y + y^2 + 3(7-2y) - 7y + 4 = 0$ so (simplifying) $3y^2 - 34y + 74 = 0$. Then

$y = \dfrac{34 \pm \sqrt{34^2 - 4(3)(74)}}{2(3)} = \dfrac{34 \pm \sqrt{268}}{6} \approx 8.398$ and 2.938.

If $y \approx 2.938$ (point D), then $x = 7 - 2y \approx 7 - 2(2.938) = 1.124$. Point D is **(1.124, 2.938)**.

Point B is $(-9.79, 8.395)$.

33.   (a) $y = (x^2 + 5)^7 (x^3 - 1)^4$.

$y' = (x^2 + 5)^7 (4)(x^3 - 1)^3 (3x^2) + (7)(x^2 + 5)^6 (2x)(x^3 - 1)^4$

$= (x^2 + 5)^6 (x^3 - 1)^3 (2x)\{6x^3 + 30x + 7x^3 - 7\} = (x^2 + 5)^6 (x^3 - 1)^3 (2x)\{13x^3 + 30x - 7\}$.

(b) $\ln(y) = 7\ln(x^2 + 5) + 4\ln(x^3 - 1)$

$\dfrac{y'}{y} = \dfrac{14x}{x^2 + 5} + \dfrac{12x^2}{x^3 - 1}$ so

$y' = y\left\{ \dfrac{14x}{x^2 + 5} + \dfrac{12x^2}{x^3 - 1} \right\} = (x^2 + 5)^7 (x^3 - 1)^4 \left\{ \dfrac{14x}{x^2 + 5} + \dfrac{12x^2}{x^3 - 1} \right\}$ and this is the

same as in part (a). (Really it is.)

35.   (a) $y = x^5 (3x + 2)^4$. $y' = x^5 D((3x + 2)^4) + (3x + 2)^4 D(x^5) = x^5 \cdot 4(3x + 2)^3 (3) + (3x + 2)^4 \cdot (5x^4)$

(b) $\ln(y) = 5\ln(x) + 4\ln(3x + 2)$.

$\dfrac{y'}{y} = \dfrac{5}{x} + \dfrac{12}{3x + 2}$ so $y' = y\left\{ \dfrac{5}{x} + \dfrac{12}{3x + 2} \right\} = \{ x^5 (3x + 2)^4 \}\left\{ \dfrac{5}{x} + \dfrac{12}{3x + 2} \right\}$

and this is the same as in part (a).

37.   (a) $y = e^{\sin(x)}$ so $y' = e^{\sin(x)} \cos(x)$.

(b) $\ln(y) = \sin(x)$ so $\dfrac{y'}{y} = \cos(x)$ and $y' = y\cos(x) = e^{\sin(x)} \cos(x)$.

39.   (a) $y = \sqrt{25 - x^2}$ so $y' = \dfrac{-x}{\sqrt{25 - x^2}}$.

(b) $\ln(y) = \frac{1}{2} \ln(25 - x^2)$ so $\frac{y'}{y} = \frac{1}{2}\frac{-2x}{25 - x^2} = \frac{-x}{25 - x^2}$. Then

$$y' = y\frac{-x}{25 - x^2} = \sqrt{25 - x^2}\ \frac{-x}{25 - x^2} = \frac{-x}{\sqrt{25 - x^2}} \ .$$

41. $y = x^{\cos(x)}$ so $\ln(y) = \cos(x) \cdot \ln(x)$ and $\frac{y'}{y} = \cos(x) \cdot \frac{1}{x} - \ln(x) \cdot \sin(x)$.

Then $y' = y \{ \cos(x) \cdot \frac{1}{x} - \ln(x) \cdot \sin(x) \} = x^{\cos(x)} \{ \cos(x) \cdot \frac{1}{x} - \ln(x) \cdot \sin(x) \}$.

43. $y = x^4(x - 2)^7 \sin(3x)$ so $\ln(y) = 4\ln(x) + 7\ln(x - 2) + \ln(\sin(3x))$ and $\frac{y'}{y} = \frac{4}{x} + \frac{7}{x - 2} + \frac{3\cos(3x)}{\sin(3x)}$.

Then $y' = y \{ \frac{4}{x} + \frac{7}{x - 2} + \frac{3\cos(3x)}{\sin(3x)} \} = x^4(x - 2)^7 \sin(3x) \{ \frac{4}{x} + \frac{7}{x - 2} + \frac{3\cos(3x)}{\sin(3x)} \}$.

45. $\ln(y) = x \cdot \ln(3 + \sin(x))$ so $\frac{y'}{y} = \frac{x \cos(x)}{3 + \sin(x)} + \ln(3 + \sin(x))$. Then

$y' = (3 + \sin(x))^x \{ \frac{x \cos(x)}{3 + \sin(x)} + \ln(3 + \sin(x)) \}$.

47.
x	f '(x)
1	1(1.2) = 1.2
2	9(1.8) = 16.2
3	64(2.1) = 134.4

49.
x	f '(x)
1	5(−1) = −5
2	2(0) = 0
3	7(2) = 14

51. $\ln(f \cdot g) = \ln(f) + \ln(g)$ so $\frac{D(f \cdot g)}{f \cdot g} = \frac{f'}{f} + \frac{g'}{g}$. Then $D(f \cdot g) = (f \cdot g) \{ \frac{f'}{f} + \frac{g'}{g} \} = f'\cdot g + g'\cdot f$.

52. $\ln(f/g) = \ln(f) - \ln(g)$ so $\frac{D(f/g)}{f/g} = \frac{f'}{f} - \frac{g'}{g}$. Then

$D(f/g) = \frac{f}{g} \{ \frac{f'}{f} - \frac{g'}{g} \} = \frac{f'}{g} - \frac{f \cdot g'}{g^2} = \frac{f \cdot g' - g \cdot f'}{g^2}$.

53. $\ln(f \cdot g \cdot h) = \ln(f) + \ln(g) + \ln(h)$ so $\frac{D(f \cdot g \cdot h)}{f \cdot g \cdot h} = \frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h}$.

Then $D(f \cdot g \cdot h) = (f \cdot g \cdot h) \{ \frac{f'}{f} + \frac{g'}{g} + \frac{h'}{h} \} = f'\cdot g \cdot h + f \cdot g'\cdot h + f \cdot g \cdot h'$.

54. $\ln(a^x) = x \ln(a)$ so $\frac{D(a^x)}{a^x} = \ln(a)$ and $D(a^x) = a^x \ln(a)$.

55.   On your own.

4.1   FINDING MAXIMUMS AND MINIMUMS

In theory and applications, we often want to maximize or minimize some quantity. An engineer may want to maximize the speed of a new computer or minimize the heat produced by an appliance. A manufacturer may want to maximize profits and market share or minimize waste. A student may want to maximize a grade in calculus or minimize the hours of study needed to earn a particular grade.

Also, many natural objects follow minimum or maximum principles, so if we want to model some natural phenomena we may need to maximize or minimize. A light ray travels along a "minimum time" path. The shape and surface texture of some animals tend to minimize or maximize heat loss. Systems reach equilibrium when their potential energy is minimized. A basic tenet of evolution is that a genetic characteristic which maximizes the reproductive success of an individual will become more common in a species.

Calculus provides tools for analyzing functions and their behavior and for finding maximums and minimums.

Methods for Finding Maximums and Minimums

We can try to find where a function f is largest or smallest by evaluating f at lots of values of x, a method which is not very efficient and may not find the exact place where f achieves its extreme value. However, if we try hundreds or thousands of values for x , then we can often find a value of f which is close to the maximum or minimum. In general, this type of exhaustive search is only practical if you have a computer do the work.

The graph of a function is a visual way of examining lots of values of f, and it is a good method, particularly if you have a computer to do the work for you. However, it is inefficient, and we still may not find the exact location of the maximum or minimum.

Calculus provides ways of drastically narrowing the number of points we need to examine to find the exact locations of maximums and minimums. Instead of examining f at thousands of values of x, calculus can often guarantee that the maximum or minimum must occur at one of 3 or 4 values of x, a substantial improvement in efficiency.

Fig. 1

A Little Terminology

Before we examine how calculus can help us find maximums and minimums, we need to define the concepts we will develop and use.
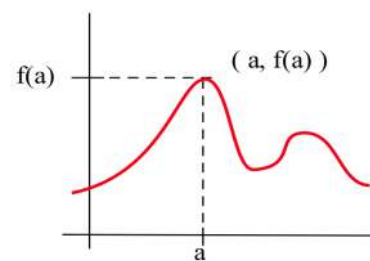
Definitions: f has a **maximum** or **global maximum** at a if $f(a) \geq f(x)$ for all x in the domain of f.

The maximum value of f **is f(a)**, and this maximum value of f **occurs at a.**

The **maximum point** on the graph of f is **(a , f(a)).** (Fig. 1)

> **Definition:**   f has a **local** or **relative maximum** at a if f(a) ≥ f(x) for all x   near a
>
>   or in some open interval which contains a.

Global and local minimums are defined similarly by replacing the **≥** with **≤** in the previous definitions.

> **Definition:**   f has a **global extreme** at a if f(a) is a **global maximum or minimum.**
>
>   f has a **local extreme** at a if f(a) is a **local maximum or minimum.**
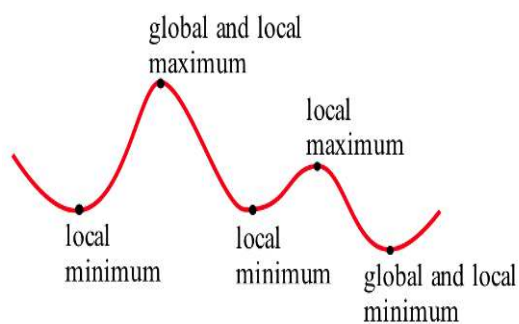
Fig. 2

The local and global extremes of the function in Fig. 2 are labeled. You should notice that every global extreme is also a local extreme, but there are local extremes which are not global extremes. If h(x) is the height of the earth above sea level at the location x, then the global maximum of h is h(summit of Mt. Everest) = 29,028 feet. The local maximum of h for the United States is h(summit of Mt. McKinley) = 20,320 feet. The local minimum of h for the United States is h(Death Valley) = – 282 feet.

Practice 1:   The table shows the annual calculus enrollments at a large university. Which years had relative maximum or minimum calculus enrollments? What were the global maximum and minimum enrollments in calculus?

year	1980	81	82	83	84	85	86	87	88	89	90
enrollment	1257	1324	1378	1336	1389	1450	1523	1582	1567	1545	1571

Finding Maximums and Minimums of a Function

One way to narrow our search for a maximum value of a function f is to eliminate those values of x which, for some reason, cannot possibly make f maximum.

> **Theorem :**   If $f'(a) > 0$ or $f'(a) < 0$,
>
>   then f(a) is not a local maximum or minimum. (Fig. 3)

Proof: Assume that f '(a) > 0. By definition, $f'(a) = \lim\limits_{\Delta x \to 0} \dfrac{f(a + \Delta x) - f(a)}{\Delta x}$,

so $\lim\limits_{\Delta x \to 0} \dfrac{f(a + \Delta x) - f(a)}{\Delta x} > 0$ and the right and left limits are both positive:

$$\lim\limits_{\Delta x \to 0^+} \frac{f(a + \Delta x) - f(a)}{\Delta x} > 0 \quad \text{and} \quad \lim\limits_{\Delta x \to 0^-} \frac{f(a + \Delta x) - f(a)}{\Delta x} > 0.$$
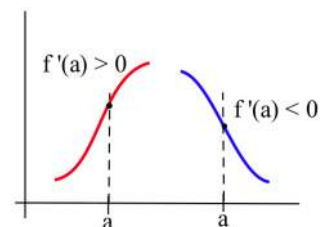
Fig. 3

Since the right limit, $\Delta x \to 0^+$, is positive, there are values of $\Delta x > 0$ so $\dfrac{f(a + \Delta x) - f(a)}{\Delta x} > 0.$

Multiplying each side of this last inequality by the positive Δx, we have $f(a + \Delta x) - f(a) > 0$ and

$f(a + \Delta x) > f(a)$ so $f(a)$ is not a maximum.

Since the left limit, $\Delta x \to 0^-$, is positive, there are values of $\Delta x < 0$ so $\dfrac{f(a + \Delta x) - f(a)}{\Delta x} > 0.$

Multiplying each side of the last inequality by the negative Δx, we have that $f(a + \Delta x) - f(a) < 0$ and

$f(a + \Delta x) < f(a)$ so $f(a)$ is not a minimum.

The proof for the "f '(a) < 0" case is similar.

When we evaluate the derivative of a function f at a point $x = a$, there are only four possible outcomes: f '(a) > 0, f '(a) < 0 , f '(a) = 0 or f '(a) is undefined. If we are looking for extreme values of f, then we can eliminate those points at which f ' is positive or negative, and only two possibilities remain: f '(a) = 0 or f '(a) is undefined.

> **Theorem :**   If   f is defined on an open interval, and f(a) is a local extreme of f ,
>
> then   either   f '(a) = 0   or   f is not differentiable at a.

Example 1:   Find the local extremes of $f(x) = x^3 - 6x^2 + 9x + 2$ for all values of x.

Solution: An extreme value of f can occur only where f '(x) = 0 or where f is not differentiable.

$f '(x) = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 1)(x - 3)$ so f '(x) = 0 only at x = 1 and x = 3. f ' is

a polynomial, so f is differentiable for all x.

The only possible locations of local extremes of f are at

x = 1 and x =3. We don't know yet whether f(1) or

f(3) is a local extreme of f, but we can be certain that no

other point is a local extreme. The graph of f (Fig. 4)

shows that $(1, f(1)) = (1, 6)$ is a local maximum and

$(3, f(3)) = (3, 2)$ is a local minimum. This function

does not have a global maximum or minimum.

Fig. 4
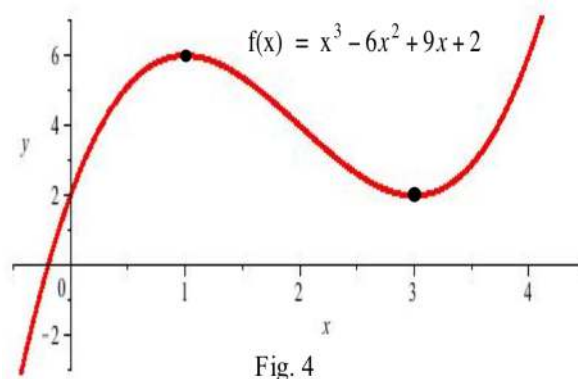
Practice 2:   Find the local extremes of $f(x) = x^2 + 4x - 5$ and $g(x) = 2x^3 - 12x^2 + 7$.

It is important to recognize that the conditions " f '(a) = 0 " or " f not differentiable at a " do **not**

guarantee that f(a) is a local maximum or minimum. They only say that f(a) **might be** a local extreme or

that f(a) is a **candidate** for being a local extreme.

Example 2:   Find all local extremes of $f(x) = x^3$.

Solution: $f(x) = x^3$ is differentiable for all x, and $f'(x) = 3x^2$. The only place where

f '(x) = 0 is at x = 0, so the only candidate is the point (0,0). But if x > 0 then

$f(x) = x^3 > 0 = f(0)$, so f(0) is not a local maximum. Similarly, if x < 0 then

$f(x) = x^3 < 0 = f(0)$ so f(0) is not a local minimum. The point (0,0) is the only

candidate to be a local extreme of f, and this candidate did not turn out to be a local

extreme of f. The function $f(x) = x^3$ does not have any local extremes. (Fig. 5)
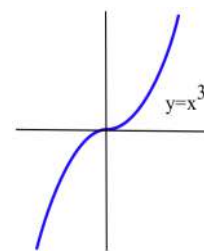
Fig. 5

> If   f '(a) = 0 or f is not differentiable at a
>
> then   the point (a , f(a)) is a <u>candidate</u> to be a local extreme and **may or may not**
>
>   be a local extreme.

Practice 3:   Sketch the graph of a differentiable function f which satisfies the conditions:

(i)   f(1) = 5, f(3) = 1, f(4) = 3 and f(6) = 7,   (ii)   f '(1) = 0, f '(3) = 0 , f '(4) = 0 and f '(6) = 0 ,

(iii)   the only local maximums of f are at (1,5) and (6,7), and the only local minimum is at   (3,1).

Is f(a) a Maximum or Minimum or Neither?
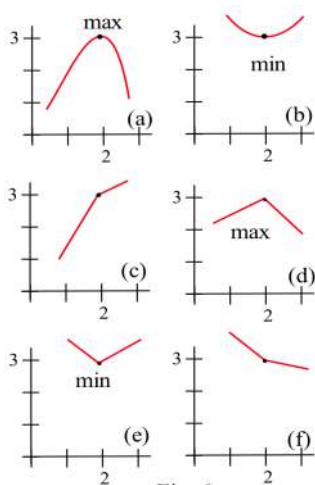
Once we have found the candidates (a, f(a)) for extreme points of f, we still have

the problem of determining whether the point is a maximum, a minimum or neither.

One method is to graph (or have your calculator graph) the function near a, and

then draw your conclusion from the graph. All of the graphs in Fig. 6 have f(2) =

3, and, on each of the graphs, f '(2) either equals 0 or is undefined. It is clear from

the graphs that the point (2,3) is a local maximum in (a) and (d), (2,3) is a local

minimum in (b) and (e), and (2,3) is not a local extreme in (c) and (f).

In sections 3.3 and 3.4, we will investigate how information about the first and

second derivatives of f can help determine whether the candidate (a, f(a)) is a

maximum, a minimum, or neither.

Fig. 6

Endpoint Extremes

So far we have been discussing finding extreme values of functions over the

entire real number line or on an open interval, but, in practice, we may need to

find the extreme of a function over some closed interval [c , d]. If the extreme

value of f occurs at x = a between c and d, c < a < d , then the previous

reasoning and results still apply: either f '(a) = 0 or f is not differentiable at a.
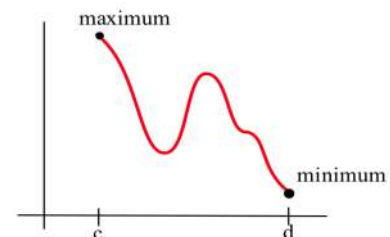
Fig. 7

On a closed interval, however, there is one more possibility: an extreme can occur at an **endpoint** of the closed interval (Fig. 7), at $x = c$ or $x = d$.

Practice 4:   List all of the local extremes $(a , f(a))$ of the function in

Fig. 8 on the interval $[1,4]$ and state whether (i) $f '(a) = 0$ or

(ii) f is not differentiable at a or (iii) a is an endpoint.
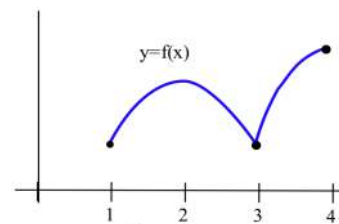
Fig. 8

Example 3:   Find the extreme values of $f(x) = x^3 - 3x^2 - 9x + 5$ for $-2 \le x \le 6$.

Solution: $f '(x) = 3x^2 - 6x - 9 = 3(x + 1)(x - 3)$. We need to find where (i) $f '(x) = 0$, (ii) f is not

differentiable, and (iii) the endpoints.

   (i) $f '(x) = 3(x + 1)(x - 3) = 0$ when $x = \mathbf{-1}$ and $x = \mathbf{3}$.

   (ii)   f is a polynomial so it is differentiable everywhere.

   (iii)   The endpoints of the interval are $x = \mathbf{-2}$ and $x = \mathbf{6}$.

Altogether we have four points in the interval to examine, and any extreme values of f can only occur when

x is one of those four points: $f(-2) = 3$, $f(-1) = 10$, $f(3) = -22$, and $f(6) = 59$. The minimum of f on $[-2, 6]$ is -22 when $x = 3$, and the maximum of f on $[-2, 6]$ is 59 when $x = 6$.

Sometimes the function we need to maximize or minimize is more complicated, but the same methods work.

Example 4:   Find the extreme values of $f(x) = \frac{1}{3}\sqrt{64 + x^2} + \frac{1}{5}(10 - x)$ for $0 \le x \le 10$.

Solution: This function comes from an application we will examine in section 3.5. The only possible

locations of extremes are where $f '(x) = 0$ or $f '(x)$ is undefined or where x is an endpoint of the

interval $[0 , 10]$.

$$f '(x) = \mathbf{D}(\frac{1}{3} (64 + x^2)^{1/2} + \frac{1}{5}(10 - x)) = \frac{1}{3}\frac{1}{2}(64 + x^2)^{-1/2} (2x) - \frac{1}{5} = \frac{x}{3\sqrt{64 + x^2}} - \frac{1}{5} .$$

To determine where $f '(x) = 0$, we need to set the derivative equal to 0 and solve for x.

If $f '(x) = \frac{x}{3\sqrt{64 + x^2}} - \frac{1}{5} = 0$ then $\frac{x}{3\sqrt{64 + x^2}} = \frac{1}{5}$ so $\frac{x^2}{576 + 9x^2} = \frac{1}{25}$.

Then $16x^2 = 576$ so $x = \pm 6$, and the only point <u>in the interval</u> $[0,10]$ where $f '(x) = 0$ is at $x = 6$.

Putting $x = 6$ into the original equation for f gives $f(6) \approx 4.13$.

We can evaluate the formula for $f '(x)$ for any value of x , so the derivative is always defined.

Finally, the interval $[0 , 10]$ has two endpoints, $x = 0$ and $x = 10$. $f(0) \approx 4.67$ and $f(10) \approx 4.27$.

The maximum of f on $[0,10]$ must occur at one of the points $(0, 4.67)$, $(6, 4.13)$ and $(10, 4.27)$, and

the minimum must occur at one of these three points.

The maximum value of f is 4.67 at $x = 0$, and the minimum

value of f is 4.13 at x = 6 . The graph of f is shown in Fig. 9.

Practice 5:   Find the extreme values of

$$f(x) = \frac{1}{3}\sqrt{64 + x^2} + \frac{1}{5}(10 - x)   \quad \text{for } \mathbf{0 \leq x \leq 5}.$$
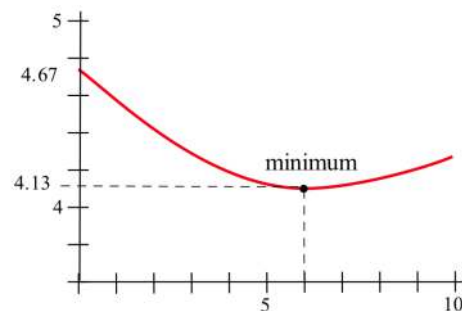
Fig. 9

Critical Numbers

The points at which a function might have an extreme value are called **critical numbers**.

> **Definitions:**   A **critical number** for a function f is a value x = **a** in the domain of f so
>
>   (i)   f '(**a**) = 0,
>
>   or   (ii)   f is not differentiable at **a**,
>
>   or   (iii)   **a**   is an endpoint

If we are trying to find the extreme values of f on an open interval c < x < d or on the entire number line, then there will not be any endpoints so there will not be any endpoint critical numbers to worry about.

We can now give a very succinct description of where to look for extreme values of a function:

> **An extreme value of f can only occur at a critical number.**

The critical numbers only give the **possible** locations of extremes, and some critical numbers are not the locations of extremes. The critical numbers are the **candidates** for the locations of maximums and minimums (Fig. 10) . Section 3.5 is devoted entirely to translating and solving maximum and minimum problems.

each (a, f(a)) is a critical point
but not an extreme point
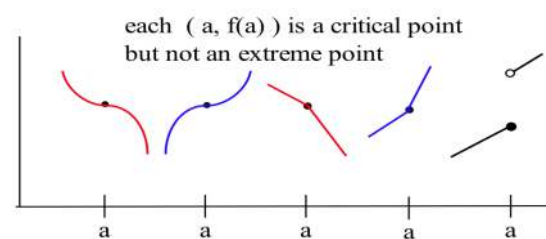
Fig. 10

Which Functions Have Extremes?

So far we have concentrated on finding the extreme values of functions, but some functions don't have extreme values. Example 2 showed that $f(x) = x^3$ did not have a maximum or minimum.

Example 5:   Find the extreme values of f(x) = x .

Solution: Since f '(x) = 1 > 0 for all x, the first theorem in this section guarantees that f has no extreme values. The function f(x) = x does not have a maximum or minimum on the real number line.

The difficulty with the previous function was that the domain was so large that we could always make the function larger or smaller than any given value. The next example shows that we can encounter the same difficulty even on a small interval.

Example 6: Show that $f(x) = \dfrac{1}{x}$ (Fig. 11) does not have a maximum or minimum on the interval $(0,1)$.

Solution: f is continuous for all $x \neq 0$ so f is continuous on the interval $(0,1)$.

For $0 < x < 1, f(x) = \dfrac{1}{x} > 0$ (Fig. 11). For any number a strictly between 0

and 1, we can show that f(a) is neither a maximum nor a minimum of f on $(0,1)$.

Pick b to be any number between 0 and a, $0 < b < a$. Then

$f(b) = \dfrac{1}{b} > \dfrac{1}{a} = f(a)$, so f(a) is not a maximum. Similarly, pick c to be

any number between a and 1, $a < c < 1$. Then $f(a) = \dfrac{1}{a} > \dfrac{1}{c} = f(c)$, so

f(a) is not a minimum. The interval $(0,1)$ is not large, but f still does not have an

extreme value in $(0,1)$.
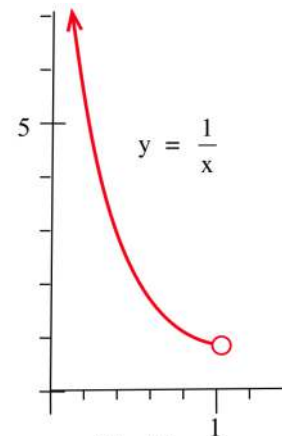
Fig. 11

The Extreme Value Theorem gives conditions so that a function is guaranteed to have a maximum and a minimum.

Extreme Value Theorem: If   f is **continuous** on a **closed** interval [a,b],   then   f attains both a maximum and minimum on [a,b].

(a)

(b)

(c)

(d)
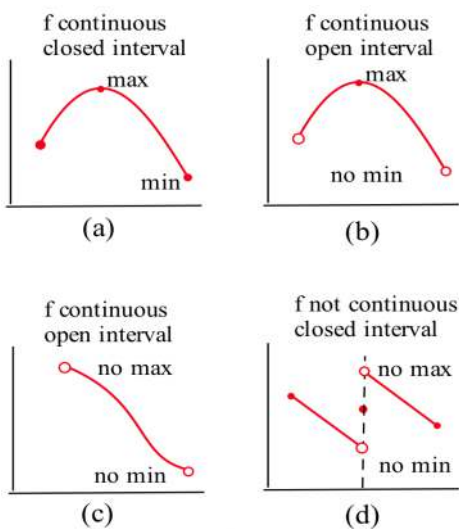
Fig. 12

The proof of this theorem is difficult and is omitted. Fig. 12 illustrates some of the possibilities for continuous and discontinuous functions on open and closed intervals. The Extreme Value Theorem guarantees that certain functions (continuous) on certain intervals (closed) must have maximums and minimums. Other functions on other intervals **may** or **may not** have maximums and minimums.

PROBLEMS

1. Label all of the local maximums and minimums of the function in Fig. 13. Also label all of the critical points.
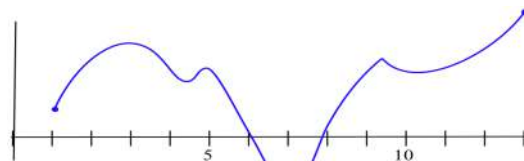
Fig. 13

2. Label all of the local maximums and minimums of the function in Fig. 14. Also label all of the critical points.
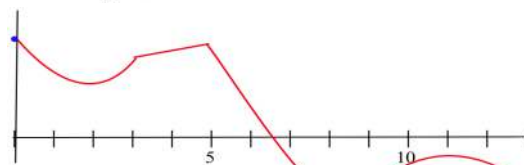
Fig. 14

In problems 3 – 14, find all of the critical points and local maximums and minimums of each function.

3. $f(x) = x^2 + 8x + 7$

4. $f(x) = 2x^2 - 12x + 7$

5. $f(x) = \sin(x)$

6. $f(x) = x^3 - 6x^2 + 5$

7. $f(x) = (x - 1)^2 (x - 3)$

8. $f(x) = \ln(x^2 - 6x + 11)$

9. $f(x) = 2x^3 - 96x + 42$

10. $f(x) = 5x - 2$

11. $f(x) = 5x + \cos(2x+1)$

12. $f(x) = \sqrt[3]{x}$

13. $f(x) = e^{-(x - 2)^2}$

14. $f(x) = |x + 5|$

15. Sketch the graph of a continuous function f so that

   (a) $f(1) = 3$, $f'(1) = 0$, and the point $(1,3)$ is a relative maximum of f.

   (b) $f(2) = 1$, $f'(2) = 0$, and the point $(2,1)$ is a relative minimum of f.

   (c) $f(3) = 5$, f is not differentiable at 3, and the point $(3,5)$ is a relative maximum of f.

   (d) $f(4) = 7$, f is not differentiable at 4, and the point $(4,7)$ is a relative minimum of f.

   (e) $f(5) = 4$, $f'(5) = 0$, and the point $(5,4)$ is not a relative minimum or maximum of f.

   (f) $f(6) = 3$, f is not differentiable at 6, and the point $(6,3)$ is not a relative minimum or maximum of f.

In problems 16 – 25 , find all critical points and local extremes of each function on the given intervals.

16. $f(x) = x^2 - 6x + 5$ on the entire real number line.

17. $f(x) = x^2 - 6x + 5$ on $[-2, 5]$.

18. $f(x) = 2 - x^3$ on the entire real number line.

19. $f(x) = 2 - x^3$ on $[-2, 1]$.

20. $f(x) = x^3 - 3x + 5$ on the entire real number line.

21. $f(x) = x^3 - 3x + 5$ on $[-2, 1]$.

22. $f(x) = x^5 - 5x^4 + 5x^3 + 7$ on the entire real number line.

23. $f(x) = x^5 - 5x^4 + 5x^3 + 7$ on $[0, 2]$.

24. $f(x) = \dfrac{1}{x^2 + 1}$ on the entire real number line.

25. $f(x) = \dfrac{1}{x^2 + 1}$ on $[1, 3]$.

26. (a) Find two numbers whose sum is 22 and whose product is as large as possible. (Suggestion: call the numbers x and 22 – x).

(b) Find two numbers whose sum is $A > 0$ and whose product is as large as possible.

27. Find the coordinates of the point in the first quadrant on the circle $x^2 + y^2 = 1$ so that the rectangle formed in Fig. 15 has the largest possible area. (Suggestion: the coordinates of a point on the circle are $(x, \sqrt{1 - x^2})$.)

Fig. 15

28. Find the coordinates of the point in the first quadrant on the ellipse $9x^2 + 16y^2 = 144$ so that the rectangle formed in Fig. 16 has the largest possible area. The smallest possible area.

Fig. 16

8-2x   8-2x
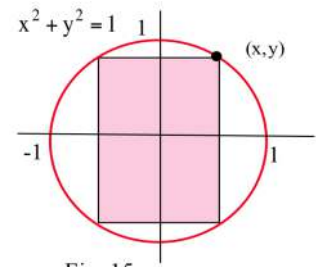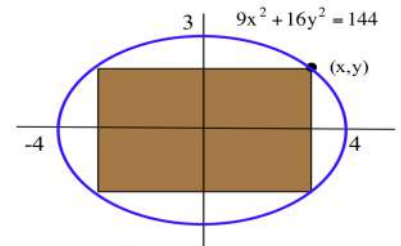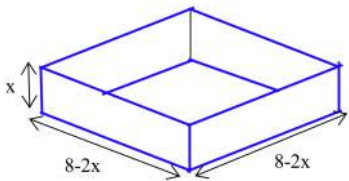
Fig. 17

29. Find the value for x so the box in Fig. 17 has the largest possible volume? The smallest volume?

30. Find the radius and height of the cylinder that has the largest volume ($V = \pi r^2 \cdot h$) if the sum of the radius and height is 9.

31. Suppose you are working with a polynomial of degree 3 on a closed interval.

(a) What is the largest number of critical points the function can have on the interval?

(b) What is the smallest number of critical points it can have?

(c) What are the patterns for the most and fewest critical points a polynomial of degree n on a closed interval can have?

32. Suppose you have a polynomial of degree 3 divided by a polynomial of degree 2 on a closed interval.

(a) What is the largest number of critical points the function can have on the interval?

(b) What is the smallest number of critical points it can have?

33. Suppose $f(1) = 5$ and $f'(1) = 0$. What can we conclude about the point (1,5) if

(a) $f'(x) < 0$ for $x < 1$, and $f'(x) > 0$ for $x > 1$?   (b) $f'(x) < 0$ for $x < 1$, and $f'(x) < 0$ for $x > 1$?

(c) $f'(x) > 0$ for $x < 1$, and $f'(x) < 0$ for $x > 1$?   (d) $f'(x) > 0$ for $x < 1$, and $f'(x) > 0$ for $x > 1$?

34. Suppose $f(2) = 3$ and f is continuous but not differentiable at $x = 2$. What can we conclude about

the point $(2,3)$ if

(a) $f'(x) < 0$ for $x < 2$, and $f'(x) > 0$ for $x > 2$?
(b) $f'(x) < 0$ for $x < 2$, and $f'(x) < 0$ for $x > 2$?

(c) $f'(x) > 0$ for $x < 2$, and $f'(x) < 0$ for $x > 2$?
(d) $f'(x) > 0$ for $x < 2$, and $f'(x) > 0$ for $x > 2$?
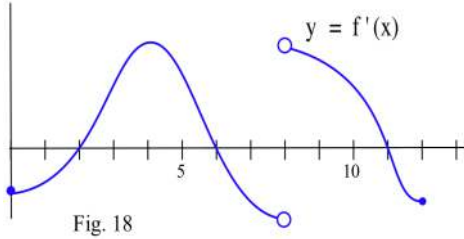
Fig. 18

35. f is a continuous function, and Fig. 18 shows the graph of f'.

(a) Which values of x are critical points.

(b) At which values of x is f a local maximum?

(c) At which values of x is f a local minimum?
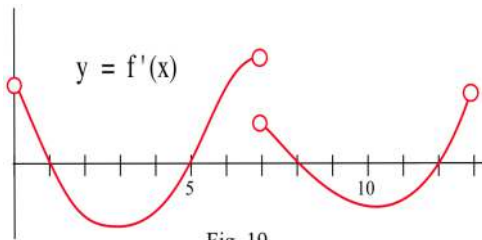
Fig. 19

36. f is a continuous function, and Fig. 19 shows the graph of f'.

(a) Which values of x are critical points.

(b) At which values of x is f a local maximum?

(c) At which values of x is f a local minimum?

