vector of constants a_i, and where $\mathbf{0}$ is the zero n-vector. In writing $(\mathbf{a}, 0)$, the t, τ and α_k dependence is assumed.

The second component $\mathbf{b}$ of the general CF $(\mathbf{a};\mathbf{b})$ may be nonzero (see App. G). However, for the moment there is no need to present the full definition of the CF's encompassing nonzero $\mathbf{b}$; for now $\mathbf{b}$ should be taken simply as an algebraic object with certain useful properties (these properties are formally justified in App. G). The reader should be aware though that in general, for nonzero $\mathbf{b}$, $(\mathbf{a};\mathbf{b})$ represents a set of expressions; i.e. $(\mathbf{a};\mathbf{b})$ does not uniquely specify a single expression. (I.e., it does not uniquely specify a function of t, τ and the α_k; see App. G.)

We represent the convolution of two CF's using brackets, so that the convolution of the CF's $(\mathbf{a};\mathbf{b})$ and $(\mathbf{c};\mathbf{d})$ is given by $[\,(\mathbf{a};\mathbf{b}),\,(\mathbf{c};\mathbf{d})\,]$. The general algebraic rule for translating this convolution into a CF is

$$[\,(\mathbf{a};\mathbf{b}),\,(\mathbf{c};\mathbf{d})\,] \subset (\mathbf{c}\;;\;\mathbf{b}\vee\mathbf{d}\vee \mathrm{nz}(\mathbf{c}-\mathbf{a}))\,, \tag{13}$$

where: i) "$\vee$" is the vector-or operator with $[\mathbf{a}\vee\mathbf{b}]_i = 1$ if $a_i = 1$ or $b_i = 1$, 0 otherwise; ii) "-" is the vector-difference operator with $[\mathbf{a}-\mathbf{c}]_i = a_i - c_i$, and iii) "nz" is the vector-valued nonzero operator, with $[\mathrm{nz}(\mathbf{a})]_i = 1$ if $a_i \neq 0$, 0 otherwise. We use the symbol "$\subset$" because Eq. (13) is a relationship between two *sets* of expressions; it means that the convolution product *may be written in the form* $(\mathbf{c}\;;\;\mathbf{b}\vee\mathbf{d}\vee\mathrm{nz}(\mathbf{c}-\mathbf{a}))$ (see App. G).

Now we state the algebra of CF's under inverse T transforms. Let $(\mathbf{a}\;;\;\mathbf{b})$ be any CF with $a_i = 0 = b_i$, for some $i \in \{1, \ldots, m\}$. Further, let $c \neq 0$. Then we have

$$T_i^{-1}\,[\,(c\mathbf{e}_i + \mathbf{a}\;;\;\mathbf{b})(x)] \subset (\mathbf{a}\;;\;\mathbf{b})\,,\ \text{and}\ T_i^{-1}\,[\,(c\,\mathbf{e}_i + \mathbf{a}\;;\;\mathbf{e}_i + \mathbf{b})(x)] \subset (\mathbf{a}\;;\;\mathbf{b})\,, \tag{14}$$

where the inverse transform is with respect to x, and $[\mathbf{e}_i]_j \equiv \delta_{ij}$, the Kronecker-delta function.

In the theorems that follow, any vector occurring as an argument of a CF has all of its components equal to either 0 or 1. Such vectors will be represented by writing a shorthand list of the indices of the non-zero values only. For example, the vector $(1, 0, 1, 1)$ will be represented by the list $\{1, 3, 4\}$. Finally, the $\{\ \}$'s will be dropped from the lists since they are burdensome, and

empty lists will be represented with the dash symbol "-". The result of a successful ordering of convolutions and inversions is the CF $(-;-)$, i.e. our goal is to find an ordering of the convolution operations such that the repeated applications of Eqs. (13) and (14) result in $(-;-)$. (See App. G for the justification of this being the condition that allows for the closed form evaluation of the inverse T transforms.)

