PRELIMINARY DRAFT: PLEASE DO NOT COPY

Probability versus Fuzziness

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1 Introduction

In a recent paper, Bart Kosko has presented fuzzy set theory from a new viewpoint [Kosko, 1990], rekindling the controversy between fuzzy theorists and probabilists. This controversy has reached the popular press, with articles appearing in newspapers and magazines on both sides of the Atlantic.[2] The purpose of this paper is to analyze the presuppositions of fuzzy set theory and fuzzy logic, as presented by Kosko.

Since the aim of the proponents of fuzzy set theory is to establish laws of thought for reasoning about uncertainty, Section 2 below considers the general question of what a law of thought is. Section 3 discusses the concept of uncertainty, and whether the postulates of fuzzy set theory are successful in capturing some or all aspects of uncertainty. In Section 4 the degree to which fuzzy logic resolves Russell's paradox is examined. The topic of Section 5 is the notion of fuzzy subsethood, and whether it can be used to show that probability theory is a special case of fuzzy set theory. Finally, Section 6 outlines some issues not examined in this paper.

2 The nature of laws of thought

A *law of thought* can be defined as a combinatorial pattern for manipulating ideas. By 'combinatorial', I mean that a law of thought describes an operation that does not delve into the internals of the ideas being manipulated. These ideas are treated as atomic for the purpose of applying the law.

Laws of thought are useful because they prevent thinking from petering out, and they make it reproducible. As an aid to thinking, a system of laws of thought can be compared to a pencil

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[2] I am aware of articles in *Business Week* (New York, May 21, 1990), the *Financial Times* (London, June 5, 1990), the *Economist* (London, June 9, 1990), and *Popular Science* (New York, June 1990). Articles have appeared in other publications also.

and paper. When a train of thought is written down, it is emancipated from the constraints posed by our limited short-term memory capacity. In addition, a written train of thought is available for inspection by ourselves at a later time, and by others: it can be duplicated and verified. If we do not record an episode of thinking, it eventually becomes impossible for us to continue because we cannot keep track of all the different ideas involved, and we cannot go over them again to check them.

Similarly, we cannot keep track of ideas combined in an indefinite variety of ways. For thinking to be precise and cumulative, ideas must be combined according to definite rules. These rules are laws of thought.

Classes of ideas. What is an idea, that it may be manipulated according to fixed patterns called laws of thought? If we adopt a somewhat naïve dualism of mind and matter, then we can say that ideas exist, that the external world also exists, and that ideas correspond to phenomena in the world. In general different phenomena interact in varying ways, so the corresponding ideas must be combined in different ways, for the result to remain meaningful. What makes laws of thought possible is that we can identify classes of phenomena that interact in similar ways. Ideas in corresponding classes can then be combined according to common patterns. A law of thought is a valid way of manipulating ideas if it mimics how the associated phenomena interact.

It is important to realize that no law of thought is self-evidently valid in the sense of being applicable to all ideas. Consider for example Aristotle's law of noncontradiction. This law states that the intersection of a set and its complement is always empty. In symbols,

$$A \cap A^c = \emptyset.$$

Consider the set of green apples. It seems extreme to claim that the intersection of this set and its complement (perhaps relative to the set of all apples) is empty. Surely some apples are both green to some extent and non-green to some extent. If we adopt the law of noncontradiction then we must disqualify green apples from being a proper set.