37. State the contrapositive form of the Extreme Value Theorem.

38. Imagine the graph of $f(x) = 1 - x$. Does f have a **maximum** value for x in the interval I?

(a) $I = [0, 2]$
(b) $I = [0, 2)$
(c) $I = (0, 2]$
(d) $I = (0,2)$
(e) $I = (1, \pi]$

39. Imagine the graph of $f(x) = 1 - x$. Does f have a **minimum** value for x in the interval I?

(a) $I = [0, 2]$
(b) $I = [0, 2)$
(c) $I = (0, 2]$
(d) $I = (0,2)$
(e) $I = (1, \pi]$

40. Imagine the graph of $f(x) = x^2$. Does f have a **maximum** value for x in the interval I?

(a) $I = [-2, 3]$
(b) $I = [-2, 3)$
(c) $I = (-2, 3]$
(d) $I = [-2, 1)$
(e) $I = (-2, 1]$

41. Imagine the graph of $f(x) = x^2$. Does f have a **minimum** value for x in the following intervals?

(a) $I = [-2, 3]$
(b) $I = [-2, 3)$
(c) $I = (-2, 3]$

(d) $I = [-2, 1)$
(e) $I = (-2, 1]$

42. Define $A(x)$ to be the **area** bounded between the x–axis,

the graph of f, and a vertical line at x (Fig. 20).

(a) At what value of x is $A(x)$ minimum?

(b) At what value of x is $A(x)$ maximum?
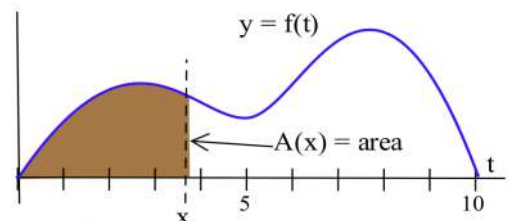
Fig. 20

43. Define $S(x)$ to be the **slope** of the line through the points $(0,0)$

and $(x, f(x))$ in Fig. 21.

(a) At what value of x is $S(x)$ minimum?

(b) At what value of x is $S(x)$ maximum?
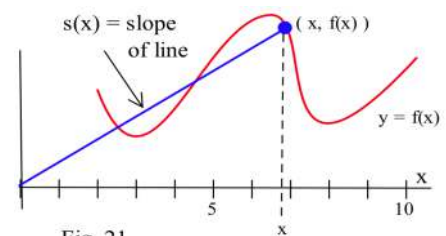
Fig. 21

Section 4.1   **PRACTICE Answers**

Practice 1:   The enrollments were relative maximums in '82, '87, and '90.

The global maximum was in '87. The enrollments were relative minimums in '80, '83, and '89. The global minimum occurred in '80.

Practice 2: $f(x) = x^2 + 4x - 5$ is a polynomial so f is differentiable for all x, and $f'(x) = 2x + 4$.

$f'(x) = 0$ when $x = -2$ so the only candidate for a local extreme is $x = -2$. Since the graph of f is a parabola opening up, the point $(-2, f(-2)) = (-2, -9)$ is a local minimum.

$g(x) = 2x^3 - 12x^2 + 7$ is a polynomial so g is differentiable for all x, and

$g'(x) = 6x^2 - 24x = 6x(x - 4)$. $g'(x) = 0$ when $x = 0, 4$ so the only candidates for a local extreme are $x = 0$ and $x = 4$. The graph of g (Fig. 22) shows that g has a local maximum at (0,7) and a local minimum at $(4, -57)$.

Practice 3:

x	f(x)	f '(x)	max/min
1	5	0	local max
2			
3	1	0	local min
4	3	0	neither
5			
6	7	0	local max

see Fig. 23

Fig. 22
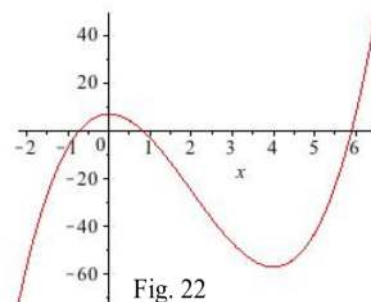
Practice 4: $(1, f(1))$ is a local minimum. $x = 1$ is an endpoint.

$(2, f(2))$ is a local maximum. $f'(2) = 0$.

$(3, f(3))$ is a local minimum. f is not differentiable at $x = 3$.

$(4, f(4))$ is a local maximum. $x = 4$ is an endpoint.

Practice 5:   This is the same function that was used in Example 4, but in this Practice problem the interval is [0, 5] instead of [0, 10] in the Example. See the Example for the calculations.

Critical points: endpoints **x = 0** and **x = 5**.

f is differentiable for all $0 < x < 5$: none.

$f'(x) = 0$: none in **[0, 5]**.

$f(0) \approx 4.67$ is the maximum of f on [0, 5]. $f(5) \approx 4.14$ is the minimum of f on [0, 5].

Fig. 23
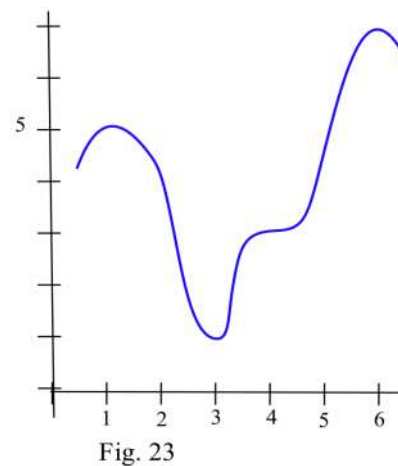
4.2   The Mean Value Theorem and Its Consequences

If you averaged 30 miles per hour during a trip, then at some instant
during the trip you were traveling exactly 30 miles per hour.

That relatively obvious statement is the Mean Value Theorem as it applies to a particular trip. It may seem strange that such a simple statement would be important or useful to anyone, but the Mean Value Theorem is important and some of its consequences are very useful for people in a variety of areas. Many of the results in the rest of this chapter depend on the Mean Value Theorem, and one of the corollaries of the Mean Value Theorem will be used every time we calculate an "integral" in later chapters. A truly delightful aspect of mathematics is that an idea as simple and obvious as the Mean Value Theorem can be so powerful.

Before we state and prove the Mean Value Theorem and examine some of its consequences, we will consider a simplified version called Rolle's Theorem.

Rolle's Theorem

Suppose we pick any two points on the x–axis and think about all of the differentiable functions which go through those two points (Fig. 1) . Since our functions are differentiable, they must be continuous and their graphs can not have any holes or breaks. Also, since these functions are differentiable, their derivatives are defined everywhere between our two points and their graphs can not have any "corners" or vertical tangents. The graphs of the functions in Fig. 1 can still have all sorts of shapes, and it may seem unlikely that they have any common properties other than the ones we have stated, but Michel Rolle (1652–1719) found one. He noticed that every one of these functions has one or more points where the tangent line is horizontal (Fig. 2), and this result is named after him.
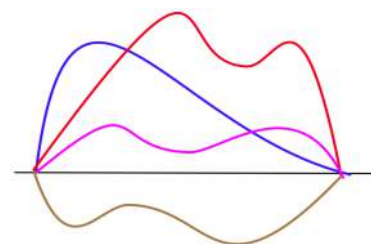
Fig. 1
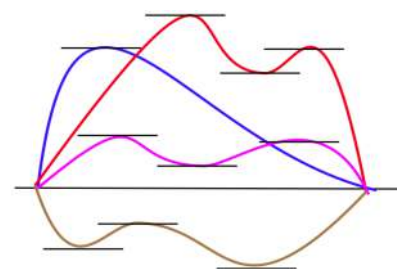
Fig. 2

Rolle's Theorem:

If $f(a) = f(b)$, and $f(x)$ is continuous for $a \leq x \leq b$ and differentiable for $a < x < b$,

then   there is at least one number c, between a and b, so that $f'(c) = 0$.

Proof: We consider two cases: when $f(x) = f(a)$ for all x in (a,b) and when $f(x) \neq f(a)$ for some x in (a,b).

  Case I, $f(x) = f(a)$ for all x in (a,b) : If $f(x) = f(a)$ for all x between a and b, then f is a
    horizontal line segment and $f'(c) = 0$ for all values of c strictly between a and b.

  Case II, $f(x) \neq f(a)$ for some x in (a,b): Since f is continuous on the closed interval [a,b], we know
    from the Extreme Value Theorem that f must have a maximum value in the closed interval [a,b] and a
    minimum value in the interval.

If $f(x) > f(a)$ for some value of x in [a,b], then the maximum of f must occur at some value c

strictly between a and b, $a < c < b$. (Why can't the maximum be at a or b?) Since f(c) is a local

maximum of f, then c is a critical number of f and $f'(c) = 0$ or $f'(c)$ is undefined. But f is

differentiable at all x between a and b, so the only possibility left is that $f'(c) = 0$.

If $f(x) < f(a)$ for some value of x in [a,b], then f has a minimum at some value $x = c$ strictly

between a and b, and $f'(c) = 0$.

In either case, there is at least one value of c between a and b so that $f'(c) = 0$.
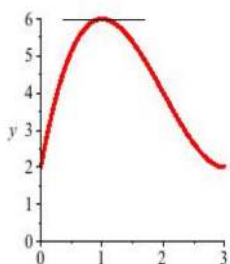
Fig. 3

Example 1:   Show that $f(x) = x^3 - 6x^2 + 9x + 2$ satisfies the hypotheses of Rolle's

Theorem on the interval [0, 3] and find the value of c which the

theorem says exists.

Solution: f is a polynomial so it is continuous and differentiable everywhere. $f(0) = 2$ and

$f(3) = 2$. $f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$ so $f'(x) = 0$ at 1 and 3.

The value $c = 1$ is between 0 and 3. Fig. 3 shows the graph of f.

Practice 1:   Find the value(s) of c for Rolle's

Theorem for the functions in Fig. 4.
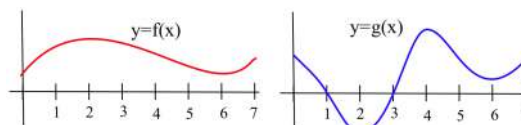
Fig. 4

The Mean Value Theorem
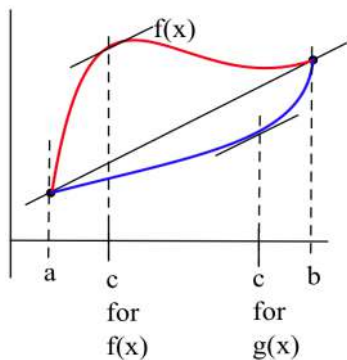
Fig. 5

Geometrically, the Mean Value Theorem is a "tilted" version of Rolle's

Theorem (Fig. 5). In each theorem we conclude that there is a number c so

that the slope of the tangent line to f at $x = c$ is the same as the slope of the

line connecting the two ends of the graph of f on the interval [a,b]. In Rolle's

Theorem, the two ends of the graph of f are at the same height, $f(a) = f(b)$, so

the slope of the line connecting the ends is zero. In the Mean Value Theorem,

the two ends of the graph of f do not have to be at the same height so the line

through the two ends does not have to have a slope of zero.

Mean Value Theorem:

If   f(x) is continuous for $a \le x \le b$ and differentiable for $a < x < b$,

then   there is at least one number c, between a and b, so the tangent line at c is parallel

to the secant line through the points (a, f(a)) and (b,f(b)): $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Proof: The proof of the Mean Value Theorem uses a tactic common in mathematics: introduce a new

function which satisfies the hypotheses of some theorem we already know and then use the conclusion

of that previously proven theorem. For the Mean Value Theorem we introduce a new function, $h(x)$,

which satisfies the hypotheses of Rolle's Theorem. Then we can be certain that the conclusion of

Rolle's Theorem is true for $h(x)$, and the Mean Value Theorem for f follows from the conclusion of

Rolle's Theorem for h.

First, let $g(x)$ be the straight line through the ends $(a, f(a))$ and $(b, f(b))$ of the graph of f. The

function g goes through the point $(a, f(a))$ so $g(a) = f(a)$. Similarly, $g(b) = f(b)$. The slope of the linear

function g is $\dfrac{f(b) - f(a)}{b - a}$ so $g'(x) = \dfrac{f(b) - f(a)}{b - a}$ for all x between a and b, and g is continuous and

differentiable. (The formula for g is $g(x) = f(a) + m(x - a)$ with $m = (f(b) - f(a))/(b - a)$.)
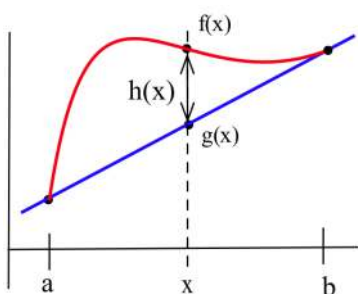
Fig. 6

Define $h(x) = f(x) - g(x)$ for $a \le x \le b$ (Fig. 6). The function h satisfies

the hypotheses of Rolle's theorem:

$h(a) = f(a) - g(a) = 0$ and $h(b) = f(b) - g(b) = 0$,

$h(x)$ is continuous for $a \le x \le b$ since both f and g are continuous there, and

$h(x)$ is differentiable for $a < x < b$ since both f and g are differentiable there,

so the conclusion of Rolle's Theorem applies to h:

there is a c, between a and b, so that $h'(c) = 0$.

The derivative of $h(x) = f(x) - g(x)$ is $h'(x) = f'(x) - g'(x)$ so we know that there is a number c, between a

and b, with $h'(c) = 0$. But $0 = h'(c) = f'(c) - g'(c)$ so $f'(c) = g'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Graphically, the Mean Value Theorem says that there
is at least one point c where the slope of the tangent
line, $f'(c)$, equals the slope of the line through the end
points of the graph segment, $(a, f(a))$ and $(b, f(b))$.
Fig. 7 shows the locations of the parallel tangent lines
for several functions and intervals.
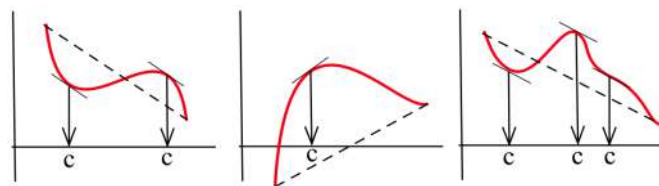
Fig. 7

The Mean Value Theorem also has a very natural interpretation if $f(x)$ represents the position of an object

at time x: $f'(x)$ represents the velocity of the object at the **instant** x, and

$\dfrac{f(b) - f(a)}{b - a}$ $= \dfrac{\text{change in position}}{\text{change in time}}$ represents the **average** (mean) velocity of the object during the time

interval from time a to time b. The Mean Value Theorem says that there is a time c, between a and b,

when the instantaneous velocity, $f'(c)$, is equal to the average velocity for the entire trip,

$\dfrac{f(b) - f(a)}{b - a}$. If your average velocity during a trip is 30 miles per hour, then at some instant during the

trip you were traveling exactly 30 miles per hour.

Practice 2:   For $f(x) = 5x^2 - 4x + 3$ on the interval $[1,3]$, calculate $m = \dfrac{f(b) - f(a)}{b - a}$ and find the

value of c so that $f'(c) = m$.

Some Consequences of the Mean Value Theorem

If the Mean Value Theorem was just an isolated result about the existence of a particular point c, it would

not be very important or useful. However, the Mean Value Theorem is the basis of several results about the

behavior of functions over entire intervals, and it is these consequences which give it an important place in

calculus for both theoretical and applied uses.

The next two corollaries are just the first of many results which follow from the Mean Value Theorem.

We already know, from the Main Differentiation Theorem, that the derivative of a constant function

$f(x) = k$ is always 0, but can a nonconstant function have a derivative which is always 0? The first

corollary says no.

Corollary 1:   If $f'(x) = 0$ for all x in an interval I, then $f(x) = K$, a constant, for all x in I.

Proof: Assume $f'(x) = 0$ for all x in an interval I, and pick any two points a and b $(a \neq b)$ in the

interval. Then, by the Mean Value Theorem, there is a number c between a and b so that

$f'(c) = \dfrac{f(b) - f(a)}{b - a}$. By our assumption, $f'(x) = 0$ for all x in I so we know that

$0 = f'(c) = \dfrac{f(b) - f(a)}{b - a}$ and we can conclude that $f(b) - f(a) = 0$ and $f(b) = f(a)$. But a and b were

any two points in I, so the value of $f(x)$ is the same for any two values of x in I, and f is a

constant function on the interval I.

We already know that if two functions are parallel (differ by a constant), then their derivatives are equal,

but can two nonparallel functions have the same derivative? The second corollary says no.

Corollary 2:   If $f'(x) = g'(x)$ for all x in an interval I,

then $f(x) - g(x) = K$, a constant, for all x in I,

so the graphs of f and g are "parallel" on the interval I.

Proof: This corollary involves two functions instead of just one, but we can imitate the proof of the

Mean Value Theorem and introduce a new function $h(x) = f(x) - g(x)$. The function h is

differentiable, and $h'(x) = f'(x) - g'(x) = 0$ for all x in I, so, by Corollary 1, $h(x)$ is a constant

function and $K = h(x) = f(x) - g(x)$ for all x in the interval. Then $f(x) = g(x) + K$.

We will use Corollary 2 hundreds of times in Sections 4 and 5 when we work with "integrals". Typically you will be given the derivative of a function, f '(x) , and asked to find **all** functions f which have that derivative. Corollary 2 tells us that if we can find **one** function f which has the derivative we want, then the only other functions which have the same derivative are f(x) + K : once you find one function with the right derivative, you have essentially found all of them.

Example 2:   (a) Find **all** functions whose derivatives equal 2x .

(b) Find a function g(x) with g '(x) = 2x and g(3) = 5.

$y = x^2 + 6$
$y = x^2 + 2$
$y = x^2$
(3, 5)
$y = x^2 - 4$
Fig. 8
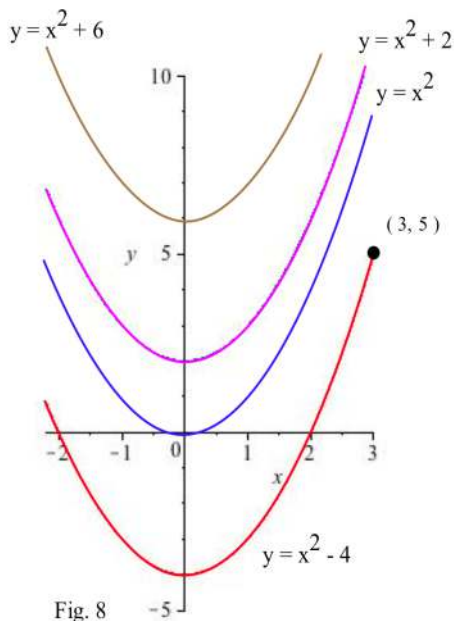
Solution: (a) We can recognize that if $f(x) = x^2$ then f '(x) = 2x so one function with the derivative we want is $f(x) = x^2$. Corollary 2 guarantees that every function g whose derivative is 2x has the form $g(x) = f(x) + K = x^2 + K$. The only functions with derivative 2x have the form $x^2 + K$.

(b) Since g '(x) = 2x, we know that g must have the form $g(x) = x^2 + K$, but this is a whole "family" of functions (Fig. 8), and we want to find one member of the family . We know that g(3) = 5 so we want to find the member of the family which goes through the point (3,5). All we need to do is replace the g(x) with 5 and the x with 3 in the formula $g(x) = x^2 + K$, and then solve for the value of K: $5 = g(3) = (3)^2 + K$ so K = − 4 . The function we want is $g(x) = x^2 - 4$.

Practice 3:   Restate Corollary 2 as a statement about the positions and velocities of two cars.

PROBLEMS

1. In Fig. 9, find the location of the number(s) "c" which Rolle's Theorem promises (guarantees).

y=f(x)
Fig. 9
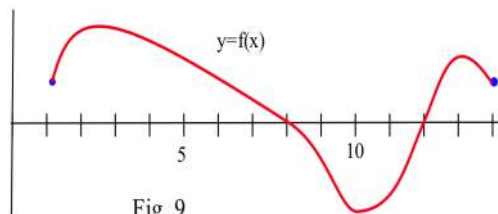
For problems 2 – 4, verify that the hypotheses of Rolle's Theorem are satisfied for each of the functions on the given intervals, and find the value of the number(s) "c" which Rolle's Theorem promises exists.

2. (a) $f(x) = x^2$ on [−2, 2]   (b) $f(x) = x^2 - 5x + 8$ on [0, 5]

3. (a) f(x) = sin(x) on [0, π]   (b) f(x) = sin(x) on [π, 5π]

4. (a) $f(x) = x^3 - x + 3$ on [−1, 1]   (b) f(x) = x·cos(x) on [0, π/2]

5. Suppose you toss a ball straight up and catch it when it comes down. If h(t) is the height of the ball at time t, then what does Rolle's Theorem say about the velocity of the ball? Why is it easier to catch a ball which someone on the ground tosses up to you on a balcony, than for you to be on the ground and catch a ball which someone on a balcony tosses down to you?

6. If $f(x) = 1/x^2$, then $f(-1) = 1$ and $f(1) = 1$ but $f'(x) = -2/x^3$ is never equal to 0. Why doesn't this function violate Rolle's Theorem?

7. If $f(x) = |x|$, then $f(-1) = 1$ and $f(1) = 1$ but $f'(x)$ is never equal to 0. Why doesn't this function violate Rolle's Theorem?

8. If $f(x) = x^2$, then $f'(x) = 2x$ is never 0 on the interval [1, 3]. Why doesn't this function violate Rolle's Theorem?

9. If I take off in an airplane, fly around for awhile and land at the same place I took off from, then my starting and stopping heights are the same but the airplane is always moving. Doesn't this violate Rolle's theorem which says there is an instant when my velocity is 0?

10. Prove the following corollary of Rolle's Theorem: If P(x) is a polynomial, then between any two roots of P there is a root of P ' .

11. Use the corollary in problem 10 to justify the conclusion that the **only** root of $f(x) = x^3 + 5x - 18$ is 2. (Suggestion: What could you conclude about f ' if f had another root?)

12. In Fig. 10, find the location(s) of the "c" which the Mean Value Theorem promises (guarantees).

Fig. 10

In problems 13–15, verify that the hypotheses of the Mean Value Theorem are satisfied for each of the functions on the given intervals, and find the value of a number(s) "c" which Mean Value Theorem guarantees.
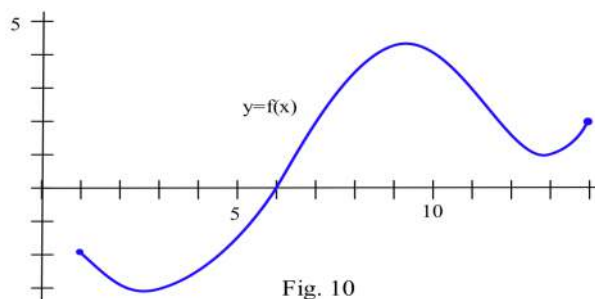
13. (a) $f(x) = x^2$ on [0, 2]   (b) $f(x) = x^2 - 5x + 8$ on [1, 5]

14. (a) $f(x) = \sin(x)$ on [0, π/2]   (b) $f(x) = x^3$ on [-1, 3]

15. (a) $f(x) = 5 - \sqrt{x}$ on [1, 9]   (b) $f(x) = 2x + 1$ on [1, 7]

16. For the quadratic functions in parts (a) and (b) of problem 13, the number c turned out to be the midpoint of the interval, c = (a + b)/2.

(a) For $f(x) = 3x^2 + x - 7$ on [1, 3] , show that $f'(2) = \dfrac{f(3) - f(1)}{3 - 1}$.

(b) For $f(x) = x^2 - 5x + 3$ on [2, 5] , show that $f'(7/2) = \dfrac{f(5) - f(2)}{5 - 2}$.

(c) For $f(x) = Ax^2 + Bx + C$ on [a, b], show that $f'(\dfrac{a+b}{2}) = \dfrac{f(b) - f(a)}{b - a}$.

17. If $f(x) = |x|$, then $f(-1) = 1$ and $f(3) = 3$ but $f'(x)$ is never equal to $\dfrac{f(3) - f(-1)}{3 - (-1)} = \dfrac{1}{2}$. Why
    doesn't this function violate the Mean Value Theorem?

In problems 18 and 19, you are a traffic court judge. In each case, a speeding ticket has been given and
you need to decide if the ticket is appropriate.

18. The toll taker says, "Your Honor, based on the elapsed time from when the car entered the toll road
    until the car stopped at my booth, I know the average speed of the car was 83 miles per hour. I did not
    actually see the car speeding, but I know it was and I gave the driver a speeding ticket."

19. The driver in the next case heard the toll taker and says, "Your Honor, my average velocity on that
    portion of the toll road was only 17 miles per hour, so I could not have been speeding. I don't deserve
    a ticket."

20. Find three different functions f, g and h so that $f'(x) = g'(x) = h'(x) = \cos(x)$.

21. Find a function f so that $f'(x) = 3x^2 + 2x + 5$ and $f(1) = 10$.

22. Find a function g so that $g'(x) = x^2 + 3$ and $g(0) = 2$.

23. Find values for A and B so that the graph of the parabola $f(x) = Ax^2 + B$ is
    (a) tangent to the line $y = 4x + 5$ at the point (1,9)
    (b) tangent to the line $y = 7 - 2x$ at the point (2,3)
    (c) tangent to the parabola $y = x^2 + 3x - 2$ at the point (0,2)

24. Sketch the graphs of several members of the "family" of functions whose derivatives always equal 3.
    Give a formula which defines every function in this family.

25. Sketch the graphs of several members of the "family" of functions whose
    derivatives always equal $3x^2$. Give a formula which defines every function
    in this family.

26. At t seconds after takeoff, the upward velocity of a helicopter was
    $v(t) = 2t - 7$ feet/second. Two seconds after takeoff, the helicopter was 8
    feet above sea level. Find a formula for the height of the helicopter at every
    time t.

Fig. 11

27. Assume that a rocket is fired from the ground and has the upward velocity
    shown in Fig. 11. Estimate the height of the rocket when $t = 1, 2$, and 5
    seconds.
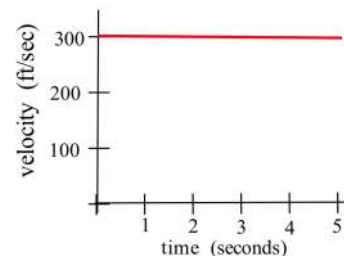
28. Fig. 12 shows the upward velocity of a rocket. Use the information in
    the graph to estimate the height of the rocket when $t = 1, 2$, and 5 seconds.
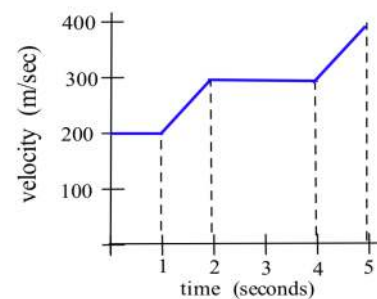
Fig. 12

29. Use the following information to determine an equation for f(x): f "(x) = 6, f '(0) = 4, and f(0) = –5.

30. Use the following information to determine an equation for g(x): g "(x) = 12x, g '(1) = 9, and g(2) = 30.

31. Define A(x) to be the **area** bounded by the x–axis, the line y = 3, and
   a vertical line at x (Fig. 13).
   (a) Find a formula for A(x)?   (b)   Determine A '(x)
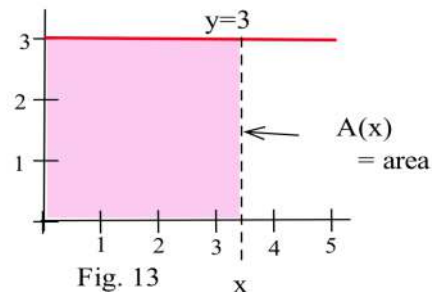
32. Define A(x) to be the **area** bounded by the x–axis, the line y = 2x,
   and a vertical line at x (Fig. 14).
   (a) Find a formula for A(x)?   (b) Determine A '(x)
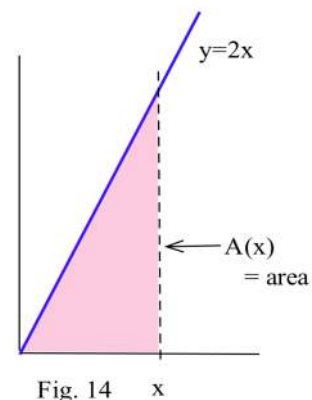
33. Define A(x) to be the **area** bounded by the x–axis, the line y = 2x + 1,
   and a vertical line at x (Fig. 15).
   (a) Find a formula for A(x)?   (b) Determine A '(x)
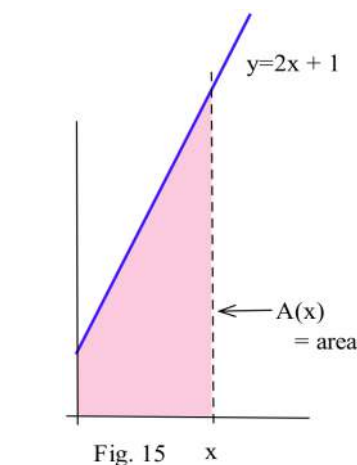
In problems 34 – 36, we have a list of numbers a_1 , a_2 , a_3 , a_4 , . . . , and the
consecutive differences between numbers in the list are $a_2 - a_1$, $a_3 - a_2$, $a_4 - a_3$, . . .

34. If a_1 = 5 and the difference between consecutive numbers in the list is
   always 0, what can you conclude about the numbers in the list?

35. If a_1 = 5 and the difference between consecutive numbers in the list is
   always 3, find a formula for a_n?

36. Suppose the "a" list starts 3, 4, 7, 8, 6, 10, 13, . . ., and there is a "b" list
   which has the same differences between consecutive numbers as the "a" list.
   (a) If b_1 = 5 , find the next six numbers in the "b" list. How is b_n related to a_n?
   (b) If b_1 = 2 , find the next six numbers in the "b" list. How is b_n related to a_n?
   (c) If b_1 = B , find the next six numbers in the "b" list. How is b_n related to a_n?

Fig. 13   x
Fig. 14   x
Fig. 15   x

Section 4.2   **PRACTICE Answers**

Practice 1: $f'(x) = 0$ when $x = 2$ and 6 so $c = 2$ and $c = 6$.

g '(x) = 0 when x = 2, 4, and 6 so c = 2, c = 4, and c = 6.

Practice 2: $f(x) = 5x^2 - 4x + 3$ on $[1, 3]$. $f(1) = 4$ and $f(3) = 36$ so

$$m = \frac{f(b) - f(a)}{b - a} \; = \; \frac{36 - 4}{3 - 1} \; = \; 16.$$

$f'(x) = 10x - 4$ so $f'(c) = 10c - 4 = 16$ if $10c = 20$ and **$c = 2$**.

The graph of f and the location of c are shown in Fig. 16.
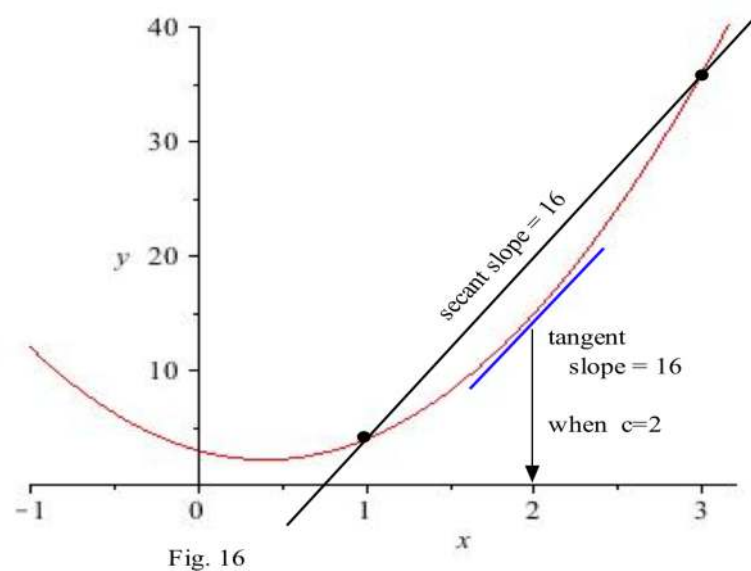
Fig. 16

Practice 3:

If   two cars have the same velocities during an interval of time (f '(t) = g '(t) for t in I)

then   the cars are always a constant distance apart during that time interval.

(Note: The "same velocity" means **same speed** and **same direction**. If two cars are traveling at the
same speed but in different directions, then the distance between them changes and is not constant.)

4.3   THE FIRST DERIVATIVE AND THE SHAPE OF f

This section examines some of the interplay between the shape of the graph of f and the behavior of f '. If we have a graph of f , we will see what we can conclude about the values of f '. If we know values of f ', we will see what we can conclude about the graph of f .

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Definitions: The function f is **increasing on (a,b)** if $a < x_1 < x_2 < b$ implies $f(x_1) < f(x_2)$.

The function f is **decreasing on (a,b)** if $a < x_1 < x_2 < b$ implies $f(x_1) > f(x_2)$.

f is **monotonic on (a,b)** if f is increasing on (a,b) or if f is decreasing on (a,b).

</div>

Graphically, f is **increasing** (decreasing) if, as we move from left to right along the graph of f, the height of the graph **increases** (decreases).

These same ideas make sense if we consider h(t) to be the height (in feet) of a rocket at time t seconds. We naturally say that the rocket is rising or that its height is increasing if the height h(t) increases over a period of time, as t increases.
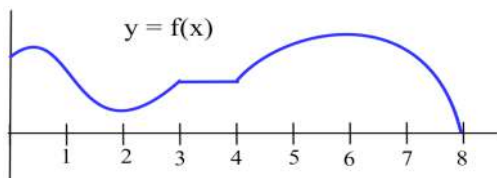
Fig. 1

Example 1:   List the intervals on which the function given in Fig. 1 is increasing or decreasing.

Solution: f is increasing on the intervals [0,1] , [2,3] and [4,6]. f is decreasing on [1,2] and [6,8]. On the interval

[3,4] the function is not increasing or decreasing, it is constant. It is also valid to say that f is increasing on the intervals [0.3, 0.8] and (0.2, 0.5) as well as many others, but we usually talk about the longest intervals on which f is monotonic.
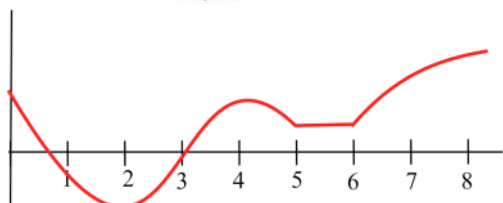
Fig. 2

Practice 1:   List the intervals on which the function given in Fig. 2 is increasing or decreasing.

If we have an accurate graph of a function, then it is relatively easy to determine where f is monotonic, but if the function is defined by an equation, then a little more work is required. The next two theorems relate the values of the derivative of f to the monotonicity of f. The first theorem says that if we know where f is monotonic, then we also know something about the values of f '. The second theorem says that if we know about the values of f ' then we can draw conclusions about where f is monotonic.