2f. MULTIPLE OVERLAP INTEGRALS.

The CF identities (Sec. 2e, Eqs. (13) and (14)) allow the efficient calculation of the needed multiple overlap integrals, as we demonstrate in Thms. 22 and 23. Theorems 24 and 25 apply Thms. 22 and 23 to evaluate the integrals appearing in the Bayes estimators for the second moments of covariance and chi-squared. In section 2g we use these results to write closed form expressions for these moments.

Theorem 22 concerns convolutions in which there are three subsets of indices $\sigma_1, \sigma_2, \sigma_3$ with i) $\sigma_{12} \neq \varnothing$ and $\sigma_{23} \neq \varnothing$, ii) $\sigma_{13} = \varnothing$, and iii) none of the intersections are equal to any of the subsets $\sigma_1, \sigma_2, \sigma_3$. Theorem 23 concerns convolutions in which there are four subsets of indices, where i) the only non-empty σ_{ij} with $i \neq j$ are $\sigma_{12}, \sigma_{23}, \sigma_{34}, \sigma_{41}$, and where ii) none of the intersections are equal to any of the subsets $\sigma_1, \sigma_2, \sigma_3, \sigma_4$.

Overlap structures of these types occur in the integrals for the second moments of covariance and chi-squared. Many mathematically equivalent forms of the results of Thms. 22 and 23 are possible; we present the result of one particular calculation in each case and note that equating different forms leads to hypergeometric identities along the lines of those discussed in footnotes 1 and 2.

In what follows "Terms 1 and 12", for example, indicates that the terms $e^{-t_1 p} p^{\alpha_1 - 1}$ and $e^{-(t_1 + t_2) p} p^{\alpha_{12} - 1}$ are being convolved. Define the shorthand notations $(\alpha, \beta, \gamma) \equiv \dfrac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha + \beta + \gamma)}$, $(\alpha, \beta) \equiv \dfrac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}$. As in App. A, the symbol $(\alpha)_i \equiv \dfrac{\Gamma(\alpha + i)}{\Gamma(\alpha)}$.

Theorem 22 states the result of a derivation that treats subsets 1 and 3 symmetrically from the outset. For future use, in Thm. 22 the expression being evaluated is defined simply as I_3, suppressing the various α and η arguments. (The inverse T transforms turn t's into η's.)

Theorem 22. If $\mathrm{Re}(\alpha_1) > 0$, $\mathrm{Re}(\alpha_{12}) > 0$, $\mathrm{Re}(\alpha_2) > 0$, $\mathrm{Re}(\alpha_{23}) > 0$, $\mathrm{Re}(\alpha_3) > 0$, and $\alpha \equiv \alpha_1 + \alpha_{12} + \alpha_2 + \alpha_{23} + \alpha_3$, then

$$I_3 \equiv T_1^{-1} T_2^{-1} T_3^{-1} [\, e^{-t_1 p} p^{\alpha_1 - 1} \otimes e^{-(t_1 + t_2) p} p^{\alpha_{12} - 1} \otimes e^{-t_2 p} p^{\alpha_2 - 1} \otimes e^{-(t_2 + t_3) p} p^{(\alpha_{23} - 1)} \otimes e^{-t_3 p} p^{\alpha_3 - 1} \,]$$

$$= \tau^{\alpha + \eta - 1} (\alpha_1, \alpha_{12})\ (\alpha_3, \alpha_{23})\ (\alpha_1 + \alpha_{12} + \eta_1, \alpha_3 + \alpha_{23} + \eta_3, \alpha_2) \times {}_{2,2,1}F_{1,1,1} [\, (\alpha_1, \alpha_1 + \alpha_{12} + \eta_1)\ ,\ (\alpha_3, \alpha_3 + \alpha_{23} + \eta_3), -\eta_2; \alpha_1 + \alpha_{12}, \alpha_3 + \alpha_{23}, \alpha + \eta_1 + \eta_3; 1, 1 \,]$$

Proof: The following derivation applies Eqs. (13) and (14) repetitively to demonstrate that a particular ordering of the convolutions results in the sucessfully inverted CF $(-;-)$.