In the development of fuzzy set theory, the law of noncontradiction is rejected. It is possible to be even more radical, and to reject even Aristotle's law of identity: the rule of thinking that $A = A$. When this law is rejected, it becomes very difficult to perform any coherent reasoning. It is difficult to articulate what commitments one makes in accepting the law of identity, without using and presupposing the law. However we can point to an important class of ideas for which the law is questionable: the ideas corresponding to individual humans [Perry, 1986]. To what extent is George Bush always George Bush? He is composed of different atoms now than ten years ago. His political views, an important part of his identity, are different. Simultaneously George Bush is George Bush, and George Bush is not George Bush. The law of identity, like every candidate law of thought, is not universally valid.[3]

[3] It is worth pointing out that the issue of personal identity is not just a puzzle for philosophers. An intelligent

Alternative laws of thought. Since any law of thought is only applicable to certain classes of ideas, it is important to consider alternative classes of ideas. The search for useful laws of thought is simultaneously a search for a useful array of classes of ideas. A useful class of ideas is one that includes many ideas all of which share some common feature that justifies manipulating all the ideas in the class according to the same patterns. The effort to discover laws of thought is simultaneously an effort to identify important commonalities between ideas.

It is often difficult to see what alternative ways to classify ideas are available, because laws of thought are so basic. It can take great effort to transcend an initial classification of phenomena and ideas. Conversely, it is difficult to imagine that familiar classes of ideas, which seem obvious now, were arrived at historically with great difficulty. For example, consider the issue of the internal structure of assertions, or basic declarative sentences. Assertions are ideas that it appears natural to label either true or false. The rules of propositional calculus capture how assertions, viewed as atomic ideas, can be combined. But it is clear that there are further laws of thought concerning assertions to be discovered: most assertions have some internal structure– they are compound ideas.

Of what more primitive ideas is a proposition composed? For centuries no good classification of the primitive ideas entering into propositions was known. One consequence of this was that no universal rules for thinking about 'some' and 'every' sentences were known. The main impediment to progress was the classification of some ideas as 'subjects' and other 'objects', following intuitions induced by the syntax of natural language.

The distinction between 'subject' and 'object' was decisively rejected only in 1879, by Gottlob Frege [Frege, 1879]. The way was then clear to identify atomic propositions as consisting of a predicate applied to some number of individuals. 'Predicates' and 'individuals' are now seen as obvious classes of ideas, and first-order predicate calculus provides laws of thought that capture how ideas in these classes combine.

The choice of a system of laws of thought. The adoption of a particular mental calculus, or system of laws of thought, is not an empirical hypothesis about the external world. The phenomena of reality do not have a single natural classification. Laws of thought admit alternatives, and which laws one uses is a *choice*.

Some systems of thought are interesting to investigate for their own sake, for their mathematical elegance perhaps. However, if a formalism is proposed for actual use in reasoning, it should be made clear which phenomena are within the scope of the formalism: what ideas can it be used to think *about*? The calculus should be presented along with some description of the classes of ideas to which it is claimed to be applicable. Of course, some formalisms are useful for operating with quite different classes of ideas. For example, group theory provides laws of

robot must be able to draw appropriate conclusions about its own identity and that of the robots and persons surrounding it.

thought for reasoning about Rubik's cube, and also for reasoning about elementary particles. As a second example, probability theory is claimed to provide rules for thinking about degrees of belief, and also for thinking about relative frequencies.

More bluntly, a calculus such as probability theory or the theory of fuzzy sets must be motivated by its usefulness: its applicability to ideas that we want to reason with. The operations of the calculus must model the behaviour of the ideas in certain classes. One way to defend a calculus is to show that it succeeds on interesting examples. A better way is to provide convincing arguments characterizing a whole domain of success.

3 What do fuzzy sets capture?

In the previous section I argued that any system of laws of thought is a choice, and that a system is a good choice if it models the way a large class of phenomena interact. The aim of this section is to investigate the class of phenomena whose ways of interaction may be captured by the postulates of fuzzy set theory or fuzzy logic

The basic motivation for fuzzy logic is clear: many ideas resemble traditional assertions, but they are not naturally either true or false. Rather, they have a degree of truth. We can call these ideas claims. Fuzzy set theory is motivated similarly: there are many ideas that resemble traditional set ideas, in that they aggregate individuals from some domain, but each individual is somehow a member of the aggregate only to a certain degree.

If a claim is true only to a certain degree, that is because of some uncertainty attached to it. Fuzzy logic is an attempt to capture valid patterns of reasoning about uncertainty.