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First Shape Theorem

For a function f which is differentiable on an interval (a,b);

(i)   if f is increasing on (a,b) , then f '(x) ≥ 0 for all x in (a,b)

(ii)   if f is decreasing on (a,b) , then f '(x) ≤ 0 for all x in (a,b)

(iii) if f is constant on (a,b) , then f '(x) = 0 for all x in (a,b).

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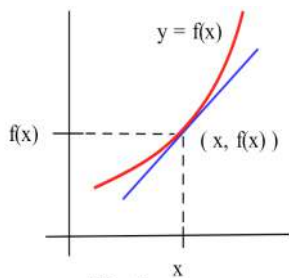
Fig. 3

Proof: Most people find a picture such as Fig. 3 to be a convincing justification of this theorem: if the graph of f increases near a point (x, f(x)) , then the tangent line is also increasing, and the slope of the tangent line is positive (or perhaps zero at a few places). A more precise proof, however, requires that we use the definitions of the derivative of f and of "increasing".

(i)   Assume that f is increasing on (a,b) . We know that f is differentiable, so if x is any number in the interval (a,b) then

$$f '(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$, and this limit exists and is a finite value.

If h is any small enough **positive** number so that x + h is also in the interval (a,b), then x < x + h and f(x) < f(x + h) . We know that the numerator, f(x + h) – f(x) , and the denominator , h , are both positive so the limiting value, f '(x) , must be positive or zero: f '(x) ≥ 0.

(ii) Assume that f is decreasing on (a,b): The proof of this part is very similar to part (i). If x < x + h, then f(x) > f(x + h) since f is decreasing on (a,b). Then the numerator of the limit , f(x + h) – f(x) , will be negative and the denominator, h , will still be positive, so the limiting value , f '(x) , must be negative or zero: f '(x) ≤ 0.

(iii) The derivative of a constant is zero, so if f is constant on (a,b) then f '(x) = 0 for all x in (a,b).

The previous theorem is easy to understand , but you need to pay attention to exactly what it says and what it does not say. It is possible for a differentiable function which is increasing on an interval to have horizontal tangent lines at some places in the interval (Fig 4). It is also possible for a continuous function which is increasing on an interval to have an undefined derivative at some places in the interval (Fig. 4). Finally, it is possible for a function which is increasing on an interval to fail to be continuous at some places in the interval (Fig. 5).
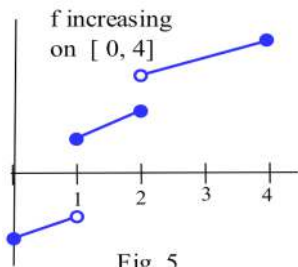
Fig. 5
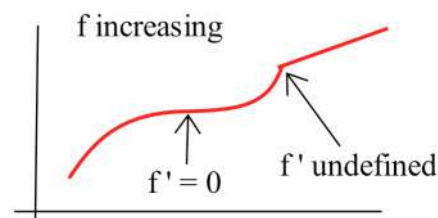
Fig. 4

The First Shape Theorem has a natural interpretation in terms of the height h(t) and upward velocity h '(t) of a helicopter at time t. If the height of the helicopter is increasing (h(t) is increasing) , then the helicopter has a positive or zero upward velocity: h '(t) ≥ 0. If the height of the helicopter is not changing, then its upward velocity is 0: h '(t) = 0.

Example 2:   Fig. 6 shows the height of a helicopter during a period of time. Sketch the graph of the upward velocity of the helicopter, dh/dt.
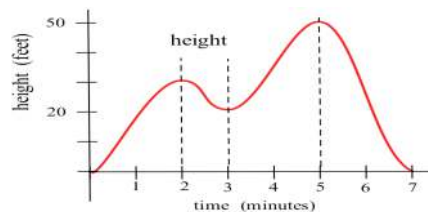
Fig. 6

Solution: The graph of v(t) = dh/dt is shown in Fig. 7. Notice that the h(t) has a local maximum when t = 2 and t = 5, and v(2) = 0 and v(5) = 0. Similarly, h(t) has a local minimum when t = 3, and v(3) = 0. When h is increasing, v is positive. When h is decreasing, v is negative.
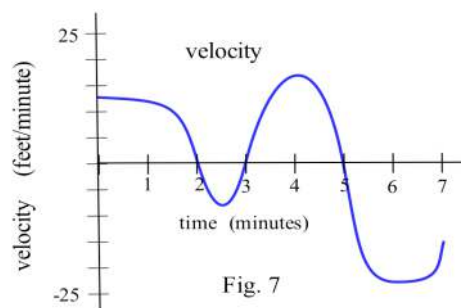
Fig. 7
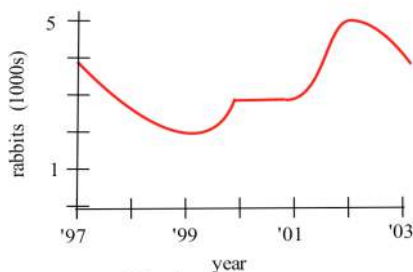
Practice 2:   Fig. 8 shows the population of rabbits on an island during 6 years. Sketch the graph of the rate of population change, dR/dt, during those years.

Fig. 8
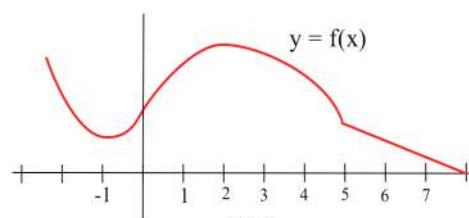
Fig. 9

Example 3: The graph of f is shown in Fig. 9 . Sketch the graph of f ' .

Solution: It is a good idea to look first for the points where f '(x) = 0 or where f is not differentiable, the critical points of f. These locations are usually easy to spot, and they naturally break the problem into several smaller pieces. The only numbers at which f '(x) = 0 are x = −1 and x = 2, so the only places the graph of f '(x) will cross the x–axis
are at x = −1 and x = 2 , and we can plot the point (−1,0) and (2,0) on the graph of f '. The only place that f is not differentiable is at the "corner" above x = 5, so the graph of f ' will not have a point for x = 5. The rest of the graph of f is relatively easy:

if x < −1 then   f(x) is decreasing   so f '(x) is negative,
if −1 < x < 2 then   f(x) is increasing   so f '(x) is positive,
if 2 < x < 5 then   f(x) is decreasing   so f '(x) is negative, and
if 5 < x then   f(x) is decreasing   so f '(x) is negative.
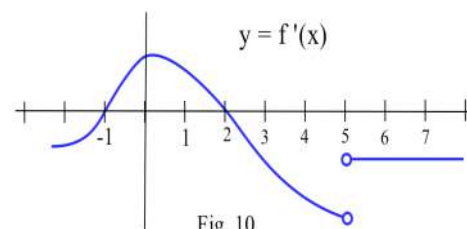
Fig. 10

The graph of f ' is shown in Fig. 10. f(x) is continuous at x =5, but
f   is not differentiable at x = 5, as is indicated by the "hole" in the graph.

Practice 3:   The graph of f is shown in Fig. 11. Sketch the

graph of f '. (The graph of f has a "corner" at x = 5.)

The next theorem is almost the converse of the First Shape

Theorem and explains the relationship between the values of the

derivative and the graph of a function from a different

perspective. It says that if we know something about the values

of f ' , then we can draw some conclusions about the shape of the graph of f .
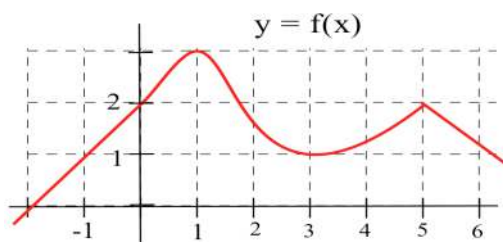
Fig. 11

Second Shape Theorem

For a function f which is differentiable on an interval I;

   (i)   if f '(x) > 0 for all x in the interval I , then f is increasing on I,

   (ii)   if f '(x) < 0 for all x in the interval I , then f is decreasing on I,

   (iii)   if f '(x) = 0 for all x in the interval I , then f is constant on I.

Proof: This theorem follows directly from the Mean Value Theorem, and part (c) is just a restatement of

the First Corollary of the Mean Value Theorem.

(a) Assume that f '(x) > 0 for all x in I and pick any points a and b in I with a < b. Then, by

the Mean Value Theorem, there is a point c between a and b so that $\frac{f(b) - f(a)}{b - a}$ = f '(c) > 0, and

we can conclude that f(b) – f(a) > 0 and f(b) > f(a). Since a < b implies that f(a) < f(b), we know

that f is increasing on I.

(b) Assume that f '(x) < 0 for all x in I and pick any points a and b in I with a < b. Then there is

a point c between a and b so that $\frac{f(b) - f(a)}{b - a}$ = f '(c) < 0, and we can conclude that

f(b) – f(a) = (b–a) f '(c) < 0 so f(b) < f(a). Since a < b implies that f(a) > f(b), we know f is

decreasing on I.

Practice 4:   Rewrite the Second Shape Theorem as a statement about the height h(t) and upward

velocity h '(t) of a helicopter at time t seconds.

The value of the function at a number x tells us the height of

the graph of f above or below the point x on the x–axis. The

value of f ' at a number x tells us whether the graph of f is

increasing or decreasing (or neither) as the graph passes through

the point (x, f(x)) on the graph of f. If f(x) is positive, it is

possible for f '(x) to be positive, negative, zero or undefined: the

value of f(x) has absolutely nothing to do with the value of f '.

Fig. 12 illustrates some of the combinations of values for f and f '.
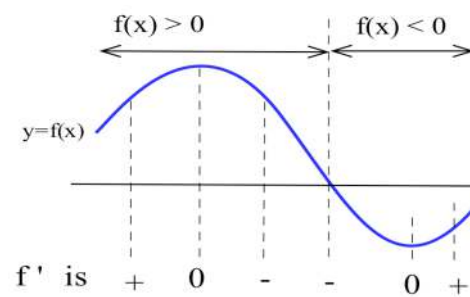
Fig. 12

Practice 5:   Graph a continuous function which satisfies the conditions on f and f ' given below:

x	−2	−1	0	1	2	3
f(x)	1	−1	−2	−1	0	2
f '(x)	−1	0	1	2	−1	1

The Second Shape Theorem is particularly useful if we need to graph a function f which is defined by an equation. Between any two consecutive critical numbers of f, the graph of f is monotonic (why?). If we can find all of the critical numbers of f, then the domain of f will be naturally broken into a number of pieces on which f will be monotonic.

Example 4:   Use information about the values of f ' to help graph $f(x) = x^3 - 6x^2 + 9x + 1$.

Solution: $f '(x) = 3x^2 - 12x + 9 = \mathbf{3(x - 1)(x - 3)}$ so f '(x) = 0 only when x = 1 or x = 3. f ' is a polynomial so it is always defined. The only critical numbers for f are x = 1 and x = 3, and they divide the real number line into three pieces on which f is monotonic: $(-\infty, 1)$, (1,3) and $(3, \infty)$.

   If x < 1, then f '(x) = 3(negative number)(negative number) > 0 so f is increasing.

   If 1 < x < 3, then f '(x) = 3(positive number)(negative number) < 0 so f is decreasing.

   If 3 < x, then f '(x) = 3(positive number)(positive number) > 0 so f is increasing.
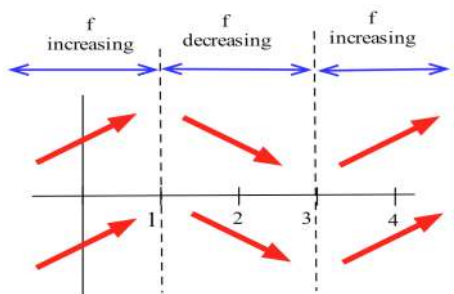
Fig. 13

Even though we don't know the value of f anywhere yet, we do know a lot about the shape of the graph of f : as we move from left to right along the x–axis, the graph of f increases until x = 1, then the graph decreases until x = 3, and then the graph increases again (Fig. 13) The graph of f makes "turns" when x = 1 and x = 3.

To plot the graph of f, we still need to evaluate f at a few values of x, but only at a very few values. f(1) = 5 , and (1,5) is a local maximum of f. f(3) = 1 , and (3,1) is a local minimum of f. The graph of f is shown in Fig. 14.
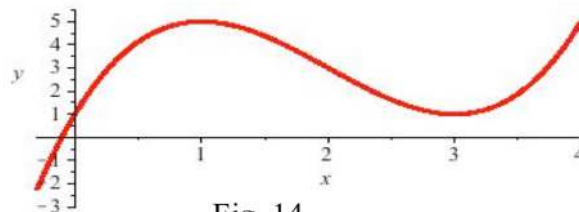
Fig. 14

Practice 6:   Use information about the values of f ' to help graph $f(x) = x^3 - 3x^2 - 24x + 5$.

Example 5:   Use the graph of f ' in Fig. 15 to sketch the shape of the graph of f. Why isn't the graph of f ' enough to completely determine the graph of f ?
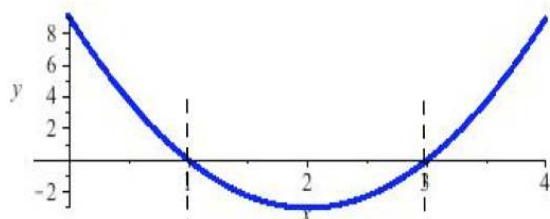
Fig. 15
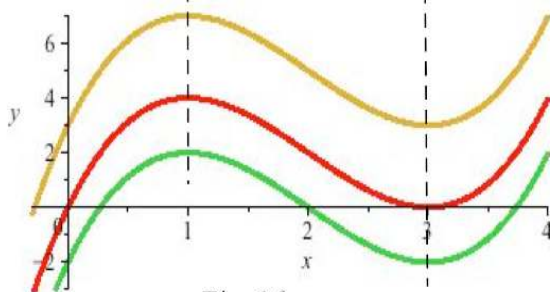
Fig. 16

Solution: Several functions which have the derivative we want are

given in Fig. 16 , and each of them is a correct answer. By the

Second Corollary to the Mean Value Theorem, we know there is

a whole family of parallel functions which have the derivative

we want, and each of these functions is a correct answer. If we

had additional information about the function such as a point it

went through, then only one member of the family would satisfy

the extra condition and that function would be the only correct

answer.

Practice 7:   Use the graph of g ' in

Fig. 17 to sketch the shape of a

graph of g.
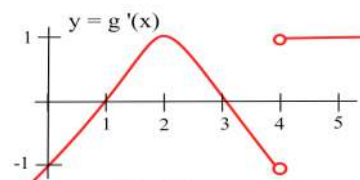
Fig. 17

Practice 8:   A weather balloon is released from the ground and sends

back its upward velocity measurements (Fig. 18). Sketch a graph

of the height of the balloon. When was the balloon highest?
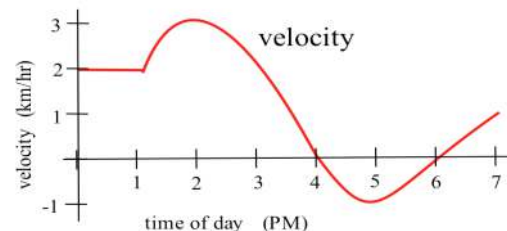
Fig. 18

Using the Derivative to Test for Extremes

The first derivative of a function tells about the general shape of the function, and we can use that shape

information to determine if an extreme point is a maximum or minimum or neither.

First Derivative Test for Local Extremes

Let f be a continuous function with $f'(a) = 0$ or $f'(a)$ is undefined.

(i)   If f '(left of a) > 0 and f '(right of a) < 0, then (a,f(a)) is a local maximum (Fig. 19a)

(ii)   If f '(left of a) < 0 and f '(right of a) > 0, then (a,f(a)) is a local minimum (Fig. 19b)

(iii)   If f '(left of a) > 0 and f '(right of a) > 0, then (a,f(a)) is **not** a local extreme (Fig. 19c)

(iv)   If f '(left of a) < 0 and f '(right of a) < 0, then (a,f(a)) is **not** a local extreme (Fig. 19d)
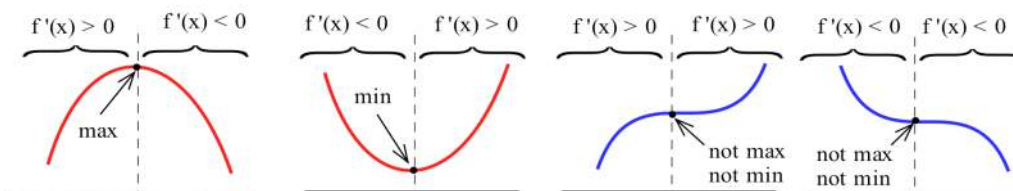
Fig. 19

Practice 9:   Find all extremes of $f(x) = 3x^2 - 12x + 7$ and use the First Derivative Test to determine
if they are maximums, minimums or neither.

A variant of the First Derivative Test can also be used to determine whether an endpoint gives a maximum
or minimum for a function.

PROBLEMS

In problems 1–3, sketch the graph of
the derivative of each function.

1.   Use Fig. 20.

2.   Use Fig. 21.

3.   Use Fig. 22

In problems 4–6, the graph of
the height of a helicopter is
shown. Sketch the graph of
the upward velocity of the
helicopter.
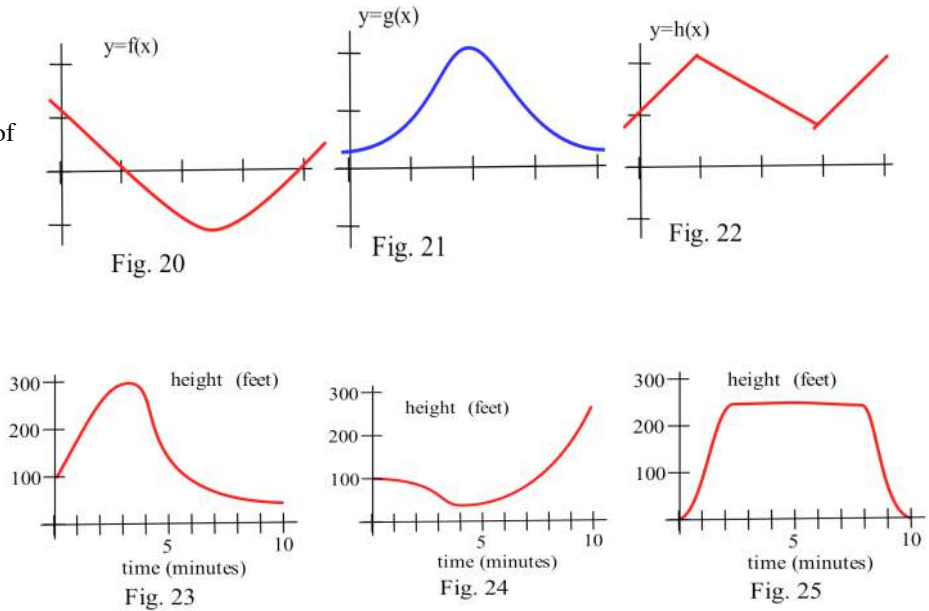
4.   Use Fig. 23.   5.   Use Fig. 24.   6.   Use Fig. 25.
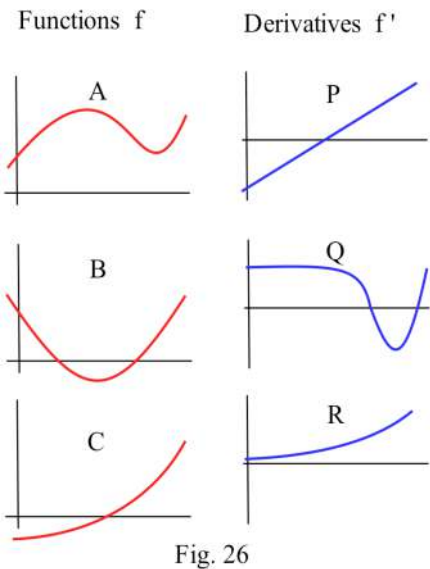
7.   In Fig. 26, match the graphs of the
functions with those of their derivatives.

8.   In Fig. 27, match the graphs showing
the heights of rockets with those
showing their velocities.

9.   Use the Second Shape Theorem to show
that $f(x) = \ln(x)$ is monotonic
increasing on $(0, \infty)$.

10. Use the Second Shape Theorem to show
that $g(x) = e^x$ is increasing on the
entire real number line.
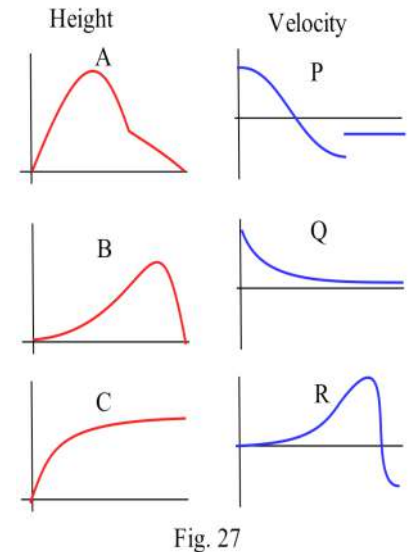
11. A student is working with a complicated function f and has shown that the derivative of f is always
positive. A minute later the student also claims that $f(x) = 2$ when $x = 1$ and when $x = \pi$. Without
checking the student's work, how can you be certain that it contains an error?

12. Fig. 28 shows the graph of the **derivative** of a continuous function f.

    (a) List the critical numbers of f.

    (b) For what values of x does f have a local maximum?

    (c) For what values of x does f have a local minimum?

y= f '(x)

Fig. 28
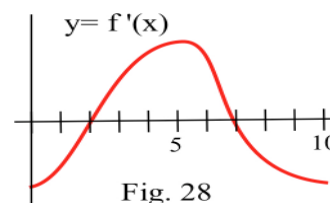
13. Fig. 29 shows the graph of the **derivative** of a continuous function g.

    (a) List the critical numbers of g.

    (b) For what values of x does g have a local maximum?

    (c) For what values of x does g have a local minimum?

y= g '(x)

Fig. 29
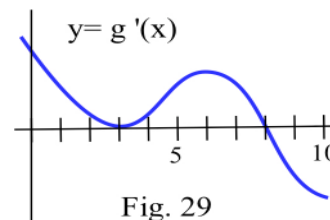
In problems 14–16, the graphs of the upward velocities of several helicopters are shown. Use each graph to determine when each helicopter was at a relatively maximum and minimum height.
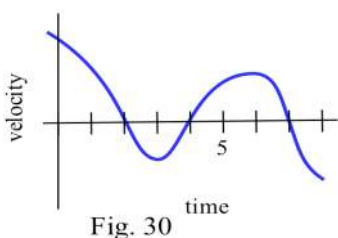
14. Use Fig. 30.   15. Use Fig. 31.   16. Use Fig. 32.

Fig. 30

Fig. 31
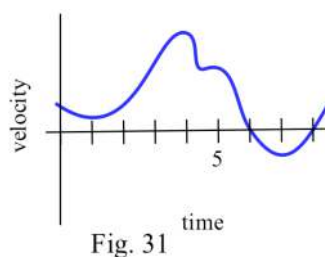
Fig. 32
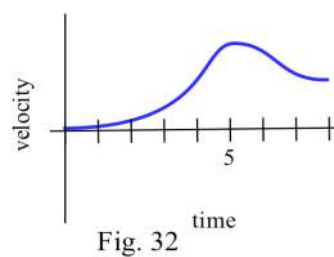
In problems 17 – 22 , use information from the derivative of each function to help you graph the function. Find all local maximums and minimums of each function.

17. $f(x) = x^3 - 3x^2 - 9x - 5$ 18. $g(x) = 2x^3 - 15x^2 + 6$ 19. $h(x) = x^4 - 8x^2 + 3$

20. $s(t) = t + \sin(t)$ 21. $r(t) = \dfrac{2}{t^2 + 1}$ 22. $f(x) = \dfrac{x^2 + 3}{x}$

23. $f(x) = 2x + \cos(x)$ so $f(0) = 1$. Without graphing the function, you can be certain that f has how many **positive** roots? (zero, one, two, more than two)

24. $g(x) = 2x - \cos(x)$ so $g(0) = -1$. Without graphing the function, you can be certain that g has how many **positive** roots? (zero, one, two, more than two)

25. $h(x) = x^3 + 9x - 10$ has a root at $x = 1$. Without graphing h, show that h has no other roots.

26. Sketch the graphs of monotonic decreasing functions which have exactly (a) zero roots, (b) one root, and (c) two roots.

27. Each of the following statements is false. Give (or sketch) a counterexample for each statement.

    (a) If f is increasing on an interval I, then f '(x) > 0 for all x in I.

    (b) If f is increasing and differentiable on I, then f '(x) > 0 for all x in I.

    (c) If cars A and B always have the same speed, then they will always be the same distance apart.

28. (a) Give the equations of several different functions f which all have the same derivative f '(x) = 2.

    (b) Give the equation of the function f with derivative f '(x) = 2 which also satisfies f(1) = 5.

    (c) Give the equation of the function g with g '(x) = 2 , and the graph of g goes through (2, 1).

29. (a) Give the equations of several different functions h which all have the same derivative h '(x) = 2x.

    (b) Give the equation of the function f with derivative f '(x) = 2x which also satisfies f(3) = 20.

    (c) Give the equation of the function g with g '(x) = 2x , and the graph of g goes through (2, 7).

30. Sketch functions with the given properties to help you determine whether each statement is True or False.

    (a) If f '(7) > 0 and f '(x) > 0 for all x near 7, then f(7) is a local maximum of f on [1,7].

    (b) If g '(7) < 0 and g '(x) < 0 for all x near 7, then g(7) is a local minimum of g on [1,7].

    (c) If h '(1) > 0 and h '(x) > 0 for all x near 1, then h(1) is a local minimum of h on [1,7].

    (d) If r '(1) < 0 and r '(x) < 0 for all x near 1, then r(1) is a local maximum of r on [1,7].

    (e) If s '(7) = 0, then s(7) is a local maximum of s on [1,7].

Section 4.3   **PRACTICE Answers**

Practice 1:   g is increasing on $[2, 4]$ and $[6, 8]$.

                      g is decreasing on $[0, 2]$ and $[4, 5]$.

                      g is constant on $[5, 6]$.
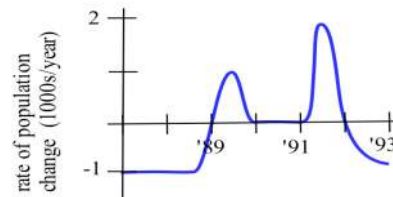
Fig. 33

Practice 2:   The graph in Fig. 33 shows the rate of population change, dR/dt.

Practice 3:   The graph of f ' is shown in Fig. 34. Notice how the graph of f ' is 0 where f has a maximum and minimum.

Practice 4:   The Second Shape Theorem for helicopters:

(i)   If   the upward velocity h ' is positive during time interval I then   the height h is increasing during time interval I.

(ii)   If   the upward velocity h ' is negative during time interval I then   the height h is decreasing during time interval I.

(iii) If   the upward velocity h ' is zero during time interval I then   the height h is constant during time interval I.
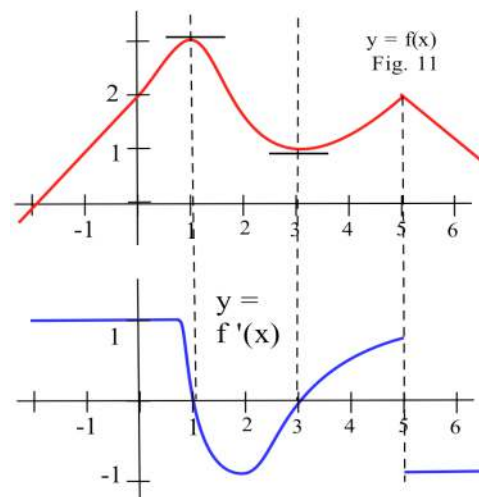
Fig. 34

Practice 5:   A graph satisfying the conditions in the table is shown
in Fig. 35.

x	−2	−1	0	1	2	3
f(x)	1	−1	−2	−1	0	2
f '(x)	−1	0	1	2	−1	1
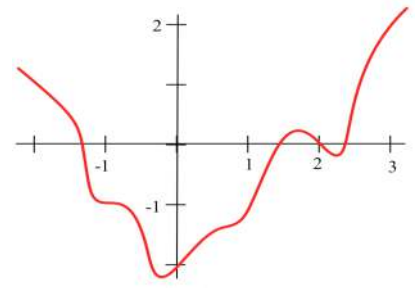
Fig. 35

Practice 6: $f(x) = x^3 - 3x^2 - 24x + 5$. $f '(x) = 3x^2 - 6x - 24 = 3(x - 4)(x + 2)$.

$f '(x) = 0$ if $x = -2, 4$.

$f(x) = x^3 - 3x^2 - 24x + 5$
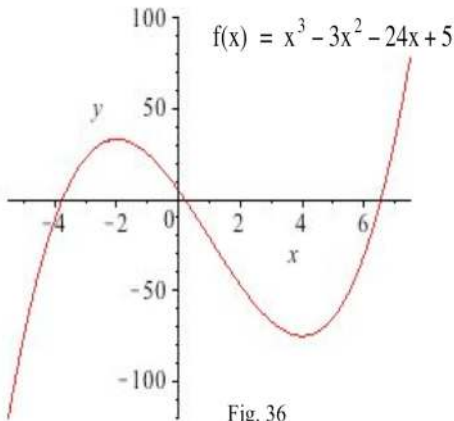
If $x < -2$,

then $f '(x) = 3(x - 4)(x + 2) = 3(\text{negative})(\text{negative}) > 0$ so f is increasing.

If $-2 < x < 4$,

then $f '(x) = 3(x - 4)(x + 2) = 3(\text{negative})(\text{positive}) < 0$ so f is decreasing.

If $x > 4$,

then $f '(x) = 3(x - 4)(x + 2) = 3(\text{positive})(\text{positive}) > 0$ so f is increasing.

f has a relative maximum at $x = -2$ and a relative minimum at $x = 4$.

The graph of f is shown in Fig. 36.

Fig. 36

Practice 7:   Fig. 37 shows several possible graphs for g. Each of the g
graphs has the correct shape to give the graph of g '. Notice that the
graphs of g are "parallel," differ by a constant.

y = g '(x)   Fig. 17

y = g (x)
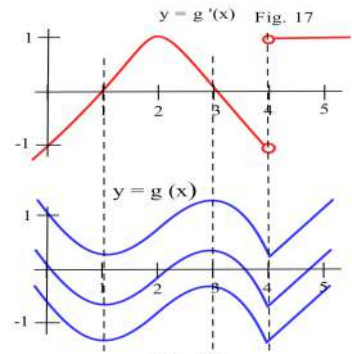
Fig. 37

Practice 8:   Fig. 38 shows the height graph for the balloon.
The balloon was highest at 4 pm and had a local
minimum at 6pm.

velocity   Fig. 18

height
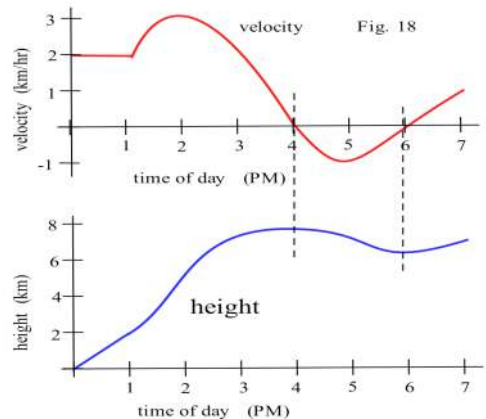
Fig. 38

Practice 9:

$f(x) = 3x^2 - 12x + 7$ so $f'(x) = 6x - 12$. $f'(x) = 0$ if $x = 2$.

If $x < 2$, then $f'(x) < 0$ and f is decreasing.

If $x > 2$, then $f'(x) > 0$ and f is increasing.

From this we can conclude that f has a minimum when $x = 2$

and has a shape similar to Fig. 19(b).

We could also notice that the graph of the quadratic

$f(x) = 3x^2 - 12x + 7$ is an upward opening parabola.

The graph of f is shown in Fig. 39.
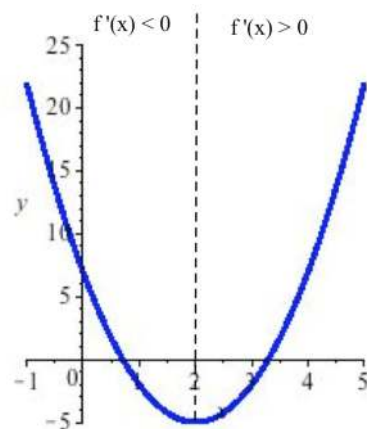
Fig. 39

4.4 SECOND DERIVATIVE AND THE SHAPE OF f

The first derivative of a function gives information about the shape of the function, so the second derivative

of a function gives information about the shape of the first derivative and about the shape of the function.

In this section we investigate how to use the second derivative and the shape of the first derivative to reach

conclusions about the shape of the function. The first derivative tells us whether the graph of f is

increasing or decreasing. The second derivative will tell us about the "concavity" of f, whether f is

curving upward or downward.

Concavity

Graphically, a function is concave up if its graph is curved with the

opening upward (Fig. 1a). Similarly, a function is concave down if its

graph opens downward (Fig. 1b). The concavity of a function can be

important in applied problems and can even affect billion–dollar decisions.

Fig. 1
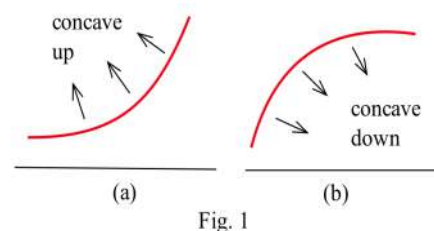
An Epidemic: Suppose an epidemic has started, and you, as a member of congress, must decide whether the

current methods are effectively fighting the spread of the disease or whether more drastic measures and

more money are needed. In Fig. 2, f(x) is the number of people who have the disease at time x, and

two different situations are shown. In both (a) and (b), the number of people with the disease, f(now),

and the rate at which new people are getting sick, f '(now), are

the same. The difference in the two situations is the concavity of

f, and that difference in concavity might have a big effect

on your decision. In (a), f is concave down at "now", and

it appears that the current methods are starting to bring the

epidemic under control. In (b), f is concave up, and it

appears that the epidemic is still out of control.

Fig. 2
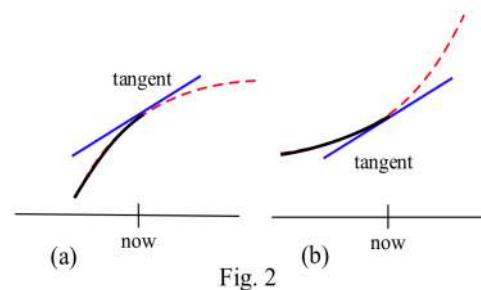
Usually it is easy to determine the concavity of a function by examining its graph, but we also need a

definition which does not require that we have a graph of the function, a definition we can apply to a

function described by a formula without having to graph the function.