Terms 1 and 12:	$[\,(1;-),\ (1, 2;-)\,]$	$\subset (1, 2;2)$	
	$T_1^{-1}\,[\,(1, 2;2)\,]$	$\subset (2;2)$	(a)
Terms 3 and 23:	$[\,(3;-),\ (2, 3;-)\,]$	$\subset (2, 3;2)$	
	$T_3^{-1}\,[\,(2, 3;2)\,]$	$\subset (2;2)$	(b)
Results (a) and (b):	$[\,(2;2),\ (2;2)\,]$	$\subset (2;2)$	(c)
Res. (c) and term 2:	$[\,(2;2),\ (2;-)\,]$	$\subset (2;2)$	(d)
Invert (d):	$T_2^{-1}\,[\,(2;2)\,]$	$\subset (-;-)$.	

Now, carry out the convolutions and inverses in the above order with the appropriate terms substituted as indicated in the left column to find

$$I_3 = \tau^{\alpha+\eta-1}\Sigma_{i=0}^{\infty}\Sigma_{j=0}^{\infty}\frac{1}{i!j!}(-\eta_2)_{i+j}(\alpha_1+i,\alpha_{12})\,(\alpha_3+j,\alpha_{23})$$

$$\times(\alpha_1+\alpha_{12}+i+\eta_1,\alpha_3+\alpha_{23}+j+\eta_3,\alpha_2)\,.$$

Rewriting this using hypergeometric notation (see App. A) gives the desired result. QED.

Theorem 23 presents the results for the relevant quadruple overlap sum convolution. As in Thm. 22, here we use the notation $(\alpha,\beta,\gamma)\equiv\frac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha+\beta+\gamma)}$ and $(\alpha,\beta)\equiv\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$. In order to simplify the resulting expressions, we do not express the result in hypergeometric notation. For future use, in Thm. 23 the expression being evaluated is defined simply as I_4, suppressing the various α and η arguments.

Theorem 23. If $\mathrm{Re}(\alpha_1)>0$, $\mathrm{Re}(\alpha_{12})>0$, $\mathrm{Re}(\alpha_2)>0$, $\mathrm{Re}(\alpha_{23})>0$, $\mathrm{Re}(\alpha_3)>0$, $\mathrm{Re}(\alpha_{34})>0$, $\mathrm{Re}(\alpha_4)>0$, and $\alpha\equiv\alpha_1+\alpha_{12}+\alpha_2+\alpha_{23}+\alpha_3+\alpha_{34}+\alpha_4+\alpha_{41}$ then

$$I_4\equiv T_1^{-1}T_2^{-1}T_3^{-1}T_4^{-1}\,[e^{-t_1p}p^{\alpha_1-1}\otimes e^{-(t_1+t_2)p}p^{\alpha_{12}-1}\otimes e^{-t_2p}p^{\alpha_2-1}$$

$$\otimes e^{-(t_2+t_3)p}p^{\alpha_{23}-1}\otimes e^{-t_3p}p^{\alpha_3-1}\otimes e^{-(t_3+t_4)p}p^{\alpha_{34}-1}\otimes e^{-t_4p}p^{\alpha_4-1}\,]$$

$$=\tau^{\alpha+\eta-1}\Sigma_{i,j,m,n,p,q=0}^{\infty}(-1)^{i+j+m+n}\frac{1}{i!j!m!n!p!q!}(-\eta_2)_{j+m+p}(-\eta_4)_{i+n+q}$$

$$\times(\alpha_{41}+i,\alpha_1,\alpha_{12}+j)\,(\alpha_{23}+m,\alpha_3,\alpha_{34}+n)$$