At least six different types of uncertainty may be identified.

Metaphysical uncertainty. Some statements are uncertain because they link different realms of existence. For example, the statement that a computer system will behave correctly because it has been formally verified is necessarily uncertain, because a formal verification is a proof, that is a mathematical object, whereas a computer system is a material entity. The applicability of the proof to the system depends on the primitive parts of the system behaving in accordance with the axioms governing the primitive symbols in the proof. At best the hypothesis that they do obey the axioms is empirically falsifiable. This type of uncertainty has been debated recently in the computer science literature [Fetzer, 1988; Cohn, 1989]. It is the type of uncertainty referred to by Einstein in his famous aphorism:

> So far as the laws of mathematics refer to reality, they are not certain. And so far as they are certain, they do not refer to reality.

Radical uncertainty. The celebrated qualification problem in AI is that no formalization of the external world can ever take into account all potentially relevant aspects of the world.

A theory of the operation of a car, for example, can always fail to predict the behaviour of a real car, because there is something about the real car, perhaps a potato in the tailpipe, that is important, but inexpressible in the theory. This type of uncertainty can be caled radical uncertainty, because it involves uncertainty about what is uncertain.

Nondeterminism. Some processes in the physical world are non-deterministic. For many centuries most scientists assumed, following Descartes, that the future evolution of the universe was fully determined by its current state. Einstein was expressing this opinion when he wrote "God does not play dice." Since the discovery of the equations of quantum mechanics, there has been a consensus among scientists, including Einstein, that the mathematics of randomness, that is probability theory, is a *sound* description of the physical world. Recent experiments have succeeded in amplifying quantum randomness to a macroscopic scale, and it is now an empirically established fact that quantum mechanics provides a *complete* theory of elementary particles: it is impossible to detect any determinism underlying quantum nondeterminism. Einstein was wrong.

Degree of confidence. If we know that certain assertions are true, but the truth of other assertions is unknown to us, then there is a third class of assertions about whose truth we have some confidence. Randomness is an objective type of uncertainty. It exists regardless of the observer. On the other hand, a person's confidence in an assertion is not an intrinsic property of the assertion, but a relation between the person and the assertion. Confidence is a subjective form of uncertainty.

Social context dependence. Some ideas are not fully defined in terms of more primitive ideas. Instead, their meaning arises from their use in communication, and depends on what contexts they are used in. 'Tall' is an example of an idea that is uncertain because its definition arises from social consensus about how it should be used in certain contexts. Wittgenstein called this process of establishing uncertain meanings for ideas through conversation "the social construction of reality."

Qualitative abstraction. Many descriptions are uncertain simply because information is omitted. In a scientific text for example, tall may be defined at one point as "over 1.83m." Much scientific progress depends on realizing that a notion is a qualitative abstraction, or socially constructed, and then discarding it in favour of a numerical measure. The new numerical concept may be arbitrary, but its precision allows statements using it to made with certainty. For example, geologists do not talk formally about mouintains. Instead, they talk about "regions at elevations over 4000m." The laws of probability theory are well-adapted for reasoning with statements of the form "The height of Mount Rainier is over 4000m."

More fundamental characterizations of different types of uncertainty, as intensional/extensional, epistemic/material, and absolute/dispellable.

3.1 Laws of thought for uncertainty

Now the question that we want to answer is whether the operations of fuzzy set theory are laws of thought for some aspect of uncertainty. In other words, we are looking for a class of uncertainty phenomena that interact in a way that is mimicked by the fuzzy set operations.

A fuzzy set is an aggregate of which objects are members to a certain degree. Let A be a fuzzy set, and x some object. Then $m_A(x)$, a real number between 0 and 1, is the degree to which x is a member of A. The basic postulates of fuzzy set theory are the following.