> **Definition:** Let f be a differentiable function.
>
> f is **concave up** at a if the graph of f is above the tangent line L to f for all x close
>
>   to a   (but not equal to a) : $\mathbf{f(x) > L(x)} = f(a) + f '(a)(x - a)$.
>
> f is **concave down** at a if the graph of f is below the tangent line L to f for all x close
>
>   to a   (but not equal to a) : $\mathbf{f(x) < L(x)} = f(a) + f '(a)(x - a)$.

Fig. 3 shows the concavity of a function at several points. The
next theorem gives an easily applied test for the concavity of a
function given by a formula.
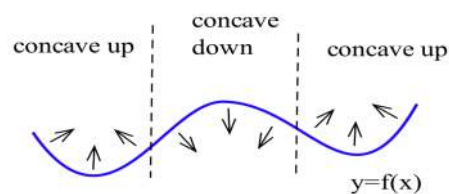
Fig. 3

The Second Derivative Condition for Concavity

    (a) If $f''(x) > 0$ on an interval I , then $f'(x)$ is increasing on I and f is concave up on I.

    (b) If $f''(x) < 0$ on an interval I , then $f'(x)$ is decreasing on I and f is concave down on I.

    (c) If $f''(a) = 0$, then $f(x)$ may be concave up or concave down or neither at a.

Proof: (a) Assume that $f''(x) > 0$ for all x in I, and let a be any point in I. We want to show that f is

concave up at a so we need to prove that the graph of f (Fig. 4) is above the

tangent line to f at a: $f(x) > L(x) = f(a) + f'(a)(x–a)$ for x close to a.
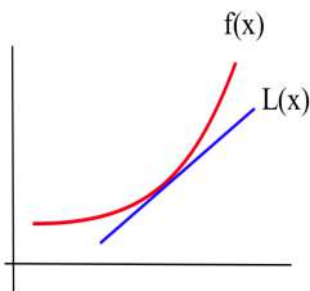
Assume that x is in I , and apply the Mean Value Theorem to f on the interval

from a to x. Then there is a number c between a and x so that

$$f'(c) = \frac{f(x) - f(a)}{x - a} \quad \text{and} \quad f(x) = f(a) + f'(c)(x–a).$$

Since $f'' > 0$ between a and x, we know from the Second Shape Theorem that

f' is increasing between a and x. We will consider two cases: x > a and x < a.

Fig. 4

If x > a, then x–a > 0 and c is in the interval [a,x] so a < c. Since f' is increasing, a < c implies

that f '(a) < f '(c). Multiplying each side of the inequality f '(a) < f '(c) by the positive amount x–a,

we get that   f '(a)(x–a) < f '(c)(x–a). Adding f(a) to each side of this last inequality, we have   L(x) =

f(a) + f '(a)(x–a) < f(a) + f '(c)(x–a) = f(x).

If x < a, then x–a < 0 and c is in the interval [x,a] so c < a. Since f' is increasing, c < a implies

that f '(c) < f '(a). Multiplying each side of the inequality f '(c) < f '(a) by the **negative** amount x–a,

we get that   f '(c)(x–a) **>** f '(a)(x–a) and f(x) = f(a) + f '(c)(x–a) > f(a) + f '(a)(x–a) = L(x).

In each case we get that the function f(x) is above the tangent line L(x). The proof of (b) is similar.

(c) Let $f(x) = x^4$, $g(x) = -x^4$, and $h(x) = x^3$ (Fig. 5). The second derivative of each of these functions is

zero at a = 0, and at (0,0) they all have the same tangent line: L(x) = 0 , the x–axis. However, at (0,0)

they all have different concavity: f is concave up, g is concave down, and h is neither concave up nor

down.

Fig. 5
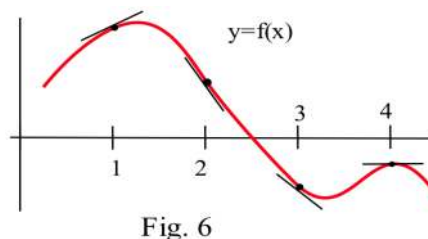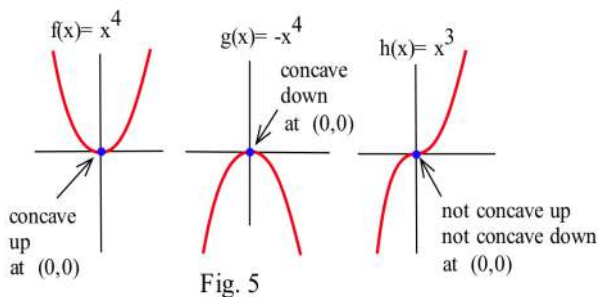
Fig. 6

Practice 1:   Use the graph of f in Fig. 6 to finish filling in the table with "+" for positive,

"−" for negative" , or "0".

x	f(x)	f '(x)	f "(x)	Concavity (up or down)
1	+	+	−	down
2	+			
3	−			
4				

Feeling the Second Derivative

Earlier we saw that if a function f(t) represents the position of a car at time t, then f '(t) is the velocity

and f "(t) is the acceleration of the car at the instant t.

If we are driving along a straight, smooth road, then what we feel is the acceleration of the car:

a large positive acceleration feels like a "push" **toward the back** of the car,

a large negative acceleration (a deceleration) feels like a "push" **toward the front** of the car, and

an acceleration of 0 for a period of time means the velocity is constant and we do not feel pushed in

either direction.

On a moving vehicle it is possible to measure these "pushes", the acceleration, and from that information to

determine the velocity of the vehicle, and from the velocity information to determine the position. Inertial

guidance systems in airplanes use this tactic: they measure front–back, left–right and up–down

acceleration several times a second and then calculate the position of the plane. They also use computers to

keep track of time and the rotation of the earth under the plane. After all, in 6 hours the earth has made a

quarter of a revolution, and Dallas has rotated more than 5000 miles!

Example 1: The upward acceleration of a rocket was $a(t) = 30$ m/s^2 for the first 6 seconds of flight,

$0 \le t \le 6$. The velocity of the rocket at t=0 was 0 m/s and the height of the rocket above the

ground at t=0 was 25 m. Find a formula for the height of the rocket at time t and determine the

height at t = 6 seconds.

Solution: $v'(t) = a(t) = 30$ so $v(t) = 30t + K$ for some constant K. We also know $v(\mathbf{0}) = 0$ so

$30(\mathbf{0}) + K = 0$ and $K = 0$. Therefore, $v(t) = 30t$.

Similarly, $h'(t) = v(t) = 30t$ so $h(t) = 15t^2 + C$ for some constant C. We know that $h(\mathbf{0}) = 25$ so

$15(\mathbf{0})^2 + C = 25$ and $C = 25$. Then $h(t) = 15t^2 + 25$. $h(6) = 15(6)^2 + 25 = 565$ m.

f '' and Extreme Values of f

The concavity of a function can also help us determine whether a critical point is a

maximum or minimum or neither. For example, if a point is at the bottom of a

concave up function (Fig. 7), then the point is a minimum.
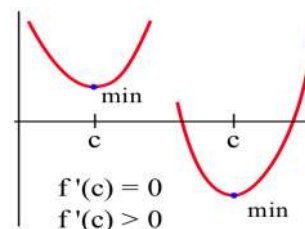
$f'(c) = 0$
$f'(c) > 0$
Fig. 7

The Second Derivative Test for Extremes:

(a) If $f'(c) = 0$ and $f''(c) < 0$ then f is concave down and has a local maximum at $x = c$.

(b) If $f'(c) = 0$ and $f''(c) > 0$ then f is concave up and has a local minimum at $x = c$.

(c) If $f'(c) = 0$ and $f''(c) = 0$ then f may have a local maximum, a minimum or neither at $x = c$.

Proof: (a) Assume that $f'(c) = 0$. If $f''(c) < 0$ then f is concave down at $x = c$ so the graph of f will

be below the tangent line $L(x)$ for values of x near c. The tangent line, however, is given by

$L(x) = f(c) + f'(c)(x - c) = f(c) + 0(x - c) = f(c)$, so if x is close to c then $f(x) < L(x) = f(c)$ and f

has a local maximum at $x = c$. The proof of (b) for a local minimum of f is similar.

(c) If $f'(c) = 0$ and $f''(c) = 0$, then we cannot immediately conclude anything about local maximums or

minimums of f at $x = c$. The functions $f(x) = x^4$, $g(x) = -x^4$, and $h(x) = x^3$ all have their first and

second derivatives equal to zero at $x = 0$, but f has a local minimum at 0, g has a local maximum at 0,

and h has neither a local maximum nor a local minimum at $x = 0$.

The Second Derivative Test for Extremes is very useful when f '' is easy to calculate and evaluate.

Sometimes, however, the First Derivative Test or simply a graph of the function is an easier way to

determine if we have a local maximum or a local minimum –– it depends on the function and on which

tools you have available to help you.

Practice 2: $f(x) = 2x^3 - 15x^2 + 24x - 7$ has critical numbers x = 1 and 4. Use the Second Derivative

Test for Extremes to determine whether f(1) and f(4) are maximums or minimums or neither.

Inflection Points

Definition: An **inflection point** is a point on the graph of a function where the concavity of the

function changes, from concave up to down or from concave down to up.

Practice 3:   Which of the labelled points in Fig. 8 are inflection points?

To find the inflection points of a function we can use the second derivative

of the function. If f "(x) > 0 , then the graph of f is concave up at the point

Fig. 8

(x, f(x)) so (x, f(x)) is not an inflection point. Similarly, if f "(x) < 0 , then

the graph of f is concave down at the point (x,f(x)) and the point is not an

inflection point. The only points left which can possibly be inflection points are the places where f "(x) is 0

or undefined (f ' is not differentiable). To find the inflection points of a function we only need to check the

points where f "(x) is 0 or undefined. If f "(c) = 0 or is undefined, then the point (c,f(c)) **may** or **may not**

be an inflection point — we would need to check the concavity of f on each side of x = c. The functions in

the next example illustrate what can happen.
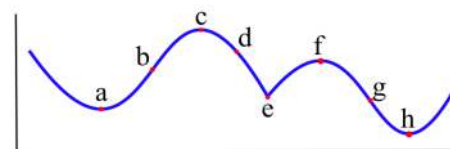
Example 2:   Let $f(x) = x^3$, $g(x) = x^4$ and $h(x) = x^{1/3}$ (Fig.

9). For which of these functions is the point

(0,0) an inflection point?

$f(x) = x^3$ $g(x) = x^4$ $h(x) = x^{1/3}$
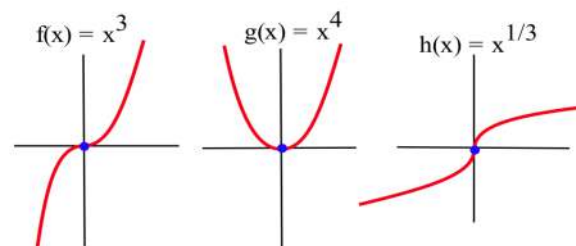
Fig. 9

Solution: Graphically, it is clear that the concavity of $f(x) = x^3$

and $h(x) = x^{1/3}$ changes at (0,0), so (0,0) is an inflection

point for f and h. The function $g(x) = x^4$ is concave up

everywhere so (0,0) is not an inflection point of g.

If $f(x) = x^3$, then $f '(x) = 3x^2$ and f "(x) = 6x . The only point at which f "(x) = 0 or is undefined

(f ' is not differentiable) is at x = 0. If x < 0, then f "(x) < 0 so f is concave down. If x > 0 , then

f "(x) > 0 so f is concave up. At x = 0 the concavity changes so the point (0,f(0)) = (0,0) is an

inflection point of x^3 .

If $g(x) = x^4$, then $g '(x) = 4x^3$ and g "(x) = $12x^2$. The only point at which g "(x) = 0 or is

undefined is at x = 0.   If x < 0, then g "(x) > 0 so g is concave up. If x > 0 , then g "(x) > 0 so g

is also concave up. At x = 0 the concavity **does not change** so the point (0, g(0)) = (0,0) is **not an**

inflection point of x^4 .

If $h(x) = x^{1/3}$, then $h'(x) = \frac{1}{3} x^{-2/3}$ and $h''(x) = -\frac{2}{9} x^{-5/3}$. h" is not defined if $x = 0$, but

h "(negative number) > 0 and h "(positive number) < 0 so h changes concavity at (0,0) and (0,0)

is an inflection point of h.

Practice 4: Find the inflection points of $f(x) = x^4 - 12x^3 + 30x^2 + 5x - 7$.

Example 3: Sketch graph of a function with $f(2) = 3, f'(2) = 1,$ and an

inflection point at (2,3) .Solution: Two solutions are given in Fig. 10.
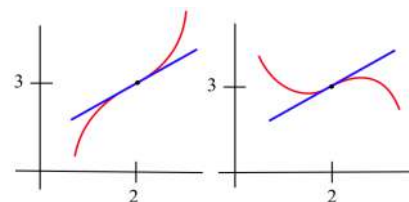
Fig. 10

PROBLEMS

In problems 1 and 2, each quotation is a statement about a quantitity of something changing over time.
Let f(t) represent the quantity at time t. For each quotation, tell what f represents and whether the first
and second derivatives of f are positive or negative.

1.   (a)   "Unemployment rose again, but the rate of increase is smaller than last month."

(b)   "Our profits declined again, but at a slower rate than last month."

(c)   "The population is still rising and at a faster rate than last year."

2.   (a)   "The child's temperature is still rising, but slower than it was a few hours ago."

(b)   "The number of whales is decreasing, but at a slower rate than last year."

(c)   "The number of people with the flu is rising and at a faster rate than last month."

3.   Sketch the graphs of functions which are defined and concave up everywhere and which have

(a) no roots.   (b) exactly 1 root.   (c) exactly 2 roots.   (d) exactly 3 roots.

4.   On which intervals is the function in Fig. 11

(a) concave up? (b) concave down?

5.   On which intervals is the function in Fig. 12

(a) concave up? (b) concave down?
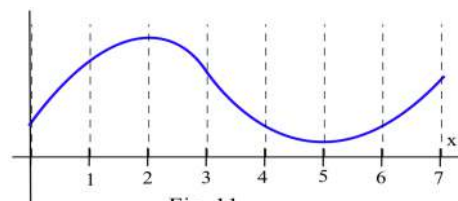
Fig. 11

In problems 6 – 10, a function and values of x so that $f'(x) = 0$
are given. Use the Second Derivative Test to determine whether each
point $(x, f(x))$ is a local maximum, a local minimum or neither

6. $f(x) = 2x^3 - 15x^2 + 6$, $x = 0$, 5 .

7. $g(x) = x^3 - 3x^2 - 9x + 7$, $x = -1$, 3 .

8. $h(x) = x^4 - 8x^2 - 2$, $x = -2, 0, 2$.

9. $f(x) = \sin^5(x)$, $x = \pi/2, \pi, 3\pi/2$
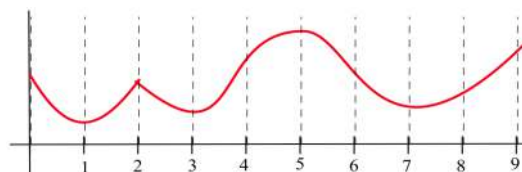
Fig. 12

10. $f(x) = x \cdot \ln(x)$, $x = 1/e$.

Fig. 13

11. At which labeled values of x in Fig. 13 is the point

   $(x, f(x))$ an inflection point?
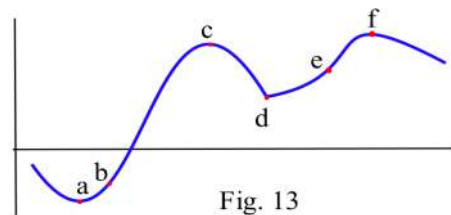
12. At which labeled values of x in Fig. 14 is the point

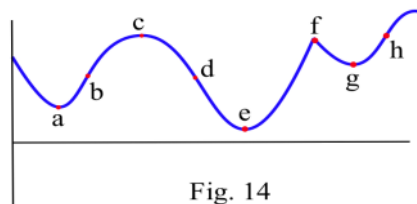 $(x, g(x))$ an inflection point?

Fig. 14

13. How many inflection points can a

   (a) quadratic polynomial have?   (b) cubic polynomial have?

   (c) polynomial of degree n have?

14. Fill in the table with "+", "−", or "0" for the function in Fig. 15.

x	f(x)	f '(x)	f "(x)
0			
1			
2			
3			
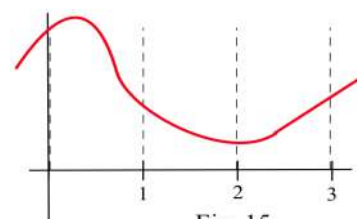
Fig. 15

15. Fill in the table with "+", "−", or "0" for the function in Fig. 16

x	g(x)	g '(x)	g "(x)
0			
1			
2			
3			
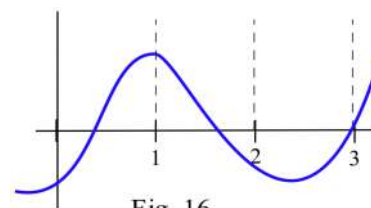
Fig. 16

16. Sketch functions f for x–values near 1 so $f(1) = 2$ and

   (a) $f '(1) = +$, $f "(1) = +$ (b) $f '(1) = +$, $f "(1) = -$

   (c) $f '(1) = -$, $f "(1) = +$

   (d) $f '(1) = +$, $f "(1) = 0$, $f "(1^-) = -$, $f "(1^+) = +$ (e) $f '(1) = +$, $f "(1) = 0$, $f "(1^-) = +$, $f "(1^+) = -$

17. Some people like to think of a concave up graph as one which will "hold water" and of a concave down

   graph as one which will "spill water." That description is accurate for a concave down graph, but it

   can fail for a concave up graph. Sketch the graph of a function which is concave up on an interval, but

   which will not "hold water".

18. The function $f(x) = \dfrac{1}{\sqrt{2\pi}} e^{-(x-c)^2/(2b^2)}$ is called the Gaussian distribution, and its graph is a bell–shaped

curve (Fig. 17) that occurs commonly in statistics.
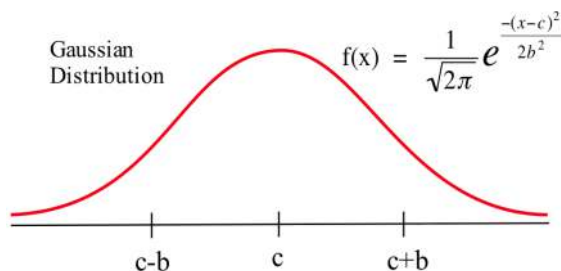
Gaussian Distribution $f(x) = \dfrac{1}{\sqrt{2\pi}} e^{\frac{-(x-c)^2}{2b^2}}$
Fig. 17

   (i)   Show that f has a maximum at $x = c$. (The

        value c is called the mean of this distribution.)

   (ii) Show that f has inflection points where

        $x = c + b$ and $x = c - b$. (The value b is called

        the standard deviation of this distribution.)

Section 4.4   **PRACTICE Answers**

Practice 1:   See Fig. 6.

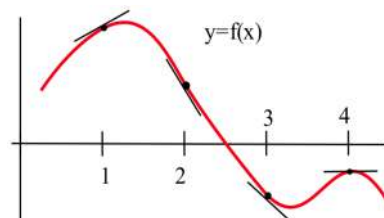| x	f(x)	f '(x)	f "(x)	Concavity (up or down)
1	+	+	–	down
2	+	–	–	**down**
3	–	–	+	**up**
4	–	**0**	–	**down**

Fig. 6

Practice 2: $f(x) = 2x^3 - 15x^2 + 24x - 7$.

$f '(x) = 6x^2 - 30x + 24$ which is defined for all x.

$f '(x) = 0$ if $x = 1, 4$ (critical values).

$f "(x) = 12x - 30$.

$f "(1) = -18$ so f is concave down at the critical

value $x = 1$ so $(1, f(1)) = (1,4)$ is a rel. max.

$f "(4) = +18$ so f is concave up at the critical

value $x = 4$ so $(4, f(4)) = (4, -23)$ is a rel. min.

Fig. 18 shows the graph of f.
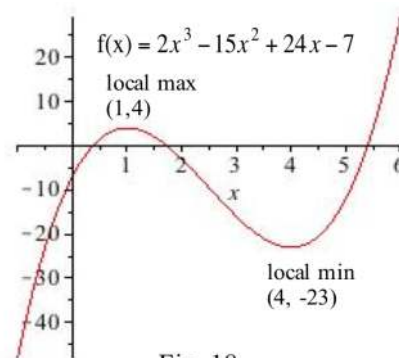
Fig. 18

Practice 3:   The points labeld (b) and (g) in Fig. 8 are inflection points.

Practice 4: $f(x) = x^4 - 12x^3 + 30x^2 + 5x - 7$. $f '(x) = 4x^3 - 36x^2 + 60x + 5$.

$f "(x) = 12x^2 - 72x + 60 = 12(x^2 - 6x + 5) = 12(x - 1)(x - 5)$.

The only candidates to be Inflection Points are $x = 1$ and $x = 5$.

If $x < 1$, then $f "(x) = 12(x - 1)(x - 5) = 12($ neg $)($ neg $)$ is positive.

If $1 < x < 5$, then $f "(x) = 12(x - 1)(x - 5) = 12($ pos $)($ neg $)$ is negative.

If $5 < x$, then $f "(x) = 12(x - 1)(x - 5) = 12($ pos $)($ pos $)$ is positive.

f changes concavity at $x = 1$ and $x = 5$ so

$x = 1$ and $x = 5$ are Inflection Points.

Fig. 19 shows the graph of f.
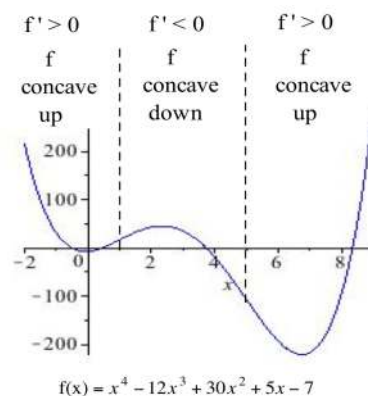
Fig. 19

4.5   APPLIED MAXIMUM AND MINIMUM PROBLEMS

We have used derivatives to help find the maximums and minimums of some functions given by equations, but it is very unlikely that someone will simply hand you a function and ask you to find its extreme values. More typically, someone will describe a problem and ask your help in maximizing or minimizing something: "What is the largest volume package which the post office will take?"; "What is the quickest way to get from here to there?"; or "What is the least expensive way to accomplish some task?" Usually these problems have some restrictions or constraints on what is allowed, and sometimes neither the problem nor the constraints are clearly stated.

Before we can use calculus or other mathematical techniques to solve the max/min problem, we need to **understand** what is really being asked. We need to **translate** the problem into a mathematical form which we can **solve**, and we need to **check** our mathematical solution to see if it is really a solution of the original problem. Often, the hardest parts of the problem are understanding the problem and translating it into a mathematical form.

In this section we examine some problems which require understanding, translation, solution, and checking. Most of these problems are not as complicated as those a working scientist, engineer or economist needs to solve, but they represent a step in developing the required skills.

Example 1:   The company you own has a large supply of 8 inch by 15 inch rectangular pieces of tin, and you decide to make them into boxes by cutting a square from each corner and folding up the sides (Fig. 1). For example, if you cut a 1 inch square from each corner the resulting 6 inch by 13 inch by 1 inch box has a volume of 78 cubic inches. The amount of money you get for a box depends on how much the box holds, so you want to make boxes with the largest possible volumes. How large a square should you cut from each corner?

Solution: First we need to understand the problem, and a diagram can be very helpful. Then we need to translate it into a mathematical problem:

 * **identify the variables**,
 * **label the variable and constant parts of the diagram**, and
 * **represent the quantity to be maximized as a function.**

Fig. 1

If we label the side of the square as x inches, then the box is x inches high, 8 – 2x inches wide, and 15 – 2x inches long, so the volume is (length)·(width)·(height) = (15 – 2x)·(8 – 2x)·(x) $= 4x^3 - 46x^2 + 120x$ cubic inches. Now we have a mathematical problem, maximize $V(x) = 4x^3 - 46x^2 + 120x$, and we can use the calculus techniques from the previous sections.
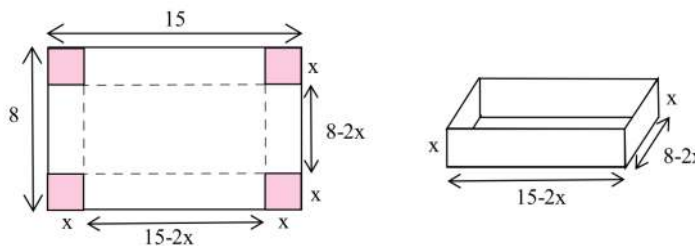
$V'(x) = 12x^2 - 92x + 120$, and we need to find the critical points. (i) We can find where $V'(x) = 0$ by factoring or using the quadratic formula: $V'(x) = 12x^2 - 92x + 120 = 4(3x - 5)(x - 6) = 0$ if $x = 5/3$ or $x = 6$, so $x = 5/3$ and $x = 6$ are critical points of V. (ii) $V'(x)$ is a polynomial so it is always defined and there are no critical points from an undefined derivative. (iii) What are the endpoints for x in this problem? A square cannot have a negative length so $x \geq 0$. We cannot remove more than half of the width, so $8 - 2x \geq 0$ and $x \leq 4$. Together, these two inequalities say that $0 \leq x \leq 4$, so the endpoints are $x = 0$ and $x = 4$. (Note that the value $x = 6$ is not in this interval, so $x = 6$ does not maximize the volume and we do not consider it further.)

The maximum volume must occur at one of the critical points $x = 0, 5/3$, or 4: $V(0) = 0$, $V(5/3) = 2450/27 \approx 90.74$ cubic inches, and $V(4) = 0$. The maximum volume of the box occurs when a 5/3 inch by 5/3 inch square is removed from each corner, and resulting box is 5/3 inches high, $8 - 2(5/3) = 14/3$ inches wide, and $15 - 2(5/3) = 35/3$ inches long.

Practice 1:   If you start with 7 inch by 15 inch pieces of tin, what size square should you remove from each corner so the box will have as large a volume as possible?
(Hint: $12x^2 - 88x + 105 = (2x - 3)(6x - 35)$)

We were fortunate in the previous example and practice problem because the functions we created to describe the volume were functions of only one variable. In some problems, the function we get will have more than one variable, and we will need to use additional information to change our function into a function of one variable. Typically the constraints will contain the additional information we need.

Example 2:   We want to fence a rectangular area in our backyard for a garden. One side of the garden is along the edge of the yard which is already fenced, so we only need to build a new fence along the other 3 sides of the rectangle (Fig. 2). If we have 80 feet of fencing available, what dimensions should the garden have in order to enclose the largest possible area?
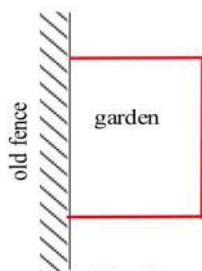
Solution: The first step is to understand the problem, and a **diagram** or picture of the situation often helps. Next, we need to **identify the variables**: in this case, the length, call it x, and width, call it y, of the garden. Fig. 3 shows the **labeled diagram** so now we can write a formula for the function which we want to maximize:

Maximize A = area of the rectangle = (length)·(width) = x·y .

Unfortunately, our function A has two variables, x and y, so we need to find a relationship between them (an equation containing both x and y) which we can solve for one of x or y. The constraint in this problem says that "we have 80 feet of fencing available" so $x + 2y = 80$ and $y = 40 - (x/2)$. Then
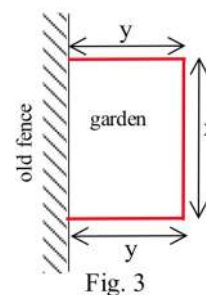
Fig. 2

garden

old fence

old fence

garden

y

x

y

Fig. 3

$A = x \cdot y = x \cdot (\, 40 - (x/2) \,) = 40x - \dfrac{x^2}{2}$, a function of one variable. We want to maximize A.

$A' = 40 - x$. The only time $A' = 0$ is when $x = 40$, so $x = 40$ so there is only one critical point of type (i). A is differentiable for all x so there are no critical numbers of the type (ii). Finally, $0 \le x \le 80$ (why?) so the only critical points of type (iii) are when $x = 0$ and $x = 80$. The only critical points of A are when $x = 0, 40$, and 80 , and the maximum area occurs at one of them:

at the critical number $x = 0$, $A = 40(0) - \dfrac{(0)^2}{2} = 0$ square feet

at the critical number $x = 40$, $A = 40(40) - \dfrac{(40)^2}{2} = 800 \ ft^2$

at the critical number $x = 80$, $A = 40(80) - \dfrac{(80)^2}{2} = 0 \ ft^2$

so the largest rectangular garden has an area of 800 square feet and dimensions $x = 40$ feet by $y = 40 - (x/2) = 40 - (40/2) = 20$ feet.

Practice 2:   Suppose you decide to fence the rectangular garden in the corner of your yard. Then two sides of the garden are bounded by the yard fence which is already there, so you only need to use the 80 feet of fencing to enclose the other two sides. What are the dimensions of the new garden of largest area? What are the dimensions of the rectangular garden of largest area in the corner of the yard if you have F feet of new fencing available?

Example 3:   You need to reach home as quickly as possible, but you are in a rowboat 4 miles from shore and your home is 2 miles up the coast (Fig. 4). If you can row at 3 miles per hour and walk at 5 miles per hour, toward which point on the shore should you row? Toward which point should you row if your home is 7 miles up the coast?
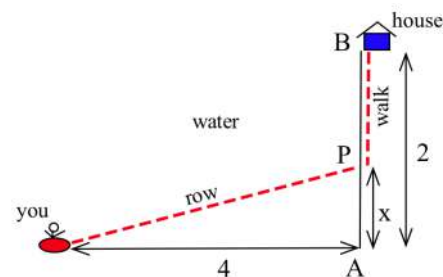

Fig. 4

Solution: Fig. 4 shows a labeled diagram with the variable x representing the distance from point A, the nearest shore point, to point P, the point you row toward. Then the total time, rowing and walking, is

$T = $ total time $ = $ (rowing time from boat to P) + (walking time from P to B)

$ = $ (distance from boat to P)/(rate from boat to P) + (distance from P to B)/(rate from P to B)

$ = \sqrt{x^2 + 4^2} / 3 + (2 - x)/5 \ = \ \dfrac{\sqrt{x^2 + 16}}{3} \ + \ \dfrac{2 - x}{5}$.

It is not reasonable to row to a point below A and then walk home, so $x \ge 0$. Similarly, we can conclude that $x \le 2$, so our interval is $0 \le x \le 2$ and the endpoints are $x = 0$ and $x = 2$.

To find the other critical numbers of T between $x = 0$ and $x = 2$, we need the derivative of T.

$$T\,'(x) = \frac{1}{3}\cdot\frac{1}{2}(x^2 + 16)^{-1/2}(2x) - \frac{1}{5} = \frac{x}{3\sqrt{x^2 + 16}} - \frac{1}{5}\;.$$

To find where T '(x) is zero, set T '(x) = 0 and solve:

$$T\,'(x) = \frac{x}{3\sqrt{x^2 + 16}} - \frac{1}{5} = 0 \text{ so } \frac{x}{3\sqrt{x^2 + 16}} = \frac{1}{5} \text{ and}$$

$5x = 3\sqrt{x^2 + 16}$ so $25x^2 = 9x^2 + 144$ and $x = \pm\,3$. Neither of these numbers, however, is in our interval $0 \le x \le 2$ so neither of them gives a minimum time.

T is differentiable for all values of x, so there are no critical numbers of type (ii).

The only critical numbers for T on this interval are x = **0** and x = **2**: $T(\mathbf{0}) = \frac{\sqrt{0 + 16}}{3} + \frac{2 - 0}{5} =$

$\frac{4}{3} + \frac{2}{5} \approx 1.73$ hours and $T(\mathbf{2}) = \frac{\sqrt{2^2 + 16}}{3} + \frac{2 - 2}{5} = \frac{\sqrt{20}}{3} + 0 \approx 1.49$ hours. The quickest

route is when P is 2 miles down the coast. You should row directly toward home.

If your home is 7 miles down the coast, then the interval for x is $0 \le x \le 7$ which has the endpoints

x = 0 and x = 7. Our function for the travel time is $T(x) = \frac{\sqrt{x^2 + 16}}{3} + \frac{7 - x}{5}$ and

$T\,'(x) = \frac{x}{3\sqrt{x^2 + 16}} - \frac{1}{5}$ so the only point in our interval where T(x) ' = 0 is at x = 3.

The only critical numbers for T in the interval are x = **0**, x = **3**, and x = **7**:

$$T(\mathbf{0}) = \frac{\sqrt{0^2 + 16}}{3} + \frac{7 - 0}{5} = \frac{4}{3} + \frac{7}{5} \approx 2.73 \text{ hours}$$

$$T(\mathbf{7}) = \frac{\sqrt{7^2 + 16}}{3} + \frac{7 - 7}{5} = \frac{\sqrt{65}}{3} + 0 \approx 2.68 \text{ hours}$$

$$T(\mathbf{3}) = \frac{\sqrt{3^2 + 16}}{3} + \frac{7 - 3}{5} = \frac{5}{3} + \frac{4}{5} \approx 2.47 \text{ hours.}$$

The quickest way home is to aim for a point P which is 3 miles down the coast, row directly to P, and then walk along the coast to home.
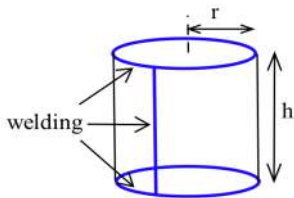

Fig. 5

One challenge of max/min problems is that they may require geometry or trigonometry or other mathematical facts and relationships.

Example 4:   Find the height and radius of the least expensive closed cylinder which has a volume of 1000 cubic inches. Assume that the materials are free, but that it costs 80¢ per inch to weld the top and bottom onto the cylinder and to weld the seam up the side of the cylinder (Fig. 5).

Solution: If we let r be the radius of the cylinder and h be its height, then the volume

$V = \pi r^2 h = 1000$. The function we want to minimize is cost, and

C   = total welding cost = (top seam cost) + (bottom seam cost) + (side seam cost)

= (top seam length)·(80¢/inch) + (bottom seam length)·(80¢/in) + (side seam length)·(80¢/in)

= (2πr)·(80) + (2πr)·(80) + (h)·(80) = 320πr + 80 h.

Unfortunately, our function C is a function of two variables, r and h , but we can use the

information in the constraint, $V = \pi r^2 h = 1000$, to solve for h and then substitute this h into the

formula for C: $1000 = \pi r^2 h$ so $h = \dfrac{1000}{\pi r^2}$ and then $C = 320\pi r + 80\,h = 320\pi r + 80(\dfrac{\mathbf{1000}}{\mathbf{\pi r^2}})$), a

function of one variable. $C' = 320\pi - \dfrac{160000}{\pi r^3}$, and C is a minimum when C ' = 0: at

$r = \sqrt[3]{\dfrac{500}{\pi^2}} \approx 3.7$ inches and $h = \dfrac{1000}{\pi r^2} \approx \dfrac{1000}{\pi (3.7)^2} \approx 23.3$ inches.