$$\times(\alpha_{41}+\alpha_1+\alpha_{12}+\eta_1+i+j,\alpha_{23}+\alpha_3+\alpha_{34}+\eta_3+m+n)$$

$$\times(\alpha-\alpha_2-\alpha_4+\eta_1+\eta_3+i+j+m+n+p,\alpha_2)$$

$$\times(\alpha-\alpha_4+\eta_1+\eta_2+\eta_3+i+n+q,\alpha_4)\,.$$

Proof: The following derivation applies Eqs. (13) and (14) repetitively to demonstrate that a particular ordering of the convolutions results in the successfully inverted CF $(-;-)$. Initially, con-

volutions are done on two non-overlapping groups - the group depending on t_1 and the group depending on t_3. Then the inverses T_1^{-1} and T_3^{-1} are taken and the remaining convolutions are done with care taken to create expressions that have simple T transform inverses.

For the t_1 group (terms 41, 1, 12),

Terms 41 and 1:	$[\,(4, 1;-)\,,\,(1;-)\,]$	$\subset (1;4)$	(a)
Term 12 and Res. (a):	$[\,(1, 2;-)\,,\,(1;4)\,]$	$\subset (1;2, 4)$	
	$T_1^{-1}\,[\,(1;2, 4)\,]$	$\subset (-;2, 4)$	(b)

Similarly for the t_3 group (terms 23, 3, 34): $\subset (-;2, 4)$ (c)

Res. (b) and (c):	$[\,(-;2, 4)\,,\,(-;2, 4)\,]$	$\subset (-;2, 4)$	(d)
Res. (d) and term 2:	$[\,(-;2, 4)\,,\,(2;-)\,]$	$\subset (2;2, 4)$	
	$T_2^{-1}\,[\,(2;2, 4)\,]$	$\subset (-;4)$	(e)
Res. (e) and term 4:	$[\,(-;4)\,,\,(4;-)\,]$	$\subset (4;4)$	
	$T_4^{-1}\,[\,(4;4)\,]$	$\subset (-;-)\,.$	

Now, carry out the convolutions and inverses in the above order with the appropriate terms substituted as indicated in the left column to find the desired result. QED.

Theorems 24 and 25 utilize Thms. 22 and 23 respectively to find the needed multiple overlap integrals. They are given without proof, as they follow immediately from these theorems, Thm. 9.1, and induction.

Theorem 24. Assume the overlap structure relevant to Thm. 22, with C_3 defined as the τ-independent factor of the I_3 defined in Thm. 22. (I.e., $I_3 = \tau^{\alpha+\eta-1}C_3$.) Define $\bar{\beta} \equiv \beta - \beta_{1+2+3}$. If $\mathrm{Re}\,(\beta_u + \eta_u) > 0$ for $u = 1, 2, 3$, then

$$I[\rho_1^{\eta_1}\rho_2^{\eta_2}\rho_3^{\eta_3}, \mathbf{n}] = \frac{\Gamma(\beta_{1+2+3}+\eta)\Gamma(\bar{\beta})}{\Gamma(\beta+\eta)} \times C_3[\beta_{1-12}, \beta_{12}, \beta_{2-12-23}, \beta_{23}, \beta_{3-23}; \eta_1, \eta_2, \eta_3].$$

Theorem 25. Assume the overlap structure relevant to Thm. 23, with C_4 defined as the τ-independent factor of the I_4 defined in Thm. 23. (I.e., $I_4 = \tau^{\alpha+\eta-1}C_4$.) Define $\bar{\beta} \equiv \beta - \beta_{1+2+3+4}$. If $\mathrm{Re}(\beta_u + \eta_u) > 0$ for $u = 1, 2, 3, 4$, then

$$I[\rho_1^{\eta_1}\rho_2^{\eta_2}\rho_3^{\eta_3}\rho_4^{\eta_4}, \mathbf{n}] = \frac{\Gamma