Definition 3.1: If $m_A(x) = a$ and $m_B(x) = b$ then

$$m_{A \bigcap B}(x) = \min\{a, b\}$$
$$m_{A \bigcup B}(x) = \max\{a, b\}$$
$$m_{A^c}(x) = 1 - a$$

Suppose our universe of objects is a collection of melons. Let R be the set of red melons, let G be the set of green melons, and let W be the set of watermelons. Suppose that $m_R(x) = 0.5$ and $m_W(x) = 0.8$. According to fuzzy set theory, $m_(x)_{R \bigcap W} = 0.5$. Is this reasonable? Watermelons are normally red. Redness and being a watermelon are mutually reinforcing facts. Intuitively, x is a red watermelon with considerably greater certainty than 0.5. On the other hand, suppose that $m_G(y) = 0.5$ and $m_W(y) = 0.8$. Is it reasonable that $m_(y)_{G \bigcap W} = 0.5$? Surely not. Watermelons are not green, and $m_(y)_{G \bigcap W}$ should be well under 0.5.

The deep issue is that intersection of sets, when membership is uncertain, is not a combinatorial operation. Degrees of membership do not combine in a fixed way. Probability theory does not provide an intersection function. It does provide a type of interval intersection operation. It is the case that

$$1 - (1 - Pr(A) + 1 - Pr(B)) \leq Pr(A \cap B) \leq \min\{Pr(A), Pr(B)\}$$

If $Pr(yellow(z)) = 0.8$, and $Pr(watermelon(z)) = 0.8$, then $Pr(yellow(z) \wedge watermelon(z))$ is at least 0.6, and at most 0.8. The actual probability depends on further aspects of the situation that are unknown at present. For example, if the two probabilities are independent, then the probability of the conjunction is $0.8 \cdot 0.8 = 0.64$.

3.2 Conclusion

The tentative conclusion of this section is that the operations of fuzzy set theory are not useful laws of thought for any wide class of ideas involving uncertainty.

4 What does fuzzy logic capture?

The laws of propositional calculus capture how propositions behave in combination. They are therefore useful for formalizing trains of thought concerning assertions. A claim is a proposition with a degree of truth. Fuzzy logic is an attempt to capture how claims behave in combination.

Formally, let x denote a proposition.h The numerical value $t(x)$ such that $0 \leq t(x) \leq 1$ is called the degree of truth of x.

Definition 4.1: The truth value of a compound claim is evaluated as follows:

$$\begin{aligned} t(A \wedge B) &= \min\{t(A), t(B)\} \\ t(A \vee B) &= \max\{t(A), t(B)\} \\ t(\neg A) &= 1 - t(A) \\ \text{if } A \text{ and } B \text{ are logically equivalent then } t(A) &= t(B). \quad \blacksquare \end{aligned}$$

In the last case of the definition above, "logically equivalent" means equivalent according to the rules of standard bivalent propositional calculus.

Do the fuzzy logic operations capture the combinatorial behaviour of degrees of truth? We shall define a logic to be *inconsistent* if it assigns the same truth value to all claims.

Theorem 4.1: Fuzzy logic is inconsistent.

Proof: Let A and B be arbitrary claims. We shall show that $t(A) = t(B)$.

Consider the claim $A \rightarrow B$. In propositional calculus,

$$A \rightarrow B \equiv \overline{A \wedge \overline{B}} \equiv B \vee (\overline{A} \wedge \overline{B}).$$

Now

$$t(\overline{A \wedge \overline{B}}) = 1 - \min\{t(A), 1 - t(B)\} = 1 + \max\{-t(A), -1 + t(B)\} = \max\{1 - t(A), t(B)\}$$

and

$$t(B \vee (\overline{A} \wedge \overline{B})) = \max\{t(B), \min\{1 - t(A), 1 - t(B)\}\}$$

Now $t(\overline{A \wedge \overline{B}}) = t(B \vee (\overline{A} \wedge \overline{B})$. The numerical expressions above are different if $t(B) < 1 - t(B) < 1 - t(A)$, i.e. $t(B) < 1 - t(B)$ and $t(A) < t(B)$, i.e. $t(A) < t(B) < 0.5$.