Practice 3:   Find the height and radius of the least expensive closed

cylinder which has a volume of 1000 cubic inches. Assume that the

only cost for this cylinder is the cost of the materials: the material

for the top and bottom costs 5¢ per square inch, and the material

for the sides costs 3¢ per square inch (Fig. 6).
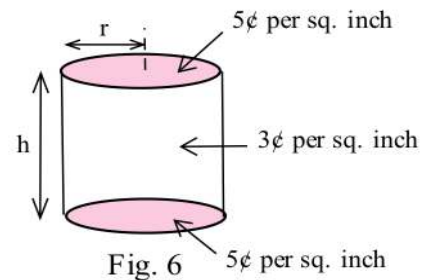
Fig. 6

Example 5:   Find the dimensions of the least expensive rectangular box which is three times as long as it

is wide and which holds 100 cubic centimeters of water. The material for the bottom costs 7¢ per cm^2,

the sides cost 5¢ per cm^2 and the top costs 2¢ per cm^2.

Solution: Label the box so w = width, l = length, and h = height. Then our cost function C is

C   = (bottom cost) + (cost of front and back) + (cost of ends) + (top cost)

= (bottom area)·(7¢) + (front and back area)·(5¢) + (ends area)·(5¢) + (top area)·(2¢)

= (wl)·(7) + (2lh)·(5) + (2wh)·(5) + (wl)·(2) = 7wl + 10lh + 10wh + 2wl = 9(wl) + 10(lh) + 10(wh).

Unfortunately, C is a function of 3 variables, w, l, and h, but we can use the other information in the

constraints to eliminate some of the variables:

the box is "three times as long as it is wide" so l = 3w and

$C = 9wl + 10lh + 10wh = 9w(\mathbf{3w}) + 10(\mathbf{3w})h + 10wh = 27w^2 + 40wh.$

We also know that the volume V is 100 in^3 and $V = lwh = 3w^2h$ (since l = 3w), so $h = \dfrac{100}{3w^2}$.

Then $C = 27w^2 + 40wh = 27w^2 + 40w(\dfrac{100}{3w^2}) = 27w^2 + \dfrac{4000}{3w}$, a function of one variable.

$$C' = 54w - \frac{4000}{3w^2} \ , \text{ and } C \text{ is minimized when } w = \sqrt[3]{\frac{4000}{162}} \approx 2.91 \text{ inches } (1 = 3w \approx 8.73 \text{ inches},$$

$$\text{and } h = \frac{100}{3w^2} \approx 3.94 \text{ inches}). \text{ The minimum cost is approximately } \$6.87 \ .$$

Problems described in words are usually more difficult to solve because we first need to understand and "translate" the problem into a mathematical problem, and, unfortunately, those skills only seem to come with practice. With practice, however, you will start to recognize patterns for understanding, translating, and solving these problems, and you will develop the skills you need. So read carefully, draw pictures, think hard, and do the best you can.

Problems

1. (a) You have 200 feet of fencing to enclose a rectangular vegetable garden. What should the dimensions of your garden be in order to enclose the largest area?

    (b) Show that if you have P feet of fencing available, the garden of greatest area is a square.

    (c) What are the dimensions of the largest rectangular garden you can enclose with P feet of fencing if one edge of the garden borders a straight river and does not need to be fenced?

    (d) Just thinking — calculus will not help with this one: What do you think is the shape of the largest garden which can be enclosed with P feet of fencing if we do not require the garden to be rectangular? What do you think is the shape of the largest garden which can be enclosed with P feet of fencing if one edge of the garden borders a river and does not need to be fenced?

2. (a) You have 200 feet of fencing available to construct a rectangular pen with a fence divider down the middle (see Fig. 7). What dimensions of the pen enclose the largest total area?

    (b) If you need 2 dividers, what dimensions of the pen enclose the largest area?

    (c) What are the dimensions in parts (a) and (b) if one edge of the pen borders on a river and does not require any fencing?

Fig. 7

Fig. 8

3. You have 120 feet of fencing to construct a pen with 4 equal sized stalls.

    (a) If the pen is rectangular and shaped like the Fig. 8, what are the dimensions of the pen of largest area and what is that area?

    (b) The square pen in Fig. 9 uses 120 feet of fencing and encloses a larger area (400 square feet) than the best design in part (a). Design a pen which uses only 120 feet of fencing and has 4 equal sized stalls but which encloses even more than 400 square feet. (Suggestion: don't use rectangles and squares.)
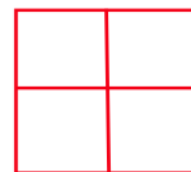
Fig. 9

4. (a) You have a 10 inch by 15 inch piece of tin which you plan to form into a box (without a top) by cutting a square from each corner and folding up the sides (see Fig. 10). How much should you cut from each corner so the resulting box has the greatest volume?

    (b) If the piece of tin is A inches by B inches, how much should you cut from each corner so the resulting box has the greatest volume?
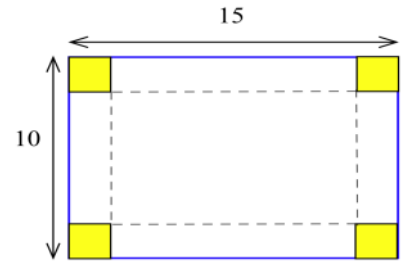
Fig. 10

5. You have a 10 inch by 10 inch piece of cardboard which you plan to cut and fold as shown in Fig. 11 to form a box with a top. Find the dimensions of the box which has the largest volume.
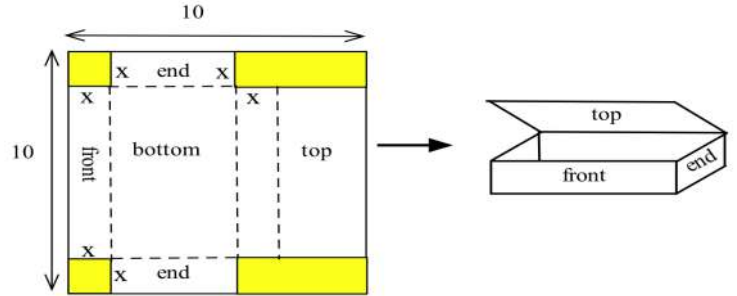
Fig. 11

6. (a) You have been asked to bid on the construction of a square-bottomed box with no top which will hold 100 cubic inches of water. If the bottom and sides are made from the same material, what are the dimensions of the box which uses the least material? (Assume that no material is wasted.)

    (b) Suppose the box in part (a) uses different materials for the bottom and the sides. If the bottom material costs 5¢ per square inch and the side material costs 3¢ per square inch, what are the dimensions of the least expensive box which will hold 100 cubic inches of water?

    (This is a "classic" problem which has many variations. We could require that the box be twice as long as it is wide, or that the box have a top, or that the ends cost a different amount than the front and back, or even that it costs some amount of money to weld each inch of edge. You should be able to set up the cost equations for these variations.)

7. (a) Determine the dimensions of the least expensive cylindrical can which will hold 100 cubic inches if the materials cost 2¢, 5¢ and 3¢ respectively for the top, bottom and sides.

    (b) How do the dimensions of the least expensive can change if the bottom material costs more than 5¢ per square inch?

8. You have 100 feet of fencing to build a pen in the shape of a circular sector, the "pie slice" in Fig. 12. The area of such a sector is rs/2.

    (a) What value of r maximizes the enclosed area?

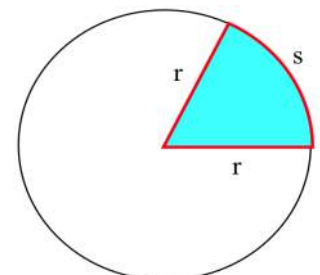   (b) What is the central angle when the area is maximized?

Fig. 12

9. You are a lifeguard standing at the edge of the water when you notice a swimmer in trouble (Fig. 13). Assuming you can run about 8 meters per second and swim about 2 m/s, how far along the shore should you run before diving into the water in order to reach the swimmer as quickly as possible?
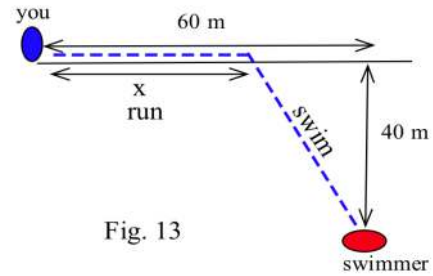

Fig. 13

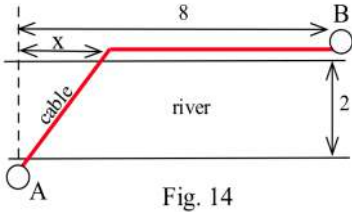10. (a) You have been asked to determine the least expensive route for a telephone cable which connects Andersonville with Beantown (see Fig. 14). If it costs $5000 per mile to lay the cable on land and $8000 per mile to lay the cable across the river and the cost of the cable is negligible, find the least expensive route.

    (b) What is the least expensive route if the cable costs $7000 per mile plus the cost to lay it.

Fig. 14

11. You have been asked to determine where a water works should be built along a river between Chesterville and Denton (see Fig. 15) to minimize the total cost of the pipe to the towns.
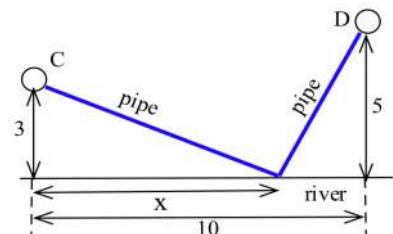
    (a) Assume that the same size (and cost) pipe is used to each town. (This part can be done quickly without using calculus.)

    (b) Assume that the pipe to Chesterville costs $3000 per mile and to Denton it costs $7000 per mile.
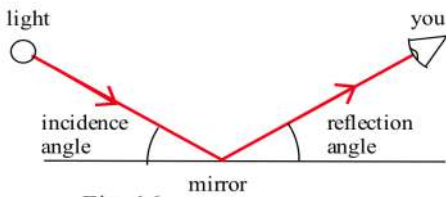

Fig. 15

Fig. 16

12. Light from a bulb at A is reflected off a flat mirror to your eye at point B (see Fig. 16). If the time (and length of the path) from A to the mirror and then to your eye is a minimum, show that the angle of incidence equals the angle of reflection. (Hint: This is similar to the previous problem.)
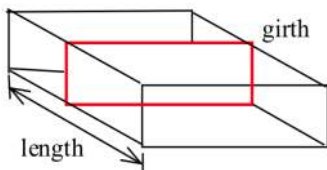
13. U.S. postal regulations state that the sum of the length and girth (distance around) of a package must be no more than 108 inches. (Fig. 17)

    (a) Find the dimensions of the acceptable box with a square end which has the largest volume.

    (b) Find the dimensions of the acceptable box which has the largest volume if its end is a rectangle twice as long as it is wide.

    (c)   Find the dimensions of the acceptable box with a circular end which has the largest volume.

Fig. 17

14. Just thinking — you don't need calculus for this problem: A spider and a fly are located on opposite corners of a cube (see Fig. 18). What is the shortest path along the surface of the cube from the spider to the fly?
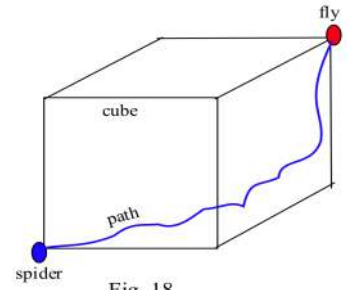

fly

cube

path

spider

Fig. 18

15. Two sides of a triangle are 7 and 10 inches respectively. What is the length of the third side so the area of the triangle will be greatest? (This problem can be done without using calculus. How? If you do use calculus, consider the angle θ between the two sides.)

16. Find the shortest distance from the point (2,0) to the curve

(a) $y = 3x - 1$ (b) $y = x^2$ (c) $x^2 + y^2 = 1$ (d) $y = \sin(x)$

17. Find the dimensions of the rectangle with the largest area if the base must be on the x-axis and its other two corners are on the graph of

(a) $y = 16 - x^2$ on $[-4, 4]$ (b) $x^2 + y^2 = 1$ on $[-1, 1]$

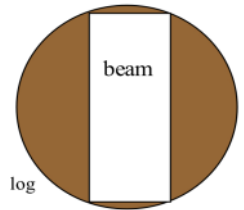(c) $|x| + |y| = 1$ on $[-1, 1]$ (d) $y = \cos(x)$ on $[-\pi/2, \pi/2]$

beam

log

Fig. 19

18. The strength of a wooden beam is proportional to the product of its width and the square of its height (Fig. 19).

(a) What are the dimensions of the strongest log which can be cut from a log with diameter 12 inches?

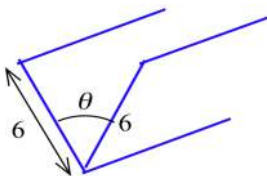(b) What are the dimensions of the strongest log which can be cut from a log with diameter d inches?

θ

6   6

Fig. 20

19. You have a long piece of 12 inch wide metal which you are going to fold along the center line to form a V–shaped gutter (Fig. 20). What angle θ will give the gutter which holds the most water (the largest cross–sectional area)?

20. You have a long piece of 8 inch wide metal which you are going to make into a gutter by bending up 3 inches on each side (Fig. 21). What angle θ will give the gutter which holds the most water (with the largest cross–sectional area)?

θ θ

3   3
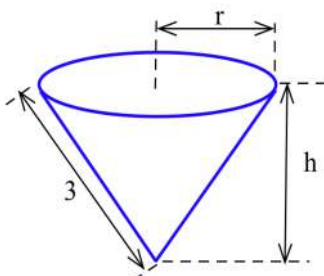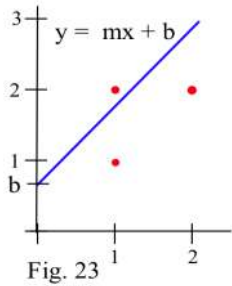
2

Fig. 21

r

h

3

Fig. 22

21. You have a 6 inch diameter circle of paper which you want to form into a drinking cup by removing a pie–shaped wedge and forming the remaining paper into a cone (Fig. 22). Find the height and top radius of so the volume of the cup is as large as possible.

Fig. 23

22. (a) What value of b minimizes the sum of the squares of the vertical distances of
the line y=2x+b from the points (1 , 1), (1 , 2) and (2 , 2)?   (Fig. 23)

(b) What slope m minimizes the sum of the squares of the vertical distances of
the line y=mx from the points (1 , 1), (1 , 2) and (2 , 2)?

(c) What slope m minimizes the sum of the squares of the vertical distances of
the line y = mx from the points (2 , 1), (4 , 3), (–2 , –2), and (–4 , –2)?

23. You own a small airplane which holds a maximum of 20 passengers. It costs you $100 per flight from
St. Thomas to St. Croix for gas and wages plus an additional $6 per passenger for the extra gas
required by the extra weight. The charge per passenger is $30 each if 10 people charter your plane (10
is the minimum number you will fly), and this charge is reduced by $1 per passenger for each
passenger over 10 who goes (that is, if 11 go they each pay $29, if 12 go they each pay $28, etc.).
What number of passengers on a flight will maximize your profits?

24. Prove: If f and g are differentiable functions and if the vertical distance
between f and g is greatest at x = c, then f '(c) = g '(c) and the tangent lines
to f and g are parallel when x = c. (Fig. 24)

Fig. 24
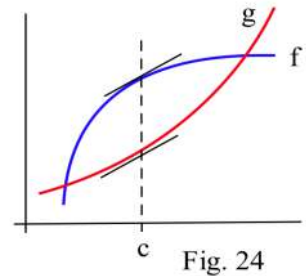
25. Profit is revenue minus expenses. Assume that revenue and expenses are
differentiable functions and show that when profit is maximized, then marginal
revenue (dR/dx) equals marginal expense (dE/dx).

26. D. Simonton claims that the "productivity levels" of people in different fields can be described as a
function of their "career age" t by $p(t) = e^{-at} - e^{-bt}$ where a and b are constants which depend on
the field of work, and career age is approximately 20 less than the actual age of the individual.

(a) Based on this model, at what ages do mathematicians (a=.03, b=.05), geologists (a=.02, b=.04),
and historians (a=.02, b=.03) reach their maximum productivity?

(b) Simonton says "With a little calculus we can show that the curve (p(t)) maximizes at
$t = \dfrac{1}{b-a} \ln(\dfrac{b}{a})$." Use calculus to show that Simonton is correct.

Note: Models of this type have uses for describing the behavior of groups, but it is dangerous and
usually invalid to apply group descriptions or comparisons to **individuals** in the group.
(Scientific Genius, by Dean Simonton, Cambridge University Press, 1988, pp. 69 – 73)

27. After the table was wiped and the potato chips dried off, the question

   remained: "Just how far could a can of cola be tipped before it fell over?"

   (i)   For a full can or an empty can the answer was easy: the center of gravity

         (cg) of the can is at the middle of the can, half as high as

         the height of the can, and we can tilt the can until the cg is directly

         above the bottom rim. (Fig. 25a) Find θ.

   (ii) For a partly filled can more thinking was needed. Some ideas you

         will see in chapter 5 let us calculate that the cg of a can containing x cm of cola is

         $C(x) = \dfrac{360 + 9.6x^2}{60 + 19.2x}$ cm above the bottom of the can. Find the height **x** of cola

         in the can which will make the cg as low as possible.

   (iii) Assuming that the cola is frozen solid (so the top of the cola stays parallel to the bottom of the can),

         how far can we tilt a can containing **x** cm of cola. (Fig. 25b)

   (iv) If the can contained **x** cm of liquid cola, could we tilt it more or less far than the frozen cola before
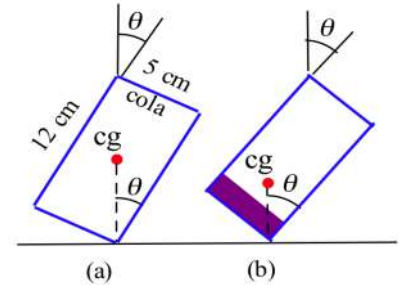
         it would fall over?

28. Just thinking — calculus will not help with this one:

   (a) Four towns are located at the corners of a square (see Fig. 26). What is the shortest length of road

        we can construct so that it is possible to travel along the road from any town to any other town?

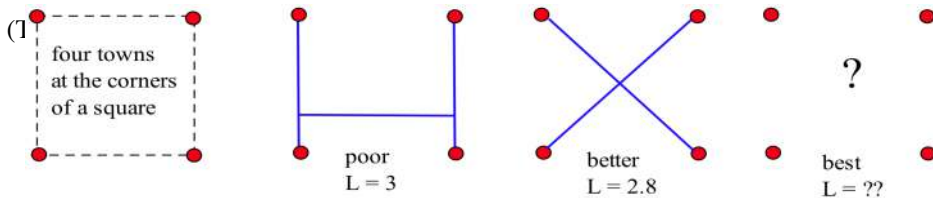   (b) What is the shortest connecting path for 5 towns located on the corners of a pentagon?

Fig. 25

Fig. 26

Generalized Max/Min Problems

The previous max/min problems were all numerical problems: the amount of fencing in problem 2 was 200

feet, the sides of the piece of tin in problem 4 were 10 and 15, and the parabola in problem 17a was

$y = 16 - x^2$. In doing those problems you might have noticed some patterns among the numbers in the

problem and the numbers in your answers, and you might have wondered if the pattern was an accident of

the numbers or if it there really was a pattern at work. Rather than trying several numerical examples to see

if the "pattern" holds, mathematicians, engineers, scientists and others sometimes resort to generalizing the

problem. We free the problem from the particular numbers by replacing the numbers with letters, and then

we solve the generalized problem. In this way, relationships between the values in the problem and those

in the solution can become more obvious. Solutions to these generalized problems are also useful if you

want to program a computer or calculator to quickly provide numerical answers.

29. (a) Find the dimensions of the rectangle with the greatest area that can be built so the base of the
rectangle is on the x–axis between 0 and 1 ($0 \leq x \leq 1$) and one corner of the rectangle is on the curve
$y = x^2$ (Fig. 27a). What is the area of this rectangle?

(b) Generalize the problem in part (a) for the parabola $y = Cx^2$ with $C > 0$ and $0 \leq x \leq 1$ (Fig. 27b).

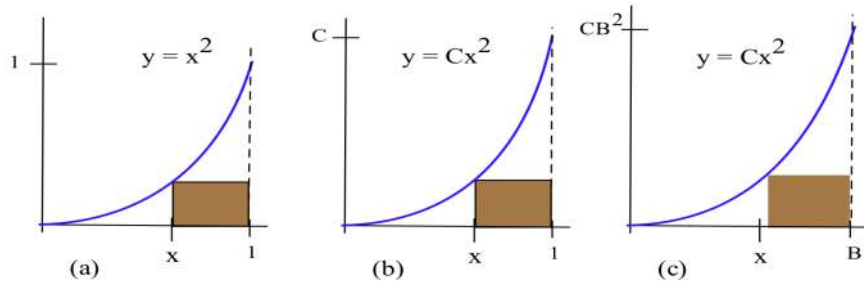(c) Generalize for the parabola $y = Cx^2$ with $C > 0$ and $0 \leq x \leq B$ (Fig. 27c).

Fig. 27

30. (a) Find the dimensions of the rectangle with the greatest area that can be built so the base of the
rectangle is on the x–axis between 0 and 1 ($0 \leq x \leq 1$) and one corner of the rectangle is on the curve
$y = x^3$. What is the area of this rectangle?

(b) Generalize the problem in part (a) for the curve $y = Cx^3$ with $C > 0$ and $0 \leq x \leq 1$.

(c) Generalize for the curve $y = Cx^3$ with $C > 0$ and $0 \leq x \leq B$.

(d) Generalize for the curve $y = Cx^n$ with $C > 0$, n a positive integer, and $0 \leq x \leq B$.

31. (a) The base of a right triangle is 50 and the height is 20 (Fig. 28a). Find the dimensions and area of
the rectangle with the greatest area that can be enclosed in the triangle if the base of the rectangle
must lie on the base of the triangle.

(b) The base of a right triangle is B and the height is H (Fig. 28b).
Find the dimensions and area of the rectangle with the greatest
area that can be enclosed in the triangle if the base of the
rectangle must lie on the base of the triangle.

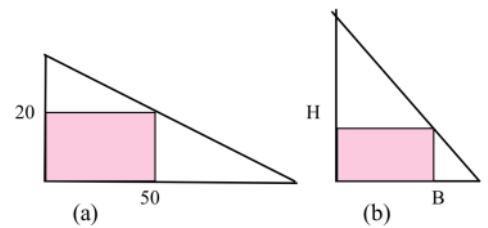(c) State your general conclusion from part (b) in words.

Fig. 28

32. (a) You have T dollars to buy fence to enclose a rectangular plot of land (Fig.
29). The fence for the top and bottom costs $5 per foot and for the sides
it costs $3 per foot. Find the dimensions of the plot with the largest area.
For this largest plot, how much money was used for the top and bottom,
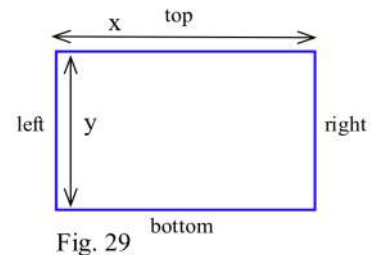and for the sides?

Fig. 29

(b) You have T dollars to buy fence to enclose a rectangular plot of land.
The fence for the top and bottom costs $A per foot and for the sides it costs $B per foot. Find the
dimensions of the plot with the largest area. For this largest plot, how much money was used for
the top and bottom (together), and for the sides (together)?

(c) You have T dollars to buy fence to enclose a rectangular plot of land. The fence costs $A per foot for the top, $B/foot for the bottom, $C/ft for the left side and $D/ft for the right side. Find the dimensions of the plot with the largest area. For this largest plot, how much money was used for the top and bottom (together), and for the sides (together)?

33. Determine the dimensions of the least expensive cylindrical can which will hold V cubic inches if the top material costs $A per square inch, the bottom material costs $B/in^2, and the side material costs $C/in^2.

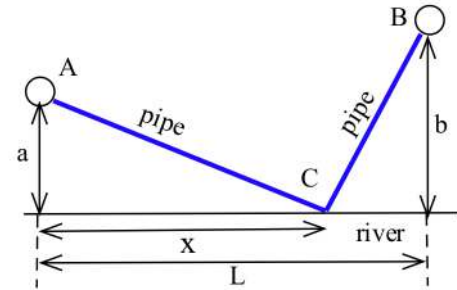34. Find the location of C in Fig. 30 so the sum of the distances from A to C and from C to B is a minimum.

Fig. 30

Section 4.5   PRACTICE Answers

Practice 1: $V(x) = x(15 - 2x)(7 - 2x) = 4x^3 - 44x^2 + 105x.$

$V'(x) = 12x^2 - 88x + 105 = (2x - 3)(6x - 35)$ which is defined for all x so the only critical numbers are the endpoints x = 0 and x = 7/2 and the places where V ' equals 0, at x = 3/2 and x = 35/6 (but 35/6 is not in the interval [0, 7/2] so it is not practical for this applied problem).

The maximum volume must occur when x = 0, x = 3/2, or x = 7/2):

$V(0) = 0 \cdot (15 - 2 \cdot 0) \cdot (7 - 2 \cdot 0) = 0$

$V(\dfrac{3}{2}) = \dfrac{3}{2} \cdot (15 - 2 \cdot \dfrac{3}{2}) \cdot (7 - 2 \cdot \dfrac{3}{2})$

$= \dfrac{3}{2}(12)(4) = 72$ max.

$V(\dfrac{7}{2}) = \dfrac{7}{2} \cdot (15 - 2 \cdot \dfrac{7}{2}) \cdot (7 - 2 \cdot \dfrac{7}{2})$

$= \dfrac{7}{2}(8)(0) = 0$
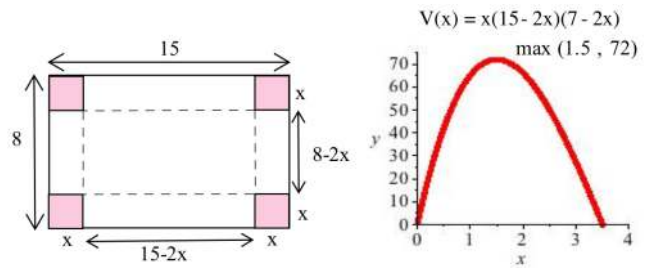
Fig. 31 shows the graph of V(x).

Fig. 31

Practice 2:   (a) We have 80 feet of fencing. (See Fig. 32). Our assignment is to maximize the area of the garden: A = x·y (two variables). Fortunately we have the constraint that x + y = 80 so y = 80 – x, and our assignment reduces to maximizing a function of one variable:

maximize $A = x \cdot y = x \cdot (80 - x) = 80x - x^2$.

A ' = 80 – 2x so A ' = 0 when x = 40.

A '' = –2 so A is concave down, and A has a

maximum at x = 40.

The maximum area is A = 40·40 = 1600 square feet when

x = 40 ft. and y = 40 ft. The maximum area garden is a square.

Fig. 32

(b) This is very similar to part (a) except we have F feet of fencing instead of 80 feet.

$x + y = F$ so $y = F - x$, and we want to maximize $A = xy = x(F - x) = Fx - x^2$.

$A' = F - 2x$ so $A' = 0$ when $x = F/2$ and $y = F/2$. The maximum area is $A = F^2/4$ square

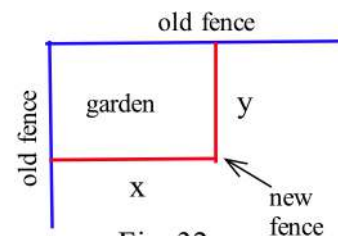feet and that occurs when the garden is a square and half of the new fence is used on each of the

two new sides.

Practice 3: Cost $C = 5(\text{area of top}) + 3(\text{area of sides}) + 5(\text{area of bottom}) = 5(\pi r^2) + 3(2\pi rh) + 5(\pi r^2)$

so our assignment is to minimize $C = 10\pi r^2 + 6\pi rh$, a function of two variables r and h.

Fortunately we also have the constraint that volume $= 1000 \text{ in}^3 = \pi r^2 h$ so $h = \dfrac{1000}{\pi r^2}$. Then

$$C = 10\pi r^2 + 6\pi r(\dfrac{1000}{\pi r^2}) = 10\pi r^2 + \dfrac{6000}{r} \quad \text{so} \quad C' = 20\pi r - \dfrac{6000}{r^2}. \quad C' = 0 \text{ if } 20\pi r - \dfrac{6000}{r^2} = 0$$

so $20\pi r^3 = 6000$ and $r = (\dfrac{6000}{20\pi})^{1/3} \approx 4.57$ in. Then $h = \dfrac{1000}{\pi r^2} \approx \dfrac{1000}{\pi(4.57)^2} \approx 15.24$ in.

($C'' = 20\pi + \dfrac{12000}{r^3} > 0$ for all $r > 0$ so C is concave up and we have found a minimum of C.)

4.6   ASYMPTOTIC BEHAVIOR OF FUNCTIONS

When you turn on an automobile or a light bulb many things happen, and some of them are uniquely part of the start up of the system. These "transient" things occur only during start up, and then the system settles down to its steady–state operation. The start up behavior of systems can be very important, but sometimes we want to investigate the steady–state or long term behavior of the system: how is the system behaving "after a long time?" In this section we consider ways of investigating and describing the long term behavior of functions and the systems they may model: how is a function behaving "when x (or –x) is arbitrarily large?"

Limits As X Becomes Arbitrarily Large ("Approaches Infinity")

The same type of questions we considered about a function f as x approached a finite number can also be asked about f as x "becomes arbitrarily large," "increases without bound," and is eventually larger than any fixed number.

Example 1:   What happens to the values of $f(x) = \dfrac{5x}{2x + 3}$ (Fig. 1) and $g(x) = \dfrac{\sin(7x + 1)}{3x}$ as

x becomes arbitrarily large, as x increases without bound?

Solution: One approach is numerical: evaluate f(x) and g(x) for some "large" values of x and see if there is a pattern to the values of f(x) and g(x). Fig. 1 shows the values of f(x) and g(x) for several large values of x. When x is very large, it appears that the values of f(x) are close to 2.5 = 5/2 and the values of g(x) are close to 0. In fact, we can guarantee that the values of f(x) are as close to 5/2 as someone wants by taking x to be "big enough." The values of $f(x) = \dfrac{5x}{2x + 3}$ may or may not ever equal 5/2 (they never do), but if x is "large," then f(x) is "close to" 5/2. Similarly, we can guarantee that the values of g(x) are as close to 0 as someone wants by taking x to be "big enough." The graphs of f and g are shown in Fig. 2 for "large" values of x.

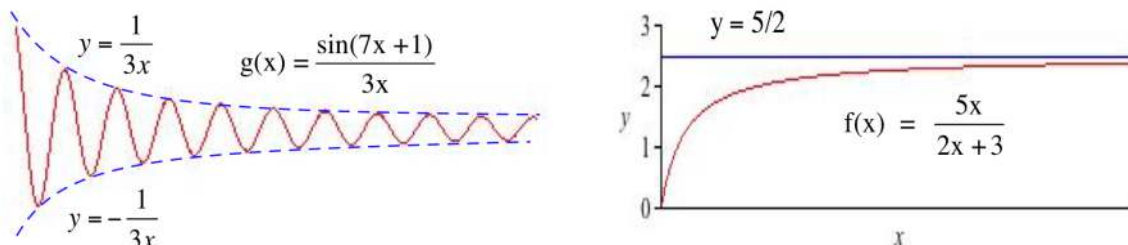| x	$\dfrac{5x}{2x + 3}$	$\dfrac{\sin(7x + 1)}{3x}$
10	2.17	0.03170
100	2.463	–0.00137
1000	2.4962	0.00033
10,000	2.4996	0.000001

Fig. 1

Fig. 2

Practice 1:   What happens to the values of $f(x) = \dfrac{3x + 4}{x - 2}$ and $g(x) = \dfrac{\cos(5x)}{2x + 7}$ as x becomes arbitrarily large?

The answers for Example 1 can be written as limit statements:

"As x becomes arbitrarily large,

the values of $\dfrac{5x}{2x + 3}$ approach $\dfrac{5}{2}$ "   can be written   " $\displaystyle\lim_{x\to\infty} \dfrac{5x}{2x + 3} = \dfrac{5}{2}$ " and

"the values of $\dfrac{\sin(7x + 1)}{3x}$ approach 0."   can be written   " $\displaystyle\lim_{x\to\infty} \dfrac{\sin(7x + 1)}{3x} = 0$."

The symbol " $\displaystyle\lim_{x\to\infty}$ " is read "the limit as x approaches infinity" and means "the limit as x becomes arbitrarily large" or as x increases without bound. (During this discussion and throughout this book, we do not treat "infinity" or "∞", as a number, but only as a useful notation. "Infinity" is not part of the real number system, and we use the common notation "x→∞" and the phrase "x approaches infinity" only to mean that "x becomes arbitrarily large." The notation "x → –∞," read as "x approaches negative infinity," means that the values of –x become arbitrarily large.)

Practice 2:   Write your answers to Practice 1 using the limit notation.

The $\displaystyle\lim_{x\to\infty} f(x)$ asks about the behavior of f(x) as the values of x get larger and larger without any bound, and one way to determine this behavior is to look at the values of f(x) at some values of x which are "large". If the values of the function get arbitrarily close to a single number as x gets larger and larger, then we will say that number is the limit of the function as x approaches infinity. A definition of the limit as "x→∞" is given at the end of this section.

x	$\dfrac{6x + 7}{3 - 2x}$	$\dfrac{\sin(3x)}{x}$
10		
200		
5000		
20,000		

Practice 3:   Fill in the table in Fig. 3 for $f(x) = \dfrac{6x + 7}{3 - 2x}$ and $g(x) = \dfrac{\sin(3x)}{x}$, and then use those values to estimate $\displaystyle\lim_{x\to\infty} f(x)$ and $\displaystyle\lim_{x\to\infty} g(x)$.

Fig. 3

Example 2:   How large does x need to be to guarantee that $f(x) = \dfrac{1}{x}$ < 0.1? 0.001? < E (assume E>0)?

Solution: If x > 10, then $\dfrac{1}{x} < \dfrac{1}{10} = 0.1$ (Fig. 4). If x > 1000, then $\dfrac{1}{x} < \dfrac{1}{1000} = 0.001$.

In general, if E is any positive number, then we can guarantee that | f(x) | < E by picking only

values of $x > \frac{1}{E} > 0$: if $x > \frac{1}{E}$, then $\frac{1}{x} < E$.

From this we can conclude that $\lim\limits_{x \to \infty} \dfrac{1}{x} = 0$.

Practice 4:   How large does x need to be to guarantee that

$f(x) = \dfrac{1}{x^2} < 0.1?$ $0.001?$ $< E$ (assume E>0)?

Evaluate $\lim\limits_{x \to \infty} \dfrac{1}{x^2}$.
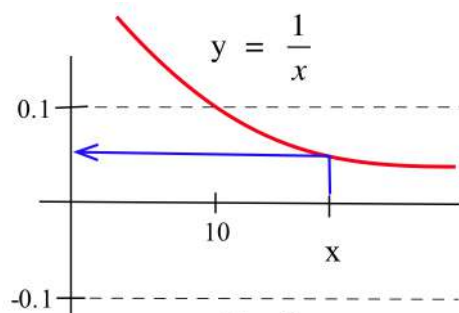

Fig. 4

The Main Limit Theorem (Section 1.2) about limits of combinations of functions is true if the limits as "x→a"
are replaced with limits as "x→∞", but we will not prove those results.

Polynomials occur commonly, and we often need the limit, as $x \to \infty$, of ratios of polynomials or
functions containing powers of x. In those situations the following technique is often helpful:

   (i) factor the highest power of x in the denominator from both the numerator and the

       denominator, and

   (ii) cancel the common factor from the numerator and denominator.

The limit of the new denominator is a constant, so the limit of the resulting ratio is easier to determine.

Example 3:   Determine $\lim\limits_{x \to \infty} \dfrac{7x^2 + 3x - 4}{3x^2 - 5}$ and $\lim\limits_{x \to \infty} \dfrac{9x + 2}{3x^2 - 5x + 1}$.

Solutions: $\lim\limits_{x \to \infty} \dfrac{7x^2 + 3x - 4}{3x^2 - 5} = \lim\limits_{x \to \infty} \dfrac{x^2(7 + 3/x - 4/x^2)}{x^2(3 - 5/x^2)}$ factoring out x^2

$= \lim\limits_{x \to \infty} \dfrac{7 + 3/x - 4/x^2}{3 - 5/x^2} = \dfrac{7}{3}$ canceling x^2 and noting $\dfrac{3}{x} , \dfrac{4}{x^2} , \dfrac{5}{x^2} \to 0$.

Similarly, $\lim\limits_{x \to \infty} \dfrac{9x + 2}{3x^2 - 5x + 1}$ $= \lim\limits_{x \to \infty} \dfrac{x^2(9/x - 2/x^2)}{x^2(3 - 5/x + 1/x^2)}$

$= \lim\limits_{x \to \infty} \dfrac{9/x - 2/x^2}{3 - 5/x + 1/x^2} = \dfrac{0}{3} = 0$.

If we have a difficult limit, as $x \to \infty$, it is often useful to algebraically manipulate the function into the
form of a ratio and then use the previous technique.

If the values of the function oscillate and do not approach a single number as x becomes arbitrarily large,
then the function does not have a limit as x approaches infinity: the limit Does Not Exist.

Example 4:   Evaluate $\lim\limits_{x\to\infty} \sin(x)$ and $\lim\limits_{x\to\infty} x - [x]$.

Solution: $f(x) = \sin(x)$ and $g(x) = x - [x]$ do not have limits as $x \to \infty$. As x grows without bound, the

values of $f(x) = \sin(x)$ oscillate between –1 and +1 (Fig. 5), and these values of sin(x) do not

approach a single number. Similarly, $g(x) = x - [x]$ continues to take on values between 0 and 1,
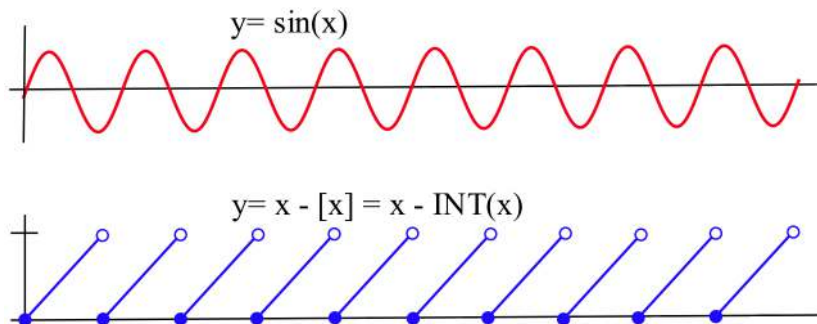
and these values are not approaching a single number.

Fig. 5

Using Calculators To Help Find Limits as "x ⟶ ∞" or "x ⟶ – ∞"

Calculators only store a limited number of digits of a number, and this is a severe limitation when we are
dealing with extremely large numbers.

Example: The value of $f(x) = (x + 1) - x$ is clearly equal to 1 for all values of x, and your calculator

will give the right answer if you use it to evaluate f(4) or f(5). Now use it to evaluate f for a big value

of x, say $x = 10^{40}$. $f(10^{40}) = (10^{40} + 1) - 10^{40} = 1$, but most calculators do not store 40 digits of a

number, and they will respond that $f(10^{40}) = 0$ which is **wrong**. In this example the calculator's error is

obvious, but the same type of errors can occur in less obvious ways when very large numbers are used on

calculators.

You need to be careful with and somewhat suspicious of the answers your calculator gives you.

Calculators can still be helpful for examining some limits as "x → ∞" and "x → –∞" as long as we do

not place too much faith in their responses.

Even if you have forgotten some of the properties of natural logarithm function ln(x) and the

cube root function $\sqrt[3]{x}$, a little experimentation on your calculator can help you determine

that $\lim\limits_{x\to\infty} \dfrac{\ln(x)}{\sqrt[3]{x}} = 0$.

The Limit is Infinite

The function $f(x) = \dfrac{1}{x^2}$ is undefined at $x = 0$, but we can still ask about the

behavior of $f(x)$ for values of x "close to" 0. Fig. 6 indicates that if x is very

small, close to 0, then $f(x)$ is very large. As the values of x get closer to 0, the

values of $f(x)$ grow larger and can be made as large as we want by picking x to

be close enough to 0. Even though the values of f are not approaching any

number, we use the "infinity" notation to indicate that the values of f are

growing without bound, and write

$$\lim_{x \to 0} = \infty .$$

The values of $\dfrac{1}{x^2}$ do not equal "infinity:" $\displaystyle\lim_{x \to 0} = \infty$ means that the values of $\dfrac{1}{x^2}$ can be made

arbitrarily large by picking values of x very close to 0.
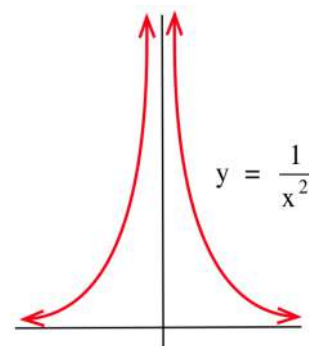
The limit, as $x \to 0$, of $\dfrac{1}{x}$ is slightly more complicated. If x is close to 0, then the value of $f(x) = 1/x$

can be a large positive number or a large negative number, depending on the sign of x.

The function $f(x) = 1/x$ does not have a (two–sided) limit as x approaches 0, but we can still ask about

one–sided limits:

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \quad \text{and} \quad \lim_{x \to 0^-} \frac{1}{x} = -\infty .$$

Example 5:   Determine $\displaystyle\lim_{x \to 3^+} \frac{x-5}{x-3}$ and $\displaystyle\lim_{x \to 3^-} \frac{x-5}{x-3}$.

Solution: (a) As $x \to 3^+$, then $x - 5 \to -2$ and $x - 3 \to 0$. Since the denominator is approaching 0 we

cannot use the Main Limit Theorem, and we need to examine the functions more carefully. If $x \to 3^+$,

then $x > 3$ so $x - 3 > 0$. If x is close to 3 and slightly larger than 3, then the ratio of $x - 5$ to

$x - 3$ is the ratio $\dfrac{\text{a number close to } -2}{\text{small positive number}}$ = large negative number. As $x > 3$ gets closer to 3, $\dfrac{x-5}{x-3}$

is $\dfrac{\text{a number closer to } -2}{\text{positive and closer to } 0}$ = larger negative number. By taking $x > 3$ closer to 3, the denominator gets

closer to 0 but is always positive, so the ratio gets arbitrarily large and negative: $\displaystyle\lim_{x \to 3^+} \frac{x-5}{x-3} = -\infty$.

 (b) As $x \to 3^-$, then $x - 5 \to -2$ and $x - 3$ gets arbitrarily close to 0, and $x - 3$ is negative. The

value of the ratio $\dfrac{x-5}{x-3}$ is $\dfrac{\text{a number close to } -2}{\text{arbitrarily small negative number}}$ = arbitrarily large positive

number: $\displaystyle\lim_{x \to 3^-} \frac{x-5}{x-3} = +\infty$.

Practice 5:   Determine $\lim\limits_{x\to 2^+} \dfrac{7}{2-x}$, $\lim\limits_{x\to 2^+} \dfrac{3x}{2x-4}$, $\lim\limits_{x\to 2^+} \dfrac{3x^2-6x}{x-2}$

Horizontal Asymptotes

The limits of f , as "x → ∞" and "x → –∞," give us information about horizontal asymptotes of f.

Definition: The line $y = K$ is a **horizontal asymptote** of f if $\lim\limits_{x\to\infty} f(x) = K$ or $\lim\limits_{x\to -\infty} f(x) = K$.

Example 6:   Find any horizontal asymptotes of $f(x) = \dfrac{2x + \sin(x)}{x}$.

Solution: $\lim\limits_{x\to\infty} \dfrac{2x + \sin(x)}{x} = \lim\limits_{x\to\infty} \dfrac{2x}{x} + \dfrac{\sin(x)}{x} = 2 + 0 = 2$ so the line **y = 2** is a

horizontal asymptote of f. The limit,
as "x → – ∞," is also 2 so y = 2 is
the only horizontal asymptote of f.
The graphs of f and y = 2 are given
in Fig. 7 . A function may or may not
cross its asymptote.
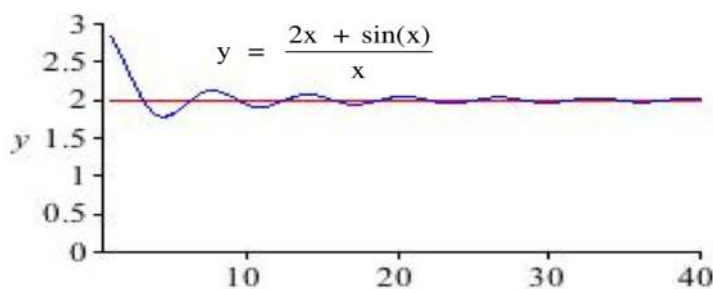
Fig. 7

Vertical Asymptotes

Definition: The vertical line $x = a$ is a **vertical asymptote** of the graph of f

if either or both of the one–sided limits, as $x \to a^-$ or $x \to a^+$, of f is infinite.

If our function f is the ratio of a polynomial P(x) and a polynomial Q(x), $f(x) = \dfrac{P(x)}{Q(x)}$, then the only
candidates for vertical asymptotes are the values of x where Q(x) = 0. However, the fact that Q(a) = 0 is
not enough to guarantee that the line x = a is a vertical asymptote of f ; we also need to evaluate P(a).
If Q(a) = 0 and P(a) ≠ 0, then the line x = a is a vertical asymptote of f. If Q(a) = 0 and P(a) = 0, then
the line x = a may or may not be a vertical asymptote.

Example 7:   Find the vertical asymptotes of $f(x) = \dfrac{x^2 - x - 6}{x^2 - x}$ and $g(x) = \dfrac{x^2 - 3x}{x^2 - x}$.

Solution: $f(x) = \dfrac{x^2 - x - 6}{x^2 - x} = \dfrac{(x-3)(x+2)}{x(x-1)}$ so the only values which make the denominator 0 are

$x = 0$ and $x = 1$, and these are the only candidates to be vertical asymptotes.

$\lim\limits_{x \to 0^+} f(x) = +\infty$ and $\lim\limits_{x \to 1^+} f(x) = -\infty$ so $x = 0$ and $x = 1$ are both vertical asymptotes of f.

$g(x) = \dfrac{x^2 - 3x}{x^2 - x} = \dfrac{x(x-3)}{x(x-1)}$ so the only candidates to be vertical asymptotes are $x = 0$ and $x = 1$.

$\lim\limits_{x \to 1^+} g(x) = \lim\limits_{x \to 1^+} \dfrac{x(x-3)}{x(x-1)} = \lim\limits_{x \to 1^+} \dfrac{x-3}{x-1} = -\infty$ so $x = 1$ is a vertical asymptote of g.

$\lim\limits_{x \to 0} g(x) = \lim\limits_{x \to 0} \dfrac{x(x-3)}{x(x-1)} = \lim\limits_{x \to 0} \dfrac{x-3}{x-1} = 3 \neq \infty$ so $x = 0$ is **not** a vertical asymptote.

Practice 6:   Find the vertical asymptotes of $f(x) = \dfrac{x^2 + x}{x^2 + x - 2}$ and $g(x) = \dfrac{x^2 - 1}{x - 1}$.

Other Asymptotes as "$x \to \infty$" and "$x \to -\infty$"

If the limit of $f(x)$ as "$x \to \infty$" or "$x \to -\infty$" is a constant K, then the graph of f gets close to the horizontal line $y = K$, and we said that $y = K$ was a horizontal asymptote of f. Some functions, however, approach other lines which are not horizontal.

Example 8:   Find all asymptotes of $f(x) = \dfrac{x^2 + 2x + 1}{x} = x + 2 + \dfrac{1}{x}$.

Solution: If x is a large positive number or a large negative number,

then $\dfrac{1}{x}$ is very close to 0, and the graph of $f(x)$ is very close to the line

$y = x + 2$ (Fig. 8). The line **$y = x + 2$** is an asymptote of the graph of f.

If x is a large positive number, then $1/x$ is positive, and the graph of f is slightly above the graph of $y = x + 2$. If x is a large negative number, then $1/x$ is negative, and the graph of f will be slightly below the graph of $y = x + 2$. The $1/x$ piece of f never equals 0 so the graph of f never crosses or touches the graph of the asymptote $y = x + 2$.

The graph of f also has a vertical asymptote at $x = 0$ since $\lim\limits_{x \to 0^+} f(x) = \infty$ and $\lim\limits_{x \to 0^-} f(x) = -\infty$.
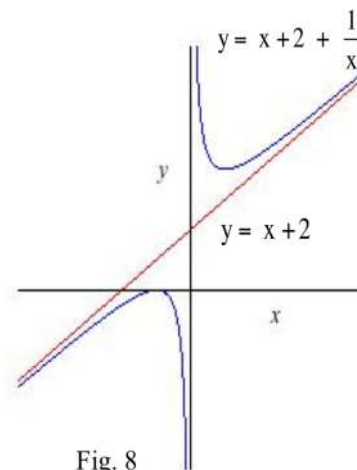

Fig. 8

Practice 7:   Find all asymptotes of $g(x) = \dfrac{2x^2 - x - 1}{x + 1} = 2x - 3 + \dfrac{2}{x + 1}$.

Some functions even have **nonlinear asymptotes**, asymptotes which are not straight lines. The graphs of these functions approach some nonlinear function when the values of x are arbitrarily large.

Example 9:   Find all asymptotes of $f(x) = \dfrac{x^4 + 3x^3 + x^2 + 4x + 5}{x^2 + 1} = x^2 + 3x + \dfrac{x + 5}{x^2 + 1}$.

Solution: When x is very large, positive or negative, then $\dfrac{x + 5}{x^2 + 1}$ is very close to 0, and the

  graph of f is very close to the graph of $g(x) = x^2 + 3x$. The function **$g(x) = x^2 + 3x$** is a nonlinear

  asymptote of f. The denominator of f is never 0, and f has no vertical asymptotes.

Practice 8:   Find all asymptotes of $f(x) = \dfrac{x^3 + 2\sin(x)}{x} = x^2 + \dfrac{2\sin(x)}{x}$.

If f(x) can be written as a sum of two other functions, $f(x) = g(x) + r(x)$, with $\displaystyle\lim_{x \to \pm\infty} r(x) = 0$, then the

graph of f is asymptotic to the graph of g , and g is an asymptote of f.

> Suppose $f(x) = g(x) + r(x)$ and $\displaystyle\lim_{x \to \pm\infty} r(x) = 0$:
>
>   if $g(x) = K$, then f has a horizontal asymptote $y = K$;
>
>   if $g(x) = ax + b$, then f has a linear asymptote $y = ax + b$; or
>
>   if $g(x) = $ a nonlinear function, then f has a nonlinear asymptote $y = g(x)$.

Definition of $\displaystyle\lim_{x \to \infty} \mathbf{f(x) = K}$

The following definition states precisely what is meant by the phrase "we can guarantee that the values of f(x) are arbitrarily close to K by using sufficiently large values of x."

Definition: $\displaystyle\lim_{x \to \infty} f(x) = K$

  means

    for every given $\varepsilon > 0$, there is a number N so that

      if   x   is larger than N

      then   f(x)   is within ε units of K.

      (equivalently; $|f(x) - K| < \varepsilon$ whenever $x > N$.)

Example 10:   Show that $\lim\limits_{x\to\infty}\dfrac{x}{2x+1}=\dfrac{1}{2}$.

Solution: Typically we need to do two things. First we need to find a value of N, usually depending on ε.

Then we need to show that the value of N we found satisfies the conditions of the definition.

(i) Assume that $|\,f(x)-K\,|$ is less than ε and solve for $x>0$.

If $\varepsilon>|\,\dfrac{x}{2x+1}\,-\dfrac{1}{2}\,|=|\,\dfrac{2x-(2x+1)}{2(2x+1)}\,|=|\,\dfrac{-1}{4x+2}\,|=\dfrac{1}{4x+2}$, then $4x+2>\dfrac{1}{\varepsilon}$ and

$x>\dfrac{1}{4}(\dfrac{1}{\varepsilon}-2)$. For any $\varepsilon>0$, take $N=\dfrac{1}{4}(\dfrac{1}{\varepsilon}-2)$.

(ii) For any $\varepsilon>0$, take $N=\dfrac{1}{4}(\dfrac{1}{\varepsilon}-2)$. (Now we can just reverse the order of the steps in part (i).)

If $x>0$ and $x>N=\dfrac{1}{4}(\dfrac{1}{\varepsilon}-2)$,

then $4x+2>\dfrac{1}{\varepsilon}$ so $\varepsilon>\dfrac{1}{4x+2}=|\,\dfrac{x}{2x+1}\,-\dfrac{1}{2}\,|=|\,f(x)-K\,|$.

We have shown that "for every given ε, there is an N" that satisfies the definition.

PROBLEMS

1. Fig. 9 shows f(x) and g(x) for $0\le x\le 5$. Let $h(x)=\dfrac{f(x)}{g(x)}$.

    (a) At what value of x does h(x) have a root?

    (b) Determine the limits of h(x) as $x\to 1^{+}, x\to 1^{-}, x\to 3^{+}$, and $x\to 3^{-}$.

    (c) Where does h(x) have a vertical asymptote?

Fig. 9

2. Fig. 10 shows f(x) and g(x) for $0\le x\le 5$. Let $h(x)=\dfrac{f(x)}{g(x)}$.

    (a) At what value(s) of x does h(x) have a root?

    (b) Where does h(x) have vertical asymptotes?

3. Fig. 11 shows f(x) and g(x) for $0\le x\le 5$. let $h(x)=\dfrac{f(x)}{g(x)}$, and determine

    the limits of h(x) as $x\to 2^{+}, x\to 2^{-}, x\to 4^{+}$, and $x\to 4^{-}$.

Fig. 10

For problems 4 – 24, calculate the limit of each expression as "$\mathbf{x\to\infty}$."

4. $\dfrac{6}{x+2}$

5. $\dfrac{28}{3x-5}$

6. $\dfrac{7x+12}{3x-2}$

7. $\dfrac{4-3x}{x+8}$

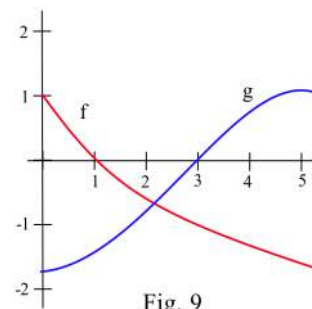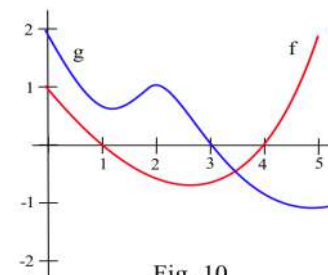8. $\dfrac{5\sin(2x)}{2x}$

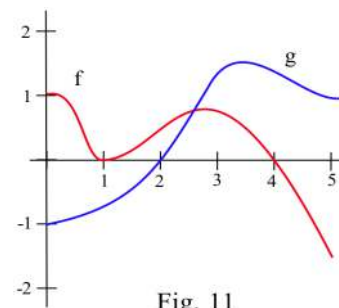9. $\dfrac{\cos(3x)}{5x-1}$

Fig. 11

10. $\dfrac{2x - 3\sin(x)}{5x - 1}$

11. $\dfrac{4 + x \cdot \sin(x)}{2x - 3}$

12. $\dfrac{x^2 - 5x + 2}{x^2 + 8x - 4}$

13. $\dfrac{2x^2 - 9}{3x^2 + 10x}$

14. $\dfrac{\sqrt{x + 5}}{\sqrt{4x - 2}}$

15. $\dfrac{5x^2 - 7x + 2}{2x^3 + 4x}$

16. $\dfrac{x + \sin(x)}{x - \sin(x)}$

17. $\dfrac{7x^2 + x \cdot \sin(x)}{3 - x^2 + \sin(7x^2)}$

18. $\dfrac{7x^{143} + 734x - 2}{x^{150} - 99x^{83} + 25}$

19. $\dfrac{\sqrt{9x^2 + 16}}{2 + \sqrt{x^3 + 1}}$

20. $\sin\left(\dfrac{3x + 5}{2x - 1}\right)$

21. $\cos\left(\dfrac{7x + 4}{x^2 + x + 1}\right)$

22. $\ln\left(\dfrac{3x^2 + 5x}{x^2 - 4}\right)$

23. $\ln(x + 8) - \ln(x - 5)$

24. $\ln(3x + 8) - \ln(2x + 1)$

25. Salt water with a concentration of 0.2 pounds of salt per gallon flows into a large tank that initially contains 50 gallons of pure water.

   (a) If the flow rate of salt water into the tank is 4 gallons per minute, what is the volume V(t) of water and the amount A(t) of salt in the tank t minutes after the flow begins?

   (b) Show that the salt concentration C(t) at time t is $C(t) = \dfrac{.8t}{4t + 50}$.

   (c) What happens to the concentration C(t) after a "long" time?

   (d) Redo parts (a) – (c) for a large tank which initially contains 200 gallons of pure water.

26. Under certain laboratory conditions, an agar plate contains $B(t) = 100\left(2 - \dfrac{1}{e^t}\right) = 100(2 - e^{-t})$

   bacteria t hours after the start of the experiment.

   (a) How many bacteria are on the plate at the start of the experiment (t = 0)?

   (b) Show that the population is always increasing. (Show B '(t) > 0 for all t > 0.)

   (c) What happens to the population B(t) after a "long" time?

   (d) Redo parts (a) – (c) for $B(t) = A\left(2 - \dfrac{1}{e^t}\right) = A(2 - e^{-t})$.

For problems 27 – 41 , calculate the limits.

27. $\lim\limits_{x \to 0} \dfrac{x + 5}{x^2}$

28. $\lim\limits_{x \to 3} \dfrac{x - 1}{(x - 3)^2}$

29. $\lim\limits_{x \to 5} \dfrac{x - 7}{(x - 5)^2}$

30. $\lim\limits_{x \to 2^+} \dfrac{x - 1}{x - 2}$

31. $\lim\limits_{x \to 2^-} \dfrac{x - 1}{x - 2}$

32. $\lim\limits_{x \to 2} \dfrac{x - 1}{x - 2}$

33. $\lim\limits_{x \to 4^+} \dfrac{x + 3}{4 - x}$

34. $\lim\limits_{x \to 1^-} \dfrac{x^2 + 5}{1 - x}$

35. $\lim\limits_{x \to 3^+} \dfrac{x^2 - 4}{x^2 - 2x - 3}$

36. $\lim\limits_{x\to 2} \dfrac{x^2 - x - 2}{x^2 - 4}$

37. $\lim\limits_{x\to 0} \dfrac{x - 2}{1 - \cos(x)}$

38. $\lim\limits_{x\to\infty} \dfrac{x^3 + 7x - 4}{x^2 + 11x}$

39. $\lim\limits_{x\to 5} \dfrac{\sin(x - 5)}{x - 5}$

40. $\lim\limits_{x\to 0} \dfrac{x + 1}{\sin^2(x)}$

41. $\lim\limits_{x\to 0^+} \dfrac{1 + \cos(x)}{1 - e^x}$

In problems 42 – 50, write the **equation** of each asymptote for each function and state whether it is a vertical or horizontal asymptote.

42. $f(x) = \dfrac{x + 2}{x - 1}$

43. $f(x) = \dfrac{x - 3}{x^2}$

44. $f(x) = \dfrac{x - 1}{x^2 - x}$

45. $f(x) = \dfrac{x + 5}{x^2 - 4x + 3}$

46. $f(x) = \dfrac{x + \sin(x)}{3x - 3}$

47. $f(x) = \dfrac{x^2 - 4}{x^2 + 1}$

48. $f(x) = \dfrac{\cos(x)}{x^2}$

49. $f(x) = 2 + \dfrac{3 - x}{x - 1}$

50. $f(x) = \dfrac{x \cdot \sin(x)}{x^2 - x}$

In problems 51 – 59, write the **equation** of each asymptote for each function.

51. $f(x) = \dfrac{2x^2 + x + 5}{x}$

52. $f(x) = \dfrac{x^2 + x}{x + 1}$

53. $f(x) = \dfrac{1}{x - 2} + \sin(x)$

54. $f(x) = x + \dfrac{x}{x^2 + 1}$

55. $f(x) = x^2 + \dfrac{x}{x^2 + 1}$

56. $f(x) = x^2 + \dfrac{x}{x + 1}$

57. $f(x) = \dfrac{x \cdot \cos(x)}{x - 3}$

58. $f(x) = \dfrac{x^3 - x^2 + 2x - 1}{x - 1}$

59. $f(x) = \sqrt{\dfrac{x^2 + 3x + 2}{x + 3}}$

Section 4.6   PRACTICE Answers

Practice 1:   As x becomes arbitrarily large, the values of f(x) approach 3 and the values of g(x) approach 0.

Practice 2: $\lim\limits_{x\to\infty} \dfrac{3x + 4}{x - 2} = 3$ and $\lim\limits_{x\to\infty} \dfrac{\cos(5x)}{2x + 7} = 0$

Practice 3:   The completed table is shown in Fig. 12.

Practice 4:   If $x > \sqrt{10} \approx 3.162$, then $f(x) = \dfrac{1}{x^2} < 0.1$.

If $x > \sqrt{1000} \approx 31.62$, then $f(x) = \dfrac{1}{x^2} < 0.001$.

If $x > \sqrt{1/E}$, then $f(x) = \dfrac{1}{x^2} < E$.

x	$\dfrac{6x + 7}{3 - 2x}$	$\dfrac{\sin(3x)}{x}$
10	−3.94117647	−0.09880311
200	−3.04030227	0.00220912
5000	−3.00160048	0.00017869
20,000	−3.00040003	0.00004787
↓	↓	↓
	−3	0

Fig. 12

Practice 5:   (a) $\lim\limits_{x \to 2^+} \dfrac{7}{2-x} = -\infty$.

As $x \to 2^+$ the values $2 - x \to 0$, and $x > 2$ so $2 - x < 0$: $2 - x$ takes small negative values.

Then the values of $\dfrac{7}{2-x} = \dfrac{7}{\text{small negative values}}$ are large negative values so we represent the

limit as "$-\infty$."

(b) $\lim\limits_{x \to 2^+} \dfrac{3x}{2x-4} = +\infty$.

As $x \to 2^+$ the values of $2x - 4 \to 0$, and $x > 2$ so $2x - 4 > 0$: $2x - 4$ takes small positive

values. As $x \to 2^+$ the values of $3x \to +6$.

Then the values of $\dfrac{3x}{2x-4} = \dfrac{\text{values near } +6}{\text{small positive values}}$ are large positive values so we represent the

limit as "$+\infty$."

(c) $\lim\limits_{x \to 2^+} \dfrac{3x^2 - 6x}{x-2} = 6$.

As $x \to 2^+$, the values of $3x^2 - 6x \to 0$ and $x - 2 \to 0$ so we need to do more work. The

numerator can be factored $3x^2 - 6x = 3x(x - 2)$ and then the rational function can be reduced (since

$x \to 2$ we know $x \neq 2$):

$$\lim\limits_{x \to 2^+} \dfrac{3x^2 - 6x}{x-2} = \lim\limits_{x \to 2^+} \dfrac{3x(x-2)}{x-2} = \lim\limits_{x \to 2^+} 3x = 6 .$$

Practice 6:   (a) $f(x) = \dfrac{x^2 + x}{x^2 + x - 2} = \dfrac{x(x+1)}{(x-1)(x+2)}$.

f has vertical asymptotes at $x = 1$ and $x = -2$.

(b) $g(x) = \dfrac{x^2 - 1}{x - 1} = \dfrac{(x+1)(x-1)}{x-1}$.

The value $x = 1$ is not in the domain of g. If $x \neq 1$, then $g(x) = x + 1$.

g has a "hole" when $x = 1$ and no vertical asymptotes.

Practice 7: $g(x) = 2x - 3 + \dfrac{2}{x+1}$.

       g has a vertical asymptote at $x = -1$.

       g has no horizontal asymptotes.

       $\displaystyle\lim_{x\to\infty} \dfrac{2}{x+1} = 0$ so g has the linear asymptote $y = 2x - 3$.

Practice 8: $f(x) = x^2 + \dfrac{2\cdot\sin(x)}{x}$.

       f is not defined at $x = 0$, so f has a vertical asymptote or a "hole" when $x = 0$.

            $\displaystyle\lim_{x\to 0} x^2 + \dfrac{2\cdot\sin(x)}{x} = 0 + 2 = 2$ so f has a "hole" when $x = 0$.

       $\displaystyle\lim_{x\to\infty} \dfrac{2\cdot\sin(x)}{x} = 0$ so f has the nonlinear asymptote $y = x^2$.

Appendix: MAPLE, infinite limits and limits as "$x \to \infty$"

command	output	comment
limit(1/x, x = 0);	undefined	Maple uses the convention "x=0" even though what we really mean is "x approaches 0"
limit(1/x, x=0, right);	∞	"x=0, right" means $0 \le x$
limit(1/x. x=0, left);	$-\infty$	"x=0, left" means $x \ge 0$
limit(sin(x)/x, x = infinity);	0	
limit(2*x, x= infinity);	∞	
limit(5-3*x, x= infinity);	$-\infty$	
limit((1=cos(x))/(1-exp(x)). x = 0, right);	$-\infty$	

4.7   L'HÔPITAL'S RULE

When we began taking limits of slopes of secant lines, $m_{sec} = \dfrac{f(x+h) - f(x)}{h}$ as h∅0 , we frequently

encountered one difficulty: both the numerator and the denominator approached 0. And since the

denominator approached 0, we could not apply the Main Limit Theorem. In each case, however, we

managed to get past this "0/0" difficulty by using algebra or geometry or trigonometry, but there was no

common approach or pattern. The algebraic steps we used to evaluate

$$\lim_{h \to 0} \frac{(2+h)^2 - 4}{h}$$ seem quite different from the trigonometric steps needed for $$\lim_{h \to 0} \frac{\sin(2+h) - \sin(2)}{h} \; .$$

In this section we consider a single technique, called l'Hôpital's Rule (pronounced Low–Pee–Tall), which

enables us to quickly and easily evaluate limits of the form "0/0" as well as several other difficult forms.

A Linear Example

Two linear functions are given in Fig. 1, and we need to find

$$\lim_{x \to 5} \frac{f(x)}{g(x)} \; .$$ Unfortunately, $\lim_{x \to 5} f(x) = 0$ and $\lim_{x \to 5} g(x) = 0$ so

we cannot apply the Main Limit Theorem. However, we know f

and g are linear, we can calculate their slopes from Fig. 1, and we

know that they both go through the point (5,0) so we can find

their equations: f(x) = **–2**(x – 5) and g(x) = **3**(x – 5).

Now the limit is easier: $$\lim_{x \to 5} \frac{f(x)}{g(x)} = \lim_{x \to 5} \frac{-2(x-5)}{3(x-5)} = \frac{-2}{3} = \frac{\text{slope of f}}{\text{slope of g}} \; .$$

In fact, this pattern works for any two linear functions:

> If f and g are linear functions with slopes **m** and **n** ≠ 0 and a common root at x = a,
>
> (f(x) – f(a) = **m**(x – a) and g(x) – g(a) = **n**(x – a) so f(x) = **m**(x – a) and g(x) = **n**(x – a))
>
> then $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{m(x-a)}{n(x-a)} = \frac{\mathbf{m}}{\mathbf{n}} = \frac{\text{slope of f}}{\text{slope of g}} \; .$$

The really powerful result, discovered by John Bernoulli and named for the Marquis de l'Hôpital who

published it in his calculus book, is that the same pattern is true for differentiable functions even if they are
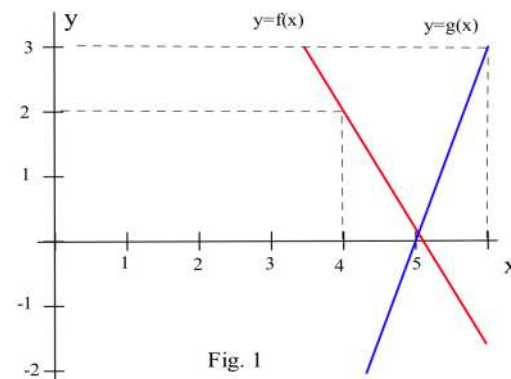
not linear.

L'Hôpital's Rule ("0/0" form)

If   f and g are differentiable at x = a ,

and f(a) = 0, g(a) = 0, and g '(a) ≠ 0 ,

then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ = $\dfrac{f'(a)}{g'(a)}$ = $\dfrac{\text{slope of f at a}}{\text{slope of g at a}}$.

Idea for a proof: Even though f and g may not be linear functions, they are differentiable so at the point

x = a they are "almost linear" in the sense that they are well approximated by their tangent lines at

that point (Fig. 2): since f(a) = g(a) = 0

f(x) ≈ f(a) + f '(a)(x–a) = f '(a)(x–a)   and g(x) ≈ g(a) + g '(a)(x–a) = g '(a)(x–a).

Then $\lim\limits_{x\to a} \dfrac{f(x)}{g(x)}$ ≈ $\lim\limits_{x\to a} \dfrac{f'(a)(x-a)}{g'(a)(x-a)}$ = $\dfrac{f'(a)}{g'(a)}$.