Now consider the seven claims

$$\begin{aligned} \overline{A} &\rightarrow B \\ A &\rightarrow \overline{B} \\ \overline{A} &\rightarrow \overline{B} \\ B &\rightarrow A \end{aligned}$$

$$\overline{B} \rightarrow A$$
$$B \rightarrow \overline{A}$$
$$\overline{B} \rightarrow \overline{A}$$

By the same reasoning, it cannot be the case that any of the following is true:

$$1 - t(A) < t(B) < 0.5$$
$$t(A) < 1 - t(B) < 0.5$$
$$1 - t(A) < 1 - t(B) < 0.5$$
$$t(B) < t(A) < 0.5$$
$$1 - t(B) < t(A) < 0.5$$
$$t(B) < 1 - t(A) < 0.5$$
$$1 - t(B) < 1 - t(A) < 0.5$$

Only if $t(A) = t(B) = 0.5$ is it possible for none of the eight inequalities derived to be satisfied. ■

Fuzzy logic is not a fully specified formal system. In particular, no rules concerning any implication connective are stated. The theorem above does not depend on $\rightarrow$ being defined as $\overline{A \wedge \overline{B}}$. Any formal system including the four postulates of Definition 4.1 is inconsistent.

5 The subsethood theorem

Kosko presents a number of theorems, the most important one of which is called the SUBSETHOOD THEOREM. Based on this theorem, he makes two arguments that fuzzy set theory is more fundamental than probability theory. The first argument is in favour of the claim that the notion of relative frequency can be understood without recourse to the idea of randomness. I do not understand this argument, and I will not comment on it.

Kosko's second argument is an attempt to refute recent claims of Lindley and Jaynes. After stating and proving the subsethood theorem, I shall consider the extent to which Kosko's arguments against Lindley and Jaynes are valid.

5.1 Degrees of subsethood

Given the imperatives of fuzzy set theory, it is clear that whether one fuzzy set is a subset of another should be a claim to which a continuous degree of truth can be attached. Let $S(A, B)$ stand for the degree to which A is a subset of B. Kosko provides a defining equation for $S(\cdot, \cdot)$, which he says is "the fundamental, unifying structure in fuzzy theory."

In fact Kosko provides two definitions of $S(\cdot,\cdot)$, alternatively founded on algebraic and geometric considerations. These definitions are called derivations, but the criteria used to justify them concern formal simplicity. The correctness of the derivations is therefore a matter of taste, and it makes no sense to argue with them.

Kosko prefers the definition based on geometric considerations. This definition is as follows.

Definition 5.1: Let A and B be fuzzy sets.

- The *distance* between A and B is

$$d(A,B) = \sum_{i=1}^{n} |m_A(x_i) - m_B(x_i)|$$

  where $\{x_1, \ldots, x_n\}$ is the domain of A and B.

- The fuzzy power set $F(2^B)$ of B is all fuzzy sets B' such that for all x_i, $m_{B'}(x_i) \leq m_B(x_i)$.

- The *distance* between A and $F(2^B)$ is

$$d(A, F(2^B)) = \inf\{d(A,B') : B' \in F(2^B)\}$$

- The degree to which A is a subset of B is

$$S(A,B) = 1 - \frac{d(A, F(2^B))}{M(A)}. \quad \blacksquare$$

Now the subsethood theorem can be stated and proved.