Fig. 2

(Unfortunately, we have ignored a couple subtle difficulties such as g(x) or

g '(x) possibly being 0 when x is close to a. A proof of l'Hôpital's
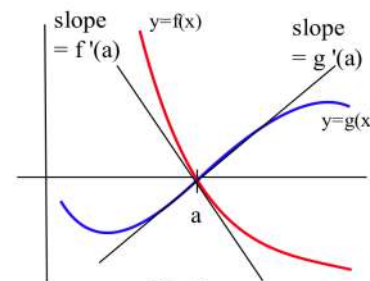
Rule is difficult and is not included.)

Example 1:   Use l'Hôpital's Rule to determine $\lim\limits_{x\to 0} \dfrac{x^2 + \sin(5x)}{3x}$ and $\lim\limits_{x\to 1} \dfrac{\ln(x)}{e^x - e}$

Solution: (a) We could evaluate this limit without l'Hôpital's Rule but let's use it. We can match the

pattern of l'Hôpital's Rule by letting a = 0, f(x) = x^2 + sin(5x) and g(x) = 3x. Then f(0) = 0, g(0)

= 0, and f and g are differentiable with f '(x) = 2x + 5cos(5x) and g '(x) = 3 so

$$\lim\limits_{x\to 0} \dfrac{x^2 + \sin(5x)}{3x} = \dfrac{f'(0)}{g'(0)} = \dfrac{2\cdot 0 + 5\cos(5\cdot 0)}{3} = \dfrac{5}{3} .$$

(b) Let a = 1, f(x) = ln(x) and g(x) = e^x – e. Then f(1) = 0, g(1) = 0, f and g are differentiable for x

near 1 (x ≠ 0), and f '(x) = 1/x and g '(x) = e^x . Then

$$\lim\limits_{x\to 1} \dfrac{\ln(x)}{e^x - e} = \dfrac{f'(1)}{g'(1)} = \dfrac{1/1}{e^1} = \dfrac{1}{e} .$$

Practice 1:   Use l'Hôpital's Rule to find $\lim\limits_{x\to 0} \dfrac{1 - \cos(5x)}{3x}$ and $\lim\limits_{x\to 2} \dfrac{x^2 + x - 6}{x^2 + 2x - 8}$.

Strong Version of L'Hôpital's Rule

L'Hôpital's Rule can be strengthened to include the case when g '(a) = 0 and the indeterminate form "∞/∞",

the case when both f and g increase without any bound.

L'Hôpital's Rule (Strong " 0/0 " and " ∞/∞ " forms)

If   f and g are differentiable on an open interval I which contains the point a,

g '(x) ≠ 0 on I except possibly at a, and

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \text{" } \frac{0}{0} \text{ "} \quad \text{or} \quad \text{" } \frac{\infty}{\infty} \text{ "}$$

then $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \lim_{x \to a} \dfrac{f'(x)}{g'(x)}$ provided the limit on the right exists.

("a" can represent a finite number or "∞.")

Example 2:   Evaluate $\lim_{x \to \infty} \dfrac{e^{7x}}{5x}$.

Solution: As "x →∞", both f(x) = e^{7x} and g(x) = 5x increase without bound so we have an "∞/∞"

indeterminate form and can use the Strong Version l'Hôpital's Rule:

$$\lim_{x \to \infty} \frac{e^{7x}}{5x} = \lim_{x \to \infty} \frac{7e^{7x}}{5} = \infty .$$

The limit of f '/g '   may also be an indeterminate form, and then we can apply l'Hôpital's Rule to the ratio

f '/g '. We can continue using l'Hôpital's Rule at each stage as long as we have an indeterminate quotient.

Example 3: $\lim_{x \to 0} \dfrac{x^3}{x - \sin(x)}$

Solution: As x → 0, f(x) = x^3 ∅ 0 and g(x) = x – sin(x) → 0 so

$$\lim_{x \to 0} \frac{x^3}{x - \sin(x)} \quad = \lim_{x \to 0} \frac{3x^2}{1 - \cos(x)} \to \text{" } \frac{0}{0} \text{ "} \quad \text{so we can use l'Hôpital's Rule again}$$

$$= \lim_{x \to 0} \frac{6x}{\sin(x)} \to \text{" } \frac{0}{0} \text{ "} \quad \text{and again}$$

$$= \lim_{x \to 0} \frac{6}{\cos(x)} = \frac{6}{1} = 6 .$$

Practice 2:   Use l'Hôpital's Rule to find $\lim_{x \to \infty} \dfrac{x^2 + e^x}{x^3 + 8x}$.

Which Function Grows Faster?

Sometimes we want to compare the asymptotic behavior of two systems or functions for large values of x, and l'Hôpital's Rule can be a useful tool. For example, if we have two different algorithms for sorting names, and each algorithm takes longer and longer to sort larger collections of names, we may want to know which algorithm will accomplish the task more efficiently for really large collections of names.

Example 4:   Algorithm A requires $n \cdot \ln(n)$ steps to sort n names and algorithm B requires $n^{1.5}$ steps. Which algorithm will be better for sorting very large collections of names?

Solution: We can compare the ratio of the number of steps each algorithm requires, $\dfrac{n \cdot \ln(n)}{n^{1.5}}$, and then

take the limit of this ratio as n grows arbitrarily large: $\displaystyle\lim_{n \to \infty} \dfrac{n \cdot \ln(n)}{n^{1.5}}$. If this limit is infinite, we

say that $n \cdot \ln(n)$ "grows faster" than $n^{1.5}$. If the limit is 0, we say that $n^{1.5}$ grows faster than

$n \cdot \ln(n)$. Since $n \cdot \ln(n)$ and $n^{1.5}$ both grow arbitrarily large when n is large, we can algebraically

simplify the ratio to $\dfrac{\ln(n)}{n^{0.5}}$ and then use L'Hopital's Rule:

$$\lim_{n \to \infty} \frac{\ln(n)}{n^{0.5}} \;=\; \lim_{n \to \infty} \frac{1/n}{0.5n^{-0.5}} \;=\; \lim_{n \to \infty} \frac{2}{\sqrt{n}} \;=\; 0 \;.$$

$n^{1.5}$ grows faster than $n \cdot \ln(n)$ so algorithm A requires fewer steps for really large sorts.

Practice 3:   Algorithm A requires e^n operations to find the shortest path connecting n towns, algorithm B requires $100 \cdot \ln(n)$ operations for the same task, and algorithm C requires n^5 operations. Which algorithm is best for finding the shortest path connecting a very large number of towns? Worst?

Other "Indeterminate Forms"

"0/0" is called an indeterminate form because knowing that f approaches 0 and g approaches 0 is **not** enough to determine the limit of f/g, even if it has a limit. The ratio of a "small" number divided by a "small" number can be almost anything as the three simple "0/0" examples show:

$$\lim_{x \to 0} 3x/x = 3 \;, \qquad\qquad \lim_{x \to 0} x^2/x = 0 \;, \text{ and} \qquad\qquad \lim_{x \to 0} 5x/x^3 = \infty \;.$$

Similarly, "∞/∞" is an indeterminate form because knowing that f and g both grow arbitrarily large is **not** enough to determine the value limit of f/g or if the limit exists:

$$\lim_{x \to \infty} 3x/x = 3 \;, \qquad\qquad \lim_{x \to \infty} x^2/x = \infty \;, \text{ and} \qquad\qquad \lim_{x \to \infty} 5x/x^3 = 0 \;.$$

Besides the indeterminate quotient forms "0/0" and "∞/∞" there are several other "indeterminate forms." In each case, the resulting limit depends not only on each function's limit but also on how quickly each function approaches its limit.

Product: If f approaches 0, and g grows arbitrarily large, the product f · g has the indeterminant form "0 · ∞."

Exponent: If f and g both approach 0, the function f^g has the indeterminant form "0^0."

If f approaches 1, and g grows arbitrarily large, the function f^g has the indeterminant form "1^∞."

If f grows arbitrarily large, and g approaches 0, the function f^g has the indeterminant form "∞^0."

Difference: If f and g both grow arbitrarily large, the function f – g has the indeterminant form "∞ – ∞."

Unfortunately, l'Hôpital's Rule can only be used directly with an indeterminate quotient ("0/0" or "∞/∞'), but these other forms can be algebraically manipulated into quotients, and then l'Hôpital's Rule can be applied to the resulting quotient.

Example 5:   Evaluate $\lim_{x\to 0^+} x \cdot \ln(x)$ (" $0 \cdot (-\infty)$ " form)

Solution: This limit involves an indeterminate product, and we need a quotient in order to apply l'Hôpital's

Rule. We can rewrite the product $x \cdot \ln(x)$ as the quotient $\dfrac{\ln(x)}{1/x}$, and then

$$\lim_{x\to 0^+} x\ln(x) \quad = \lim_{x\to 0^+} \frac{\ln(x)}{1/x} \;\to\; \frac{\infty}{\infty} \qquad\qquad \text{so apply l'Hôpital's Rule}$$

$$= \lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0 \;.$$

> A product f·g with the indeterminant form "0·∞" can be rewritten as a quotient, $\dfrac{f}{1/g}$ or $\dfrac{g}{1/f}$,
> and then l'Hôpital's Rule can be used.

Example 6:   Evaluate $\lim_{x\to 0^+} x^x$ (" 0^0 " form)

Solution: An indeterminate exponent can be converted to a product by recalling a property of exponential and

logarithm functions: for any positive number a, $a = e^{\ln(a)}$ so $f^g = e^{\ln(f^g)} = e^{g\ln(f)}$.

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln(x^x)} = \lim_{x\to 0^+} e^{x\cdot\ln(x)} \quad \text{and this last limit involves an indeterminate}$$

product $x \cdot \ln(x) \to 0 \cdot (-\infty)$ which we converted to a quotient and evaluated to be 0 in Example 5.

Our final answer is then $e^0 = 1$:

$$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{\ln(x^x)} = \lim_{x\to 0^+} e^{x\cdot\ln(x)} = e^0 = 1 \;.$$

An indeterminate form involving exponents, f^g with the form $"0^0,"$ $"1^\infty,"$ or $"\infty^0,"$ can be converted to an indeterminate product by recognizing that $f^g = e^{g \cdot \ln(f)}$ and then determining the limit of $g \cdot \ln(f)$. The final result is $e^{(\text{limit of } g \cdot \ln(f))}$.

Example 7:   Evaluate $\lim\limits_{x \to \infty} \left(1 + \dfrac{a}{x}\right)^x$ ($"1^\infty"$ form)

Solution: $\lim\limits_{x \to \infty} \left(1 + \dfrac{a}{x}\right)^x = \lim\limits_{x \to \infty} e^{x \cdot \ln(1 + a/x)}$ so we need $\lim\limits_{x \to \infty} x \cdot \ln(1 + \frac{a}{x})$.

$\qquad \lim\limits_{x \to \infty} x \cdot \ln(1 + \frac{a}{x}) \rightarrow "\infty \cdot 0"$ an indeterminate product so rewrite it as a quotient

$\qquad = \lim\limits_{x \to \infty} \dfrac{\ln\left(1 + \dfrac{a}{x}\right)}{1/x} \rightarrow "\dfrac{0}{0}"$ an indeterminate quotient so use l'Hôpital's Rule

$\qquad = \lim\limits_{x \to \infty} \dfrac{\left(\dfrac{-a/x^2}{1 + \dfrac{a}{x}}\right)}{\left(-\dfrac{1}{x^2}\right)} = \lim\limits_{x \to \infty} \dfrac{a}{1 + \dfrac{a}{x}} \rightarrow \dfrac{a}{1} = a$.

Finally, $\lim\limits_{x \to \infty} \left(1 + \dfrac{a}{x}\right)^x = \lim\limits_{x \to \infty} e^{x \cdot \ln(1 + a/x)} = e^a$.

PROBLEMS

Determine the limits in problems 1 – 15.

1. $\lim\limits_{x \to 1} \dfrac{x^3 - 1}{x^2 - 1}$
2. $\lim\limits_{x \to 2} \dfrac{x^4 - 16}{x^5 - 32}$
3. $\lim\limits_{x \to 0} \dfrac{\ln(1 + 3x)}{5x}$

4. $\lim\limits_{x \to \infty} \dfrac{e^x}{x^3}$
5. $\lim\limits_{x \to 0} \dfrac{x \cdot e^x}{1 - e^x}$
6. $\lim\limits_{x \to 0} \dfrac{\cos(a + x) - \cos(a)}{x}$

7. $\lim\limits_{x \to \infty} \dfrac{\ln(x)}{x}$
8. $\lim\limits_{x \to \infty} \dfrac{\ln(x)}{\sqrt{x}}$
9. $\lim\limits_{x \to \infty} \dfrac{\ln(x)}{x^p}$ (p is any positive number)

10. $\lim\limits_{x \to 0} \dfrac{e^{3x} - e^{2x}}{4x}$
11. $\lim\limits_{x \to 0} \dfrac{1 - \cos(3x)}{x^2}$
12. $\lim\limits_{x \to 0} \dfrac{1 - \cos(2x)}{x}$

13. $\lim\limits_{x \to a} \dfrac{x^m - a^m}{x^n - a^n}$
14. $\lim\limits_{x \to 0} \dfrac{2^x - 1}{x}$
15. $\lim\limits_{x \to 0} \dfrac{1 - \cos(x)}{x \cdot \cos(x)}$

16. Find a value for p so $\lim\limits_{x\to\infty} \dfrac{3x}{px+7} = 2$.

17. Find a value for p so $\lim\limits_{x\to 0} \dfrac{e^{px}-1}{3x} = 5$.

18. $\lim\limits_{x\to\infty} \dfrac{\sqrt{3x+5}}{\sqrt{2x-1}}$ has the indeterminate form "∞/∞". Why doesn't l'Hôpital's Rule work with this

    limit? (Hint: Apply l'Hôpital's Rule twice and see what happens.) Evaluate the limit without using

    l'Hôpital's Rule.

19. (a) Evaluate $\lim\limits_{x\to\infty} \dfrac{e^x}{x}$, $\lim\limits_{x\to\infty} \dfrac{e^x}{x^2}$, $\lim\limits_{x\to\infty} \dfrac{e^x}{x^5}$.

    (b) An algorithm is "exponential" if it requires $a \cdot e^{bn}$ steps (a and b are positive constants). An

        algorithm is "polynomial" if it requires $c \cdot n^d$ steps (c and d are positive constants). Show that

        polynomial algorithms require fewer steps than exponential algorithms for large problems.

20. The problem $\lim\limits_{x\to 0} \dfrac{x^2}{3x^2+x}$ appeared on a test.

    One student determined the limit was an indeterminate "0/0" form, and applied l'Hôpital's Rule to get:

    $$\text{"}\lim\limits_{x\to 0} \dfrac{x^2}{3x^2+x} = \lim\limits_{x\to 0} \dfrac{2x}{6x+1} = \lim\limits_{x\to 0} \dfrac{2}{6} = \tfrac{1}{3} \text{.''}$$

    Another student also determined the limit was an indeterminate "0/0" form and wrote,

    $$\text{"}\lim\limits_{x\to 0} \dfrac{x^2}{3x^2+x} = \lim\limits_{x\to 0} \dfrac{2x}{6x+1} = \dfrac{0}{0+1} = 0 \text{.''}$$

    Which student is correct? Why?

Determine the limits in problems 21 – 29.

21. $\lim\limits_{x\to 0^+} \sin(x)\cdot\ln(x)$ 22. $\lim\limits_{x\to\infty} x^3\cdot e^{-x}$ 23. $\lim\limits_{x\to 0^+} \sqrt{x}\cdot\ln(x)$

24. $\lim\limits_{x\to 0^+} x^{\sin(x)}$ 25. $\lim\limits_{x\to\infty} (1-3/x^2)^x$ 26. $\lim\limits_{x\to\infty} (1-\cos(3x))^x$

27. $\lim\limits_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin(x)}\right)$ 28. $\lim\limits_{x\to\infty} (x-\ln(x))$ 29. $\lim\limits_{x\to\infty} \left(\dfrac{x+5}{x}\right)^{1/x}$

30. $\lim\limits_{x\to\infty} (1+3/x)^{2/x}$

Section 4.7   **PRACTICE Answers**

Practice 1:   (a) $\lim\limits_{x\to 0} \dfrac{1-\cos(5x)}{3x}$. The numerator and denominator are both differentiable and both

equal 0 when x = 0, so we can apply l'Hôpital's Rule:

$$\lim_{x\to 0} \frac{1-\cos(5x)}{3x} \;=\; \lim_{x\to 0} \frac{5\cdot\sin(5x)}{3} \;\to\; \frac{0}{3} = \mathbf{0} \;.$$

(b) $\lim\limits_{x\to 2} \dfrac{x^2 + x - 6}{x^2 + 2x - 8}$.   The numerator and denominator are both differentiable functions

and they both equal 0 when x = 0, so we can apply l'Hôpital's Rule:

$$\lim_{x\to 2} \frac{x^2 + x - 6}{x^2 + 2x - 8} \;=\; \lim_{x\to 2} \frac{2x + 1}{2x + 2} \;=\; \mathbf{\frac{5}{6}} \;.$$

Practice 2: $\lim\limits_{x\to\infty} \dfrac{x^2 + e^x}{x^3 + 8x}$. The numerator and denominator are both differentiable

and both become arbitrarily large as x becomes large, so we can apply l'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{x^2 + e^x}{x^3 + 8x} \;=\; \lim_{x\to\infty} \frac{2x + e^x}{3x^2 + 8} \;\to\; "\frac{\infty}{\infty}" . \text{ Using l'Hôpital's Rule again:}$$

$$\lim_{x\to\infty} \frac{2x + e^x}{3x^2 + 8} \;=\; \lim_{x\to\infty} \frac{2 + e^x}{6x} \;\to\; "\frac{\infty}{\infty}" \text{ and again:}$$

$$\lim_{x\to\infty} \frac{2 + e^x}{6x} \;=\; \lim_{x\to\infty} \frac{e^x}{6} \;\to\; \infty \;.$$

Practice 3:   Comparing A with e^n operations to B with $100\cdot\ln(n)$ operations.

$$\lim_{n\to\infty} \frac{e^n}{100\cdot\ln(n)} \;\to\; "\frac{\infty}{\infty}" \text{ so use L'Hopital's Rule:}$$

$$\lim_{n\to\infty} \frac{e^n}{100\cdot\ln(n)} \;=\; \lim_{n\to\infty} \frac{e^n}{100/n} \;=\; \lim_{n\to\infty} \frac{n\cdot e^n}{100} = \infty \text{ so B requires fewer operations than A.}$$

Comparing B with $100\cdot\ln(n)$ operations to C with n^5 operations.

$$\lim_{n\to\infty} \frac{100\cdot\ln(n)}{n^5} \;\to\; \frac{\infty}{\infty} \;. \; \lim_{n\to\infty} \frac{100\cdot\ln(n)}{n^5} \;=\; \lim_{n\to\infty} \frac{100/n}{5n^4} \;=\; \lim_{n\to\infty} \frac{100}{5n^5} = 0$$

so B requires fewer operations than C. B requires the fewest operations of the three algorithms.

Comparing A with e^n operations to C with n^5 operations. Using l'Hôpital's Rule several times:

$$\lim_{n\to\infty} \frac{e^n}{n^5} \;=\; \lim_{n\to\infty} \frac{e^n}{5n^4} \;=\; \lim_{n\to\infty} \frac{e^n}{20n^3} \;=\; \lim_{n\to\infty} \frac{e^n}{60n^2} \;=\; \lim_{n\to\infty} \frac{e^n}{120n} \;=\; \lim_{n\to\infty} \frac{e^n}{120} = \infty$$

so A requires more operations than C. A requires the most operations of the three algorithms.

Section Four

Section 4.1

1. Local maximums at x = 3, x = 5, x = 9, and x = 13. Global maximums at x = 3 and x = 13.
    Local minimums at x = 1, x = 4.5, x = 7, and x = 10.5 . Global minimun at x = 7 .

3. $f(x) = x^2 + 8x + 7$ so $f'(x) = 2x + 8$ which is defined for all values of x. $f'(x) = 0$ when
    x = –4 so x = –4 is a critical number. There are no endpoints.
    The only critical number is x = –4 , and the only critical point is (–4, f(–4)) = (–4, –9) which is the
    global (and local) minimum.

5. $f(x) = \sin(x)$ so $f'(x) = \cos(x)$ which is defined for all values of x. $f'(x) = 0$ when $x = \dfrac{\pi}{2} + n\pi$ so

    the values $x = \dfrac{\pi}{2} + n\pi$ are critical numbers. There are no endpoints.

    $f(x) = \sin(x)$ has local and global maximums at $x = \dfrac{\pi}{2} + 2n\pi$, and global and local minimums at

    $x = \dfrac{3\pi}{2} + 2n\pi$.

7. $f(x) = (x – 1)^2(x – 3)$ so $f'(x) = (x – 1)^2 + 2(x – 1)(x – 3) = (x – 1)(3x – 7)$ which is defined for all
    values of x. $f'(x) = 0$ when x = 1 and x = 7/3 so x = 1 and x = 7/3 are critical numbers. There are
    no endpoints.
    The only critical points are (1, 0) which is a local maximum and (7/3, –32/27) which is a local
    minimum. When the interval is the entire real number line, this function does not have a global
    maximum or global minimum.

9. $f(x) = 2x^3 – 96x + 42$ so $f'(x) = 6x^2 – 96$ which is defined for all values of x.
    $f'(x) = 6(x + 4)(x – 4) = 0$ when x = –4 and x = 4 so x = –4 and x = 4 are critical numbers. There
    are no endpoints. The only critical points are (–4, 298) which is a local maximum and (4,–214) which
    is a local minimum.   When the interval is the entire real number line, this function does not have a
    global maximum or global minimum.

11. $f(x) = 5x + \cos(2x + 1)$ so $f'(x) = 5 – 2\sin(2x + 1)$ which is defined for all values of x. $f'(x)$ is always
    positve (why?) so $f'(x)$ is never equal to 0. There are no endpoints. The function
    $f(x) = 5x + \cos(2x + 1)$ is always increasing and has no critical numbers, no critical points, no local or
    global maximums or minimums.

13. $f(x) = e^{-(x–2)^2}$ so $f'(x) = –2(x – 2)\, e^{-(x–2)^2}$ which is defined for all values of x. $f'(x) = 0$ when
    x = 2 so x = 2 is a critical number. There are no endpoints. The only critical point is (2, 1) which is
    a local and global maximum. When the interval is the entire real number line, this function does not
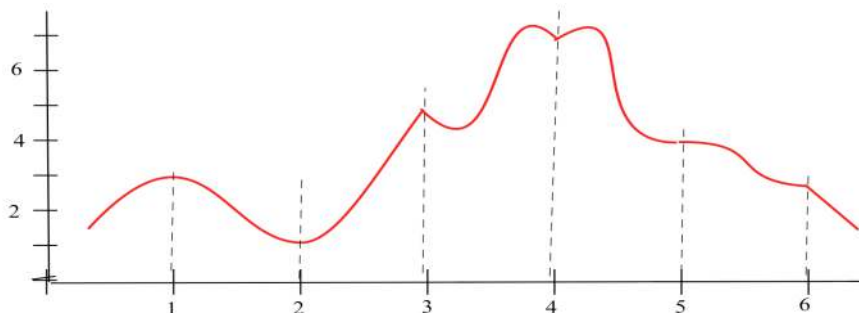   have a local or global minimum.

15. See Fig. 3.1P15

Fig. 3.1P15

17. $f(x) = x^2 - 6x + 5$ on $[-2, 5]$ so $f'(x) = 2x - 6$ which is defined for all values of x. $f'(x) = 0$ when x = 3 so x = 3 is a critical number. The endpoints are x = −2 and x = 5 which are also critical numbers. The critical points are $(3, -4)$ which is the local and global minimum, $(-2, 21)$ which is a local and global maximum, and $(5, 0)$ which is a local maximum.

19. $f(x) = 2 - x^3$ on $[-2, 1]$ so $f'(x) = -3x^2$ which is defined for all values of x. $f'(x) = 0$ when x = 0 so x = 0 is a critical number. The endpoints are x = −2 and x = 1 which are also critical numbers. The critical points are $(-2, 10)$ which is a local and global maximum, $(0, 2)$ which is not a local or global maximum or minimum, and $(1, 1)$ which is a local and global minimum.

21. $f(x) = x^3 - 3x + 5$ on $[-2, 1]$ so $f'(x) = 3x^2 - 3 = 3(x - 1)(x + 1)$ which is defined for all values of x. $f'(x) = 0$ when x = −1 and x = +1 so these are critical numbers. The endpoints x = −2 and x = 1 are also critical numbers. The critical points are $(-2, 3)$ which is a local and global minimum on $[-2, 1]$, the point $(-1, 7)$ which is a local and global maximum on $[-2, 1]$, and the point $(1, 3)$ which is a local and global minimum on $[-2, 1]$.

23. $f(x) = x^5 - 5x^4 + 5x^3 + 7$ so $f'(x) = 5x^4 - 20x^3 + 15x^2 = 5x^2(x^2 - 4x + 3) = 5x^2(x - 3)(x - 1)$ which is defined for all values of x. $f'(x) = 0$ when x = 0 and x = 1 in the interval $[0, 2]$ so each of these values is a critical number. The endpoints x = 0 and x = 2 are also critical numbers. The critical points are $(0, 7)$ which is a local minimum, $(1, 8)$ which is a local and global maximum, and $(2, -1)$ which is a local and global minimum. ($f'(3) = 0$ too, but x = 3 is not in the interval $[0, 2]$.)

25. $f(x) = \dfrac{1}{x^2 + 1}$ so $f'(x) = \dfrac{-2x}{(x^2 + 1)^2}$ which is defined for all values of x. $f'(x) = 0$ when x = 0 but x = 0 is not in the interval $[1, 3]$ so x = 0 is a not a critcal number. The endpoints x = 1 and x = 3 are critical numbers. The critical points are $(1, 1/2)$ which is a local and global maximum, and $(3, 1/10)$ which is a local and global minimum.

27. $A(x) = 4x \sqrt{1 - x^2}$ (0<x<1) .

$$A'(x) = 4 \left[\frac{-x^2}{\sqrt{1 - x^2}} + \sqrt{1 - x^2} \right] = 4 \frac{1 - 2x^2}{\sqrt{1 - x^2}} \quad \begin{cases} > 0 & \text{if } 0 < x < 1/\sqrt{2} \\ < 0 & \text{if } 1/\sqrt{2} < x < 1 \end{cases}$$

A maximum is attained when $x = 1/\sqrt{2}$: $A(1/\sqrt{2}) = 4\dfrac{1}{\sqrt{2}} \sqrt{1 - \dfrac{1}{2}} = 4\dfrac{1}{\sqrt{2}}\dfrac{1}{\sqrt{2}} = 2.$

29. $V = x(8 - 2x)^2$ for $0 < x < 4$.

$V' = x(2)(8 - 2x)(-2) + (8 - 2x)^2 = (8 - 2x)(-4x + 8 - 2x) = (8 - 2x)(8 - 6x) = 4(4 - x)(4 - 3x)$ so

$$V' \begin{cases} < 0 & \text{if } 4/3 < x \\ > 0 & \text{if } 0 < x < 4/3. \end{cases}$$

$V(4/3) = \dfrac{4}{3}(8 - \dfrac{8}{3})^2 = \dfrac{4}{3}(\dfrac{16}{3})^2 = \dfrac{1024}{27} \approx 37.926$ cubic units is the largest volume.

Smallest volume is 0 which occurs when x = 0 and x = 4.

31. (a) 4. The endpoints and two values of x for which $f'(x) = 0$.
    (b) 2. The endpoints.
    (c) At most n + 1. The 2 endpoints and the n − 1 interior points x for which $f'(x) = 0$.
         At least 2. The 2 endpoints.

33. (a) local minimum at (1, 5)   (b) no extrema at (1, 5)
    (c) local maximum at (1, 5)   (d) no extrema at (1, 5)

35. (a) 0, 2, 6, 8, 11, 12   (b) 0, 6, 11   (c) 2, 8, 12

37. If f does not attain a maximum on [a,b] or f does not attain a mimimum on [a,b], then f must have a discontinuity on [a,b].

39 (a) yes, −1   (b) no   (c) yes, −1   (d) no   (e) yes, 1 − π

41. (a) yes, 0   (b) yes, 0   (c) yes, 0   (d) yes, 0   (e) yes, 0

43. (a) $S(x)$ is minimum when $x \approx 8$.   (b) $S(x)$ is maximum when $x = 2$.

Section 4.2

1. $c \approx 3, 10,$ and 13.   3. (a) $c = \pi/2$ (b) $c = 3\pi/2, 5\pi/2, 7\pi/2, 9\pi/2$

5. Rolle's Theorem asserts that the velocity $h'(t)$ will equal 0 at some point between the time the ball is tossed and the time it comes back down. The ball is not moving as fast when it reaches the balcony from below.

7. The function does not violate Rolle's Thm. because the function does not satisfy the hypotheses of the theorem: f is not differentiable at 0, a point in the interval $-1 < x < 1$.

9. No. The velocity is not the same as the rate of change of altitude, since altitude is only one of the components of position. Rolle's Theorem only says there was a time when my altitude was not changing.

11. Since $f'(x) = 3x^2 + 5$, $f'(x) = 0$ has no real roots. If $f(x) = 0$ for a value of x other than 2, then by the corollary from Problem 8, we would have an immediate contradiction.

13. (a) $f(0) = 0, f(2) = 4, f'(c) = 2c$. $\frac{4-0}{2-0} = 2c$ implies that $c = 1$.

    (b) $f(1) = 4, f(5) = 8, f'(c) = 2c - 5$. $\frac{8-4}{5-1} = 2c - 5$ implies that $c = 3$.

14. (a) $f(0) = 0, f(\pi/2) = 1, f'(c) = \cos(c)$. $\frac{1-0}{\pi/2-0} = \cos(c)$ implies $c = \arccos(2/\pi) \approx 0.88$.

    (b) $f(-1) = -1, f(3) = 27, f'(c) = 3c^2$. $\frac{27+1}{3+1} = 3c^2$ implies $c^2 = 7/3$ so $c = \sqrt{7/3}$ (since $c > -1$)

15. (a) $f(1) = 4, f(9) = 2$, $f'(c) = \frac{-1}{2\sqrt{c}}$. $\frac{2-4}{1-9} = \frac{-1}{2\sqrt{c}}$ implies $\frac{-1}{4} = \frac{-1}{2\sqrt{c}}$ so $c = 4$.

    (b) $f(1) = 3, f(7) = 15$, $f'(c) = 2$. $\frac{15-3}{7-1} = 2$ so any c between 1 and 7 will do.

17. The hypotheses are not all satisfied since $f'(x)$ does not exist at $x = 0$ which is between -1 and 3.

19. Guilty. All we know is that $f'(c) = 17$ at some point, but this does not prove that the motorist "could not have been speeding."

21. $f(x) = x^3 + x^2 + 5x + c$. $f(1) = 7 + c = 10$ when $c = 3$. Therefore, $f(x) = x^3 + x^2 + 5x + 3$.

23. (a) $f'(x) = 2Ax$. We need $A(1)^2 + B = 9$ and $2A(1) = 4$ so $A = 2$ and $B = 7$ and $f(x) = 2x^2 + 7$.

    (b) $A(2)^2 + B = 3$ and $2A(2) = -2$ so $A = -1/2$ and $B = 5$ and $f(x) = \frac{-1}{2}x^2 + 5$.

    (c) $A(0)^2 + B = 2$ and **$2A(0) = 3$.** There is no such A. The point $(0,2)$ is not on the parabola $y = x^2 + 3x - 2$.

25. $f(x) = x^3 + C$, a family of "parallel" curves for different values of C.

27. $v(t) = 300$. Assuming the rocket left the ground at $t = 0$, we have $y(1) = 300$ ft, $y(2) = 600$ ft, $y(5) = 1500$ ft.

29. $f''(x) = 6, f'(0) = 4, f(0) = -5$. $f(x) = 3x^2 + 4x - 5$.

31. (a) $A(x) = 3x$ (b) $A'(x) = 3$.   33. (a) $A(x) = x^2 + x$ (b) $A'(x) = 2x + 1$.

35. $a_1 = 5, a_2 = a_1 + 3 = 5 + 3 = 8, a_3 = a_2 + 3 = (5+3) + 3 = 11, a_4 = a_3 + 3 = (5+3) + 3 + 3 = 14$.
    In general, $a_n = 5 + 3(n-1) = 2 + 3n$.

Section 4.3

1. See Fig. 3.3P1.

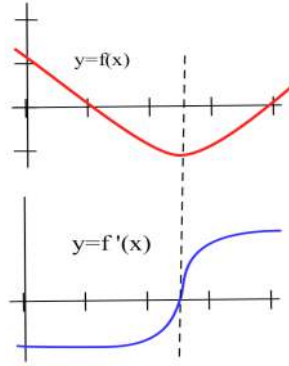3. See Fig. 3.3P2.

4. See Fig. 3.3P3.

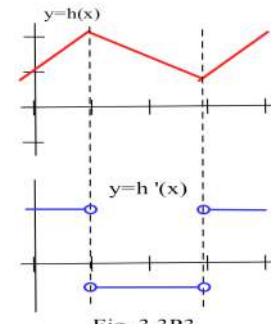5. See Fig. 3.3P4.

6. See Fig. 3.3P5.
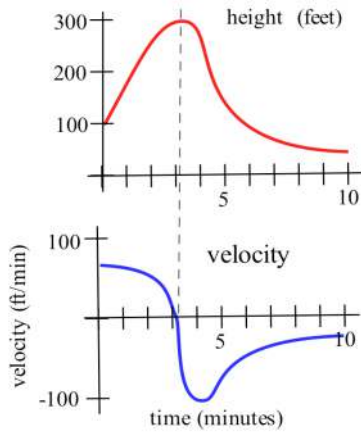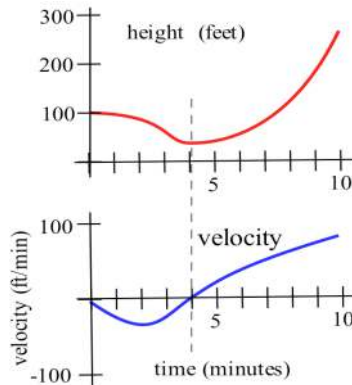
7. A–Q, B–P, C–R

Fig 3.3P1
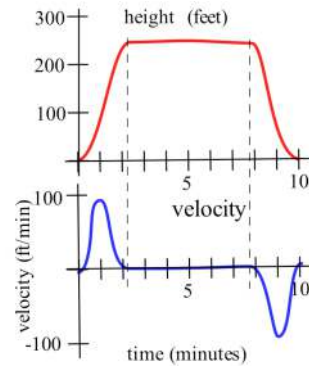
Fig. 3.3P3

Fig. 3.3P4

Fig. 3.3P5

Fig. 3.3P6

9. $f'(x) = \dfrac{1}{x} > 0$ for $x > 0$ so $f(x) = \ln(x)$ is increasing on $(0, \infty)$.

11. If f is increasing then $f(1) < f(\pi)$ so $f(1)$ and $f(\pi)$ cannot both equal 2.

13. (a) $x = 3$, $x = 8$ (b) maximum at $x = 8$ (c) none (or only at right endpoint)

14. Relative maximum height at $x = 2$ and $x = 7$. Relative minimum height at $x = 4$.

15. Relative maximum height at $x = 6$. Relative minimum height at $x = 8$.

17. $f(x) = x^3 - 3x^2 - 9x - 5$ has a relative minimum at $(3, -32)$ and a relative maximum at $(-1, 0)$.

19. $h(x) = x^4 - 8x^2 + 3$ has a relative maximum at $(0, 3)$ and relative minimums at $(2, -13)$ and $(-2, -13)$.

21. $r(t) = 2(t^2 + 1)^{-1}$ has a relative maximum at $(0, 2)$ and no relative minimums.

23. No positive roots. $f(x) = 2x + \cos(x)$ is continuous. $f(0) = 1 > 0$. Since $f'(x) = 2 - \sin(x) > 0$ for all x, f is increasing and never decreases back to the x–axis (a root).

24. One positive root. $g(x) = 2x - \cos(x)$ is continuous. $g(0) = -1 < 0$ and $g(1) = 2 - \cos(1) > 0$ so by the Intermediate Value Theorem g has a root between 0 and 1. Since $g'(x) = 2 + \sin(x) > 0$ for all x, g is increasing and can have only that 1 root.

25. $h(x) = x^3 + 9x - 10$ and $h(1) = 0$. $h'(x) = 3x^2 + 9 = 3(x^2 + 3) > 0$ for all x so h is always increasing and can cross the x–axis at most at one place. Since the graph of h crosses the x–axis at x = 1, that is the only root of h.
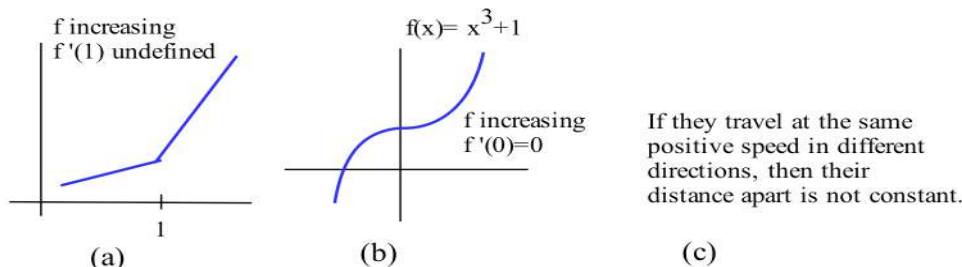
27.

Fig. 3.3P27

29. (a) $h(x) = x^2, x^2 + 1, x^2 - 7$, or, in general, $x^2 + C$ for any constant C.

(b) $f(x) = x^2 + C$ for some value C and $20 = f(3) = 3^2 + C$ so $C = 20 - 9 = 11$. $f(x) = x^2 + 11$.

(c) $g(x) = x^2 + C$ for some value C and $7 = g(2) = 2^2 + C$ so $C = 7 - 4 = 3$. $g(x) = x^2 + 3$.

Section 4.4

1. (a) f(t) = number of workers unemployed at time t. $f'(t) > 0$ and $f''(t) < 0$
   (b) f(t) = profit at time t. $f'(t) < 0$ and $f''(t) > 0$.
   (c) f(t) = population at time t. $(f'(t) > 0$ and $f''(t) > 0$.
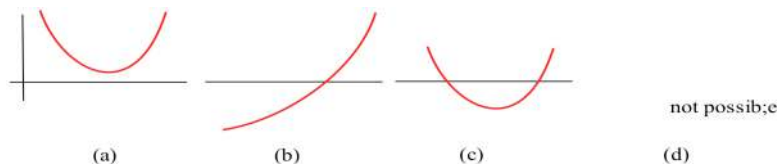
3. See Fig. 3.4P3.

Fig. 3.3P3

5. (a) Concave up on $(0, 2), (2, 3+), (6, 9)$. Concave down on $(3+, 6)$. (A small technical note: we have defined concavity only at points where the function is differentiable, so we exclude the endpoints and points where the function is not differentiable from the intervals of concave up and concave down.)

7. $g(x) = x^3 - 3x^2 - 9x + 7$. $g''(x) = 6x - 6$. $g''(-1) < 0$ so $(-1, 12)$ is a local maximum. $g''(3) > 0$ so $(3, -20)$ is a local minimum.

9. $f(x) = \sin^5(x)$. $f''(x) = 5\{ -\sin^5(x) + 4\sin^3(x) \cdot \cos^2(x) \}$. $f''(\pi/2) < 0$ so $(\pi/2, 1)$ is a local maximum. $f''(3\pi/2) > 0$ so $(3\pi/2, -1)$ is a local minimum.
$f''(\pi) = 0$ and f changes concavity at $x = \pi$ so $(\pi, 0)$ is an inflection point.

11. d and e.   13. (a) 0   (b) at most 1   (c) at most n – 2.

15.

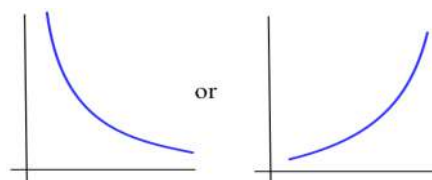| x	g(x)	g'(x)	g''(x)
0	–	+	+
1	+	0	–
2	–	–	+
3	0	+	+

17. See Fig. 3.4P17.

or

Fig. 3.4P17

Section 4.5

1. (a) $2x + 2y = 200$ so $y = 100 – x$. Maximize $A = x \cdot y = x \cdot (100 – x) = 100x – x^2$.
$A' = 100 – 2x$ and $A' = 0$ when $x = 50$ ($y = 100 – x = 50$). $A'' = – 2 < 0$ so
$x = 50$ yields the maximum enclosed area. When $x = 50$,
$A = 50(100 – 50) = 2500$ square feet.

(b) $2x + 2y = P$ so $y = P/2 – x$. Maximize $A = x \cdot y = x \cdot (P/2 – x) = (P/2)x – x^2$.
$A' = P/2 – 2x$ and $A' = 0$ when $x = P/4$ (then $y = P/2 – x = P/4$). $A'' = –2 < 0$
so $x = P/4$ yields the maximum enclosed area.
This garden is a P/4 by P/4 square.

(c) $2x + y = P$ so $y = P – 2x$. Maximize $A = xy = x(P – 2x) = Px – 2x^2$.
$A' = P – 4x$ and $A' = 0$ when $x = P/4$ (then $y = P – 2x = P/2$).

(d) A circle. A semicircle.

(a) & (b)

(c) Fig. 3.5P1
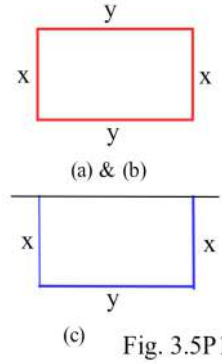
3. (a) $120 = 2x + 5y$ so $y = 24 – \frac{2}{5} x$. Maximize $A = xy = x(24 – \frac{2}{5} x) = 24x – \frac{2}{5} x^2$.

$A' = 24 – \frac{4}{5} x$ and $A' = 0$ when $x = 30$ (then $y = 12$). $A'' = – 4/5 < 0$ so
$x = 30$ yields the maximum enclosed area. Area is $(30 \text{ ft})(12 \text{ ft}) = 360$ square feet.

(b) A circular pen divided into 4 equal stalls by two diameters shown in
diagram (a) does a better job than a square with 400 square feet. If the
radius is r, then $4r + 2\pi r = 120$ so $r = 120/(4 + 2\pi) \approx 11.67$.
The resulting enclosed area is $A = \pi r^2 \approx \pi(11.67)^2 \approx 427.8$ sq. ft.
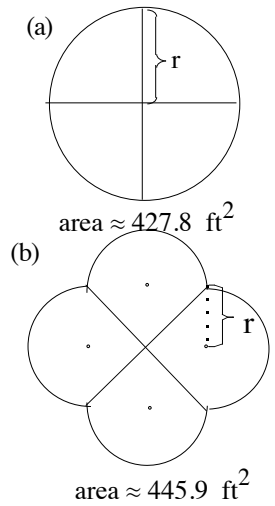
The pen shown in diagram (b) does even better. If each semicircle
has radius r, then the figure uses $4\sqrt{2}\ r + 4\pi r = 120$ feet of fence
so $r = 120/(4\sqrt{2} + 4\pi) \approx 6.585$. The resulting enclosed area is

$A = (\text{square}) + (\text{four semicircles}) = (2r)^2 + 4(\frac{1}{2} \pi r^2) \approx 445.90$ sq. ft.

(a)

area $\approx$ 427.8 ft^2

(b)

area $\approx$ 445.9 ft^2

5. $2x + 2y = 10$ so $y = 5 – x$.

Maximize $V = xy(10 – 2x) = x(5 – x)(10 – 2x) = 50x – 20x^2 + 2x^3$.

$V' = 50 – 40x + 6x^2 = 2(3x – 5)(x – 5)$ and $V' = 0$ when $x = 5$ and $x = 5/3$. When $x = 5$, then $V = 0$, clearly

not a maximum, so $x = 5/3$. The dimensions of the box with the largest volume are $5/3, 10/3$, and $20/3$.

7. (a) $V = \pi r^2 h = 100$ so $h = \dfrac{100}{\pi r^2}$.

Minimize $C = 2(\text{top area}) + 5(\text{bottom area}) + 3(\text{side area})$

$= 2(\pi r^2) + 5(\pi r^2) + 3(2\pi r h) = 7\pi r^2 + 6\pi r (\dfrac{100}{\pi r^2}) = 7\pi r^2 + \dfrac{600}{r}$.

$C' = 14\pi r – \dfrac{600}{r^2}$ and $C' = 0$ when $r = \sqrt[3]{600/(14\pi)} \approx 2.39$ (then $h = \dfrac{100}{\pi r^2} \approx 5.57$).

(b) Let $k = \text{top} + \text{bottom rate} = 2\text{¢} + \text{the bottom rate} > 2\text{¢} + 5\text{¢} = 7\text{¢}$. Minimize $C = k\pi r^2 + \dfrac{600}{r}$.

$C' = 2k\pi r – \dfrac{600}{r^2}$ and $C' = 0$ when $r = \sqrt[3]{600/(2k\pi)}$. If $k = 8$, then $r \approx 2.29$.

If $k = 9$, then $r \approx 2.20$. If $k = 10$, then $r \approx 2.12$. As the cost of the bottom material increases,
the radius of the least expensive cylindrical can decreases: the least expensive can becomes
narrower and taller

9. Time = distance/rate. Run distance = x   ($0 \le x \le 60$ Why?)   so   run time = x/8.

Swim distance = $\sqrt{40^2 + (60–x)^2}$ so   swim time = $\frac{1}{2}\sqrt{40^2 + (60–x)^2}$ and the total time is

$$T = \frac{x}{8} + \frac{1}{2}\sqrt{40^2 + (60–x)^2} \ .$$

$$T' = \frac{1}{8} + \frac{1}{2}\frac{1}{2}(40^2 + (60–x)^2)^{-1/2} \cdot 2\cdot(60 – x)\cdot(–1) = \frac{1}{8} - \frac{60 – x}{2\sqrt{40^2 + (60–x)^2}} \ .$$

$T' = 0$ when $x = 60 \pm \dfrac{40}{\sqrt{15}}$. The value $x = 60 + \dfrac{40}{\sqrt{15}} > 60$ so the least total time occurs when

$x = 60 - \dfrac{40}{\sqrt{15}} \approx 49.7$ meters. In this situation, the lifeguard should run about 5/6 of the way

along the beach before going into the water.

11.   (a)   Consider a similar problem with a new town D* located at the "mirror image" of D across the
river (Fig. 3.5P11a). If the water works is built at any location W along the river, then the
distances are the same from W to D and to D*: dist(W,D) = dist(W,D*).
Then dist(C,W) + dist(W,D) = dist(C,W) + dist(W,D*). The shortest distance from C to D* is a
straight line (Fig. 3.5P11b), and this
straight line gives similar triangles with

equal side ratios: $\frac{x}{3} = \frac{10 – x}{5}$ so

x = 15/4 = 3.75 miles. A consequence of
this "mirror image" view of the problem
is that "at the best location W the angle
of incidence α equals the angle of
reflection β "

Fig. 3.5P11

(a)   D*   b   D*
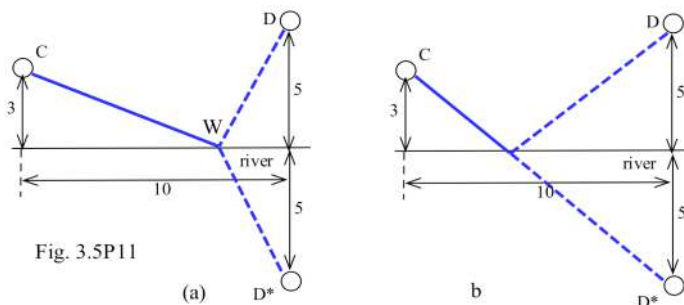
   (b)   Minimize $C = 3000\,\text{dist}(C,W) + 7000\,\text{dist}(W,D) = 3000\sqrt{x^2 + 9} + 7000\sqrt{(10 – x)^2 + 25}$.

$$C' = \frac{3000x}{\sqrt{x^2 + 9}} + \frac{–7000(10 – x)}{\sqrt{(10 – x)^2 + 25}} \quad \text{so}$$

$C' = 0$ when $\dfrac{3x}{\sqrt{x^2 + 9}} = \dfrac{7(10 – x)}{\sqrt{(10 – x)^2 + 25}}$ and $x \approx 7.82$ miles.

As it becomes relatively more expensive to build the pipe from a point W on the river to D, the
cheapest route tends to shorten the distance from W to D.

13. (a)   Let x be the length of one edge of the square end. Then $V = x^2(108 – 4x) = 108x^2 – 4x^3$.

$V' = 216x – 12x^2 = 6x(18 – x)$ so $V' = 0$ when x = 0 or x = 18. The dimensions of the

greatest volume acceptable box with a square end are 18 by 18 by 36 inches: $V = 11{,}664\ \text{in}^3$.

   (b)   Let x be the length of the shorter edge of the end. Then $V = 2x^2(108 – 6x) = 216x^2 – 12x^3$.

$V' = 432x – 36x^2 = 36x(12 – x)$ so $V' = 0$ when x = 0 or x = 12. The dimensions of the

largest box acceptable box with this shape are 12 by 24 by 36 inches: $V = 10{,}368\ \text{in}^3$.

   (c)   Let x be the radius of the circular end. Then $V = \pi x^2(108 – 2\pi x) = 108\pi x^2 – 2\pi^2 x^3$.

$V' = 216\pi x – 6\pi^2 x^2 = 6\pi x(36 – \pi x)$ so $V' = 0$ when x = 0 or $x = 36/\pi \approx 11.46$ inches. The

dimensions of the largest box acceptable box with a circular end are a radius of $36/\pi \approx 11.46$ and

a length of 36 inches: $V \approx 14{,}851\ \text{in}^3$.

15. Without calculus: The area of the triangle is $\frac{1}{2}$ (base)(height) $= \frac{1}{2}$ (7)(height)   and the height is

maximum when the angle between the sides is a right angle.

Using calculus: Let θ be the angle between the sides. Then the area of the triangle is

$A = \frac{1}{2}$ (base)(height) $= \frac{1}{2}$ (7)(height) $= \frac{1}{2}$ (7)(10 sin θ) = 35 sin θ . A ' = 35 cos θ so A ' = 0

when $\theta = \pi/2$, and the triangle is a right triangle with sides 7 and 10.

Using either approach, the maximum area of the triangle is $\frac{1}{2}$ (7)(10) = 35 square inches, and the other

side is the hypotenuse with length $\sqrt{7^2 + 10^2}$ $= \sqrt{149}$ $\approx$ 12.2 inches.

17. (a) $A = 2x(16 - x^2) = 32x - 2x^3$. Then $A ' = 32 - 6x^2$ so A ' = 0 when $x = \sqrt{32/6}$ $\approx$ 2.31.
      The dimensions are $2\sqrt{32/6}$ $\approx$ 4.62 and $16 - (\sqrt{32/6})^2$ = 64/6 $\approx$ 10.67.

   (b) $A = 2x(\sqrt{1 - x^2})$. Then $A ' = 2(\sqrt{1 - x^2}$ $- \dfrac{x^2}{\sqrt{1 - x^2}}$) so A ' = 0 when

      $x = 1/\sqrt{2}$ $\approx$ 0.707 . The dimensions are $2(1/\sqrt{2}) \approx$ 1.414 and $1/\sqrt{2}$ $\approx$ 0.707 .

   (c)   The graph of |x| + |y| = 1 is a "diamond" (a square) with corners at (1,0), (0,1), (–1,0) and (0,–1).
      For $0 \leq x \leq 1$, $A = 2x\cdot 2(1 - x)$ so $A = 4x - 4x^2$. Then A ' = 4 – 8x and A ' = 0 when
      x = 1/2. A " = – 8 so we have a local max. The dimensions are 2(1/2) = 1 and 2(1 – 1/2) = 1.

   (d)   A = 2x cos(x) (0 $\leq$ x $\leq$ π/2). Then A ' = 2 cos(x) – 2x$\cdot$sin(x) so A ' = 0 when x $\approx$ 0.86 . The
      dimensions are 2(0.86) = 1.72 and cos(0.86) $\approx$ 0.65 .

19. A = 6$\cdot$sin(θ/2)$\cdot$6$\cdot$cos(θ/2) = 36$\cdot$$\frac{1}{2}$ sin(θ) = 18 sin(θ) and this is a maximum when $\theta = \pi/2$. Then

the maximum area is A = 18 sin(π/2) = 18 square inches. (This problem is similar to problem

15.)

21. $V = \frac{1}{3} \pi r^2 h$ and $h = \sqrt{9 - r^2}$ so $V = \frac{1}{3} \pi r^2 \sqrt{9 - r^2}$ $= \frac{\pi}{3}\sqrt{9r^4 - r^6}$. Then

$V ' = \frac{\pi}{3} \dfrac{36r^3 - 6r^5}{\sqrt{9r^4 - r^6}}$, and V ' = 0 when $36r^3 = 6r^5$ so $r = \sqrt{6}$ $\approx$ 2.45 inches and

$h = \sqrt{9 - r^2}$ $= \sqrt{3}$ $\approx$ 1.73 inches.

23. Let n $\geq$ 10 be the number of passengers. The income is I = n(30 – (n–10)) = $40n - n^2$. The cost is
   C = 100 + 6n so the profit is P = Income – Cost = $(40n - n^2) - (100 + 6n) = 34n - n^2 - 100$.
   P ' = 34 – 2n and P ' = 0 when n = 17. 17 passengers on the flight maximize your profit.
   (This is an example of treating a naturally discrete variable, the number of passengers, as a continuous variable.)

25. Apply the result of problem 24 with R = f and E = g.

27. (i)   Let D = diameter of the base of the can, and let H = the height of the can.

      Then $\theta = \arctan(\dfrac{\text{radius of can}}{\text{height of cg}})$ $= \arctan(\dfrac{D/2}{H/2})$.

      For this can, D = 5 cm and H = 12 cm (sorry this should be in the statement of the problem) so

      $\theta = \arctan(2.5/6) = \arctan(0.42) \approx 0.395$ which is about 22.6$^{\text{O}}$. The can can be tilted about 22.6$^{\text{O}}$
      before it falls over.

   (ii) $C(x) = \dfrac{360 + 9.6x^2}{60 + 19.2x}$ so $C '(x) = \dfrac{(60 + 19.2x)(19.2x) - (360 + 9.6x^2)(19.2)}{(0 + 19.2x)^2}$. C '(x) = 0 when
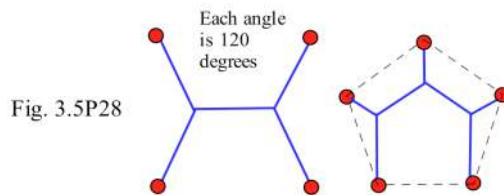
(19.2)($9.6x^2 + 60x - 360$) $= 0$ so $x = 3.75$: the height of the cola is $h = 3.75$ cm.

(iii) $C(3.75) = 3.75$ (The center of gravity is exactly at the top edge of the cola. It turns out that when the cg of a can and liquid system is as low as possible then the cg is at the top edge of the liquid.) Then $\theta = \arctan($ radius/(height of cg)) $= \arctan($ 2.5/3.75) $= \arctan($ 0.667) ≈ 0.588 which is about 33.7^O . In this situation, the can can be tilted about 33.7^O before it falls over.

(iv) Less.

28. (a) & (b) See Fig. 3.5P28. In the solutions to these "shortest path" problems, the roads all meet at 120° angles.

Each angle is 120 degrees

Fig. 3.5P28

29. (a) A=(base)(height)= $(1 - x)(x^2) = x^2 - x^3$ for $0 \le x \le 1$. $A' = 2x - 3x^2 = 0$ if $x = 2/3$.

(Clearly the endpoints x=0 and x=1 will not give the largest area.) Then $A = \left(\dfrac{1}{3}\right)\left(\dfrac{2}{3}\right)^2 = \dfrac{4}{27}$.

(b) A=(base)(height)= $(1 - x)(Cx^2) = Cx^2 - Cx^3$ for $0 \le x \le 1$.

$A' = 2Cx - 3Cx^2 = 0$ if $x = 2/3$. Then $A = \left(\dfrac{1}{3}\right)(C)\left(\dfrac{2}{3}\right)^2 = \dfrac{4C}{27}$.

(c) A=(base)(height)= $(B - x)(Cx^2) = BCx^2 - Cx^3$ for $0 \le x \le 1$.

$A' = 2BCx - 3Cx^2 = 0$ if $x = \dfrac{2}{3}B$. Then $A = \left(\dfrac{B}{3}\right)(C)\left(\dfrac{2B}{3}\right)^2 = \dfrac{4}{27}B^3C$.

31. (a) $y = 20 - \dfrac{20}{50}x$. $A = (base)height) = xy = x(20 - \dfrac{2}{5}x) = 20x - \dfrac{2}{5}x^2$.

$A' = 20 - \dfrac{4}{5}x = 0$ when $x = 25$. Then $y = 10$ and $Area = 250$.

(b) $y = H - \dfrac{H}{B}x$. $A = (base)height) = x(H - \dfrac{H}{B}x) = Hx - \dfrac{H}{B}x^2$.

$A' = H - \dfrac{2H}{B}x = 0$ when $x = \dfrac{B}{2}$. Then $y = \dfrac{H}{2}$ and $Area = \dfrac{BH}{4}$.

33. $F = \cos t = (top \ \cos t) + (bottom) + (sides) = (\pi r^2)A + (\pi r^2)B + (2\pi rh)C$

But we know $V = \pi r^2 h$ so $h = \dfrac{V}{\pi r^2}$ so $F = \pi r^2(A + B) + \dfrac{2CV}{r}$.

Then $F' = 2\pi r(A + B) - \dfrac{2CV}{r^2} = 0$ when $r = \sqrt[3]{\dfrac{CV}{\pi(A + B)}}$. Now you can find h and F.

Section 4.6

1. (a) h has a root at $x = 1$.
   (b) limits of h(x) = f(x)/g(x): as x $\to 1^+$ is 0: as x $\to 1^-$ is 0: as x $\to 3^+$ is $-\infty$: as x $\to 3^-$ is $+\infty$
   (c) h has a vertical asynptote at $x = 3$.

3. limits of h(x) = f(x)/g(x): as x $\to 2^+$ is $+\infty$: as x $\to 2^-$ is $-\infty$: as x $\to 4^+$ is 0 : as x $\to 4^-$ is 0

5. 0   7. –3   9. 0   11. DNE

13. 2/3   15. 0   17. –7   19. 0

21. $\cos(0) = 1$ 23. $\ln(1) = 0$

25. (a) $V(t) = 50 + 4t$ gallons, and $A(t) = 0.8t$ pounds of salt

(b) $C(t) = \dfrac{\text{amount of salt}}{\text{total amount of liquid}} = \dfrac{A(t)}{V(t)} = \dfrac{0.8t}{50 + 4t}$

(c)   "after a long time" (as $t \to \infty$), $C(t) \to 0.8/4 = 0.2$ pounds of salt per gallon.

(d) $V(t) = 200 + 4t$, $A(t) = 0.8t$, $C(t) = \dfrac{0.8t}{200 + 4t} \to 0.8/4 = 0.2$ pounds of salt per gallon.

27. $+\infty$ 29. $-\infty$ 31. $-\infty$ 33. $-\infty$

35. $+\infty$ 37. $-\infty$ 39. 1 41. $-\infty$

42. Horizontal: $y = 1$. Vertical: $x = 1$.   43. Horizontal: $y = 0$. Vertical: $x = 0$.

44. Horizontal: $y = 0$. Vertical: $x = 0$ (a "hole" at $x = 1$).   45. Horizontal: $y = 0$. Vertical: $x = 3$ and $x = 1$.

46. Horizontal: $y = 1/3$. Vertical: $x = 1$.   47. Horizontal: $y = 1$.

48. Horizontal: $y = 0$. Vertical: $x = 0$.   49. Horizontal: $y = 1$. Vertical: $x = 1$.

50. Horizontal: $y = 0$. Vertical: $x = 1$. (and a "hole" at $x = 0$)

51. $y = 2x + 1$. $x = 0$ 52. $y = x$ 53. $y = \sin(x)$. $x = 2$ 54. $y = x$

55. $y = x^2$ 56. $y = x^2 + 1$. $x = -1$ 57. $y = \cos(x)$. $x = 3$.   58. $y = x^2 + 2$. $x = 1$.

59. $y = \sqrt{x}$. $x = -3$.

Section 4.7

1. $3/2$ 3. $3/5$ 5. -1 7. 0

9. 0 11. $9/2$

13. For $a \neq 0$: $\dfrac{f'}{g'} = \dfrac{m}{n} x^{(m-n)} \rightarrow \dfrac{m}{n} a^{(m-n)}$.

For $a = 0$, $\lim\limits_{x \to a} \dfrac{x^m - a^m}{x^n - a^n} = \begin{cases} 0 & \text{if } m > n \\ 1 & \text{if } m = n \\ +\infty & \text{if } m < n \text{ and } (m - n) \text{ is even} \\ \text{DNE} & \text{if } m < n \text{ and } (m - n) \text{ is odd} \end{cases}$

15. 0 $\in$ 17. $\dfrac{f'}{g'} = \dfrac{pe^{px}}{3} \rightarrow \dfrac{p}{3}$ so $p = 3(5) = 15$.

19. (a) All three limits are $+\infty$.

(b) After applying L'Hopital's Rule d times, $\dfrac{f^{(d)}}{g^{(d)}} = \dfrac{a \cdot b^n \cdot e^{bn}}{c(d)(d-1)(d-2) \ldots (2)(1)} = \dfrac{\text{constant} \cdot e^{bn}}{\text{another constant}} \to +\infty$.

21. 0 23. 0 24. 1 25. 1 26. 1

27. 0 28. $+\infty$ 29. 1 30. 1

Calculus

Derivative Reference Facts

Notation: $\mathbf{D}(\, f(x)\,) = \dfrac{d}{dx}\, f(x) = f\,'(x)$ represents the derivative with respect to x of the function $f(x)$

General Derivative Patterns:

$\mathbf{D}(\, k \,) = 0$ k a constant $\mathbf{D}(\, f + g \,) = \mathbf{D}f + \mathbf{D}g$ $\mathbf{D}(\, f{\cdot}g \,) = f{\cdot}\mathbf{D}g + g{\cdot}\mathbf{D}f$

$\mathbf{D}(\, k{\cdot}f \,) = k{\cdot}\mathbf{D}f$ k a constant $\mathbf{D}(\, f - g \,) = \mathbf{D}f - \mathbf{D}g$ $\mathbf{D}(\, f/g \,) = \dfrac{g{\cdot}\mathbf{D}f - f{\cdot}\mathbf{D}g}{g^2}$

$\mathbf{D}(\, f^n(x)) = n\, f^{n-1}(x){\cdot}\mathbf{D}f$ $\mathbf{D}(\, f(\, g(x)\,)\,) = f\,'(\, g(x)\,){\cdot}\mathbf{D}g$ (Chain Rule !!)

For Parametric Equations:

$$slope \;=\; \frac{dy/dt}{dx/dt} \qquad\qquad speed \;=\; \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

For Particular Functions:

$\mathbf{D}(\, \sin(\, x \,)\,) = \cos(\, x \,)$ $\mathbf{D}(\, \tan(\, x \,)\,) = \sec^2(\, x \,)$ $\mathbf{D}(\, \sec(\, x \,)\,) = \sec(\, x \,){\cdot}\tan(\, x \,)$

$\mathbf{D}(\, \cos(\, x \,)\,) = -\sin(\, x \,)$ $\mathbf{D}(\, \cot(\, x \,)\,) = -\csc^2(\, x \,)$ $\mathbf{D}(\, \csc(\, x \,)\,) = -\csc(\, x \,){\cdot}\cot(\, x \,)$

$\mathbf{D}(\, e^x \,) = e^x$ $\mathbf{D}(\, a^x \,) = a^x \ln(a)$ for any positive constant a

$\mathbf{D}(\, \ln|\, x \,|\,) = \dfrac{1}{x}$

$\mathbf{D}(\, \arcsin(\, x \,)\,) = \dfrac{1}{\sqrt{1 - x^2}}$ $\mathbf{D}(\, \arctan(\, x \,)\,) = \dfrac{1}{1 + x^2}$ $\mathbf{D}(\, \text{arcsec}(\, x \,)\,) = \dfrac{1}{|\, x \,|\sqrt{x^2 - 1}}$

$\mathbf{D}(\, \arccos(\, x \,)\,) = \dfrac{-1}{\sqrt{1 - x^2}}$ $\mathbf{D}(\, \text{arccot}(\, x \,)\,) = \dfrac{-1}{1 + x^2}$ $\mathbf{D}(\, \text{arccsc}(\, x \,)\,) = \dfrac{-1}{|\, x \,|\sqrt{x^2 - 1}}$

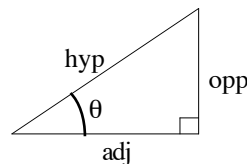# TRIGONOMETRY FACTS

Right Angle Trigonometry

$$\sin(\theta) = \frac{opp}{hyp} \qquad \cos(\theta) = \frac{adj}{hyp}$$

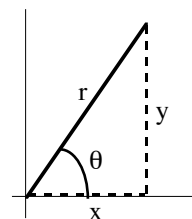$$\tan(\theta) = \frac{opp}{adj} \qquad \cot(\theta) = \frac{adj}{opp} \qquad \sec(\theta) = \frac{hyp}{adj} \qquad \csc(\theta) = \frac{hyp}{opp}$$

Trigonometric Functions

$$\sin(\theta) = \frac{y}{r} \qquad \cos(\theta) = \frac{x}{r}$$

$$\tan(\theta) = \frac{y}{x} \qquad \cot(\theta) = \frac{x}{y} \qquad \sec(\theta) = \frac{r}{x} \qquad \csc(\theta) = \frac{x}{y}$$

Fundamental Identities

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)} \qquad \cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)} \qquad \sec(\theta) = \frac{1}{\cos(\theta)} \qquad \csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\mathbf{\sin^2(\theta) + \cos^2(\theta) = 1} \qquad \tan^2(\theta) + 1 = \sec^2(\theta) \qquad 1 + \cot^2(\theta) = \csc^2(\theta)$$

$$\sin(-\theta) = -\sin(\theta) \qquad \cos(-\theta) = \cos(\theta) \qquad \tan(-\theta) = -\tan(\theta)$$

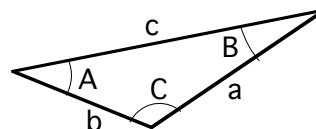$$\sin(\tfrac{\pi}{2} - \theta) = \cos(\theta) \qquad \cos(\tfrac{\pi}{2} - \theta) = \sin(\theta) \qquad \tan(\tfrac{\pi}{2} - \theta) = \cot(\theta)$$

Law of Sines: $\dfrac{\sin(A)}{a} = \dfrac{\sin(B)}{b} = \dfrac{\sin(C)}{c}$

Law of Cosines: $a^2 = b^2 + c^2 - 2bc\cdot\cos(A)$

$$b^2 = a^2 + c^2 - 2ac\cdot\cos(B)$$

$$c^2 = a^2 + b^2 - 2ab\cdot\cos(C)$$

Angle Addition and Subtraction Formulas

$$\sin(x + y) = \sin(x)\cdot\cos(y) + \cos(x)\cdot\sin(y)$$
$$\cos(x + y) = \cos(x)\cdot\cos(y) - \sin(x)\cdot\sin(y)$$
$$\tan(x + y) = \frac{\tan(x) + \tan(y)}{1 - \tan(x)\cdot\tan(y)}$$

$$\sin(x - y) = \sin(x)\cdot\cos(y) - \cos(x)\cdot\sin(y)$$
$$\cos(x - y) = \cos(x)\cdot\cos(y) + \sin(x)\cdot\sin(y)$$
$$\tan(x - y) = \frac{\tan(x) - \tan(y)}{1 + \tan(x)\cdot\tan(y)}$$

Function Product Formulas

$$\sin(x)\cdot\sin(y) = \frac{1}{2}\cos(x - y) - \frac{1}{2}\cos(x + y)$$
$$\cos(x)\cdot\cos(y) = \frac{1}{2}\cos(x - y) + \frac{1}{2}\cos(x + y)$$
$$\sin(x)\cdot\cos(y) = \frac{1}{2}\sin(x + y) + \frac{1}{2}\sin(x - y)$$

Function Sum Formulas

$$\sin(x) + \sin(y) = 2\sin(\tfrac{x + y}{2})\cdot\cos(\tfrac{x - y}{2})$$
$$\cos(x) + \cos(y) = 2\cos(\tfrac{x + y}{2})\cdot\cos(\tfrac{x - y}{2})$$
$$\tan(x) + \tan(y) = \frac{\sin(x + y)}{\cos(x)\cdot\cos(y)}$$

Double Angle Formulas

$$\sin(2x) = 2\sin(x)\cdot\cos(x)$$

$$\cos(2x) = \cos^2(x) - \sin^2(x) = 2\cos^2(x) - 1$$

$$\tan(2x) = \frac{2\tan(x)}{1 - \tan^2(x)}$$

Half Angle Formulas

$$\sin(\tfrac{x}{2}) = \pm\sqrt{\tfrac{1}{2}(1 - \cos(x))} \qquad \pm \text{ depends on}$$

$$\cos(\tfrac{x}{2}) = \pm\sqrt{\tfrac{1}{2}(1 + \cos(x))} \qquad \text{quadrant of } x/2$$

$$\tan(\tfrac{x}{2}) = \frac{1 - \cos(x)}{\sin(x)}$$