Theorem 5.1:

$$S(A,B) = \frac{M(A \cap B)}{M(A)}.$$

Proof: What must be proved is that

$$1 - \frac{d(A, F(2^B))}{M(A)} = \frac{M(A \cap B)}{M(A)}.$$

Now

$$\begin{aligned}
d(A, F(2^B)) &= \inf\{d(A,B') : \forall\ x_i\ m_{B'}(x_i) \leq m_B(x_i)\} \\
&= \inf\{\sum_{i=1}^{n} |m_A(x_i) - m_{B'}(x_i)| : \forall\ x_i\ m_{B'}(x_i) \leq m_B(x_i)\} \\
&= \sum_{i=1}^{n} \inf\{|m_A(x_i) - m_{B'}(x_i)| : m_{B'}(x_i) \leq m_B(x_i)\} \\
&= \sum_{i=1}^{n} \max\{0, m_A(x_i) - m_B(x_i)\}.
\end{aligned}$$

Therefore

$$\begin{aligned}
1 - \frac{d(A, F(2^B))}{M(A)} &= \frac{M(A) - d(A, F(2^B))}{M(A)} \\
&= \frac{\sum_{i=1}^n m_A(x_i) - \sum_{i=1}^n \max\{0, m_A(x_i) - m_B(x_i)\}}{M(A)} \\
&= \frac{\sum_{i=1}^n m_A(x_i) + \sum_{i=1}^n \min\{0, -m_A(x_i) + m_B(x_i)\}}{M(A)} \\
&= \frac{\sum_{i=1}^n \min\{m_A(x_i), m_B(x_i)\}}{M(A)} \\
&= \frac{M(A \cap B)}{M(A)}. \quad \blacksquare
\end{aligned}$$

The expression $M(A \cap B)/M(A)$ is syntactically similar to the expression $Pr(A \cap B)/Pr(A)$ that is taken as the definition of the conditional probability $Pr(B|A)$ in some presentations of probability theory. According to Kosko,

> The ratio form of the subsethood measure $S(A, B)$ is familiar. It is the same as the ratio form of the conditional probability $P(B|A)$. The fundamental difference is that the ratio form is *derived* for the subsethood measure $S(A, B)$ but *assumed* for the conditional probability $P(B|A)$. This is the difference between showing and telling. The inability to derive conditional probability further suggests that probability is not real. For every probability is a conditional probability, $P(A) = P(A|X)$.

It is difficult to disentangle the misconceptions in this paragraph. First, it is not true that the equation $Pr(B|A) = Pr(A \cap B)/Pr(A)$ is always assumed in probability theory. In some presentations this equation is taken as a definitional axiom, but in other presentations of probability theory, it is proved to follow from other axioms.

In order for the subsethood theorem to be a derivation of $Pr(B|A) = Pr(A \cap B)/Pr(A)$, we must make the notational identification $S(A, B) = P(B|A)$. In addition, we must identify fuzzy sets with events, the intersection of fuzzy sets with the conjunction of events, and the probability measure $Pr(\cdot)$ with the sigma-count measure $M(\cdot)$.

5.2 Lindley's arguments

Let $A, B, \ldots, H, \ldots$ denote events. To each ordered pair of events, we can assign a number as their conditional probability. Lindley stated three defining relationships that any assignment of conditional probabilities should satisfy. Lindley claims that any procedure for reasoning about uncertainty that does not adopt these axioms is incoherent.

Kosko shows that if $A, B, \ldots, H, \ldots$ are fuzzy sets, then defining $P(A|H)$ etc. as $S(H, A)$ etc. satisfies Lindley's postulates. He argues that since a measure that satisfies the axioms of conditional probability can be derived from the postulates of fuzzy set theory, "rather than establish the primacy of axiomatic probability, Lindley seems to argue that it is fuzziness in disguise."

Exactly the opposite is true. Kosko's calculus satisfies Lindley's postulates: therefore Kosko's calculus is a special case of Lindley's. Any conclusion that can be drawn from Lindley's axioms will be true in Kosko's calculus, but not every consequence of Kosko's theory is a consequence of Lindley's theory.

Lindley's theory is sound in the sense that if a conclusion can be drawn from it about a collection of events, then that conclusion must be true about those events. The theory is complete in the sense that if a conclusion is necessarily true given certain facts about a collection of events, then the conclusion can be proved using Lindley's axioms. Kosko's theory is not sound.

6 The logic of probability

7 Conclusion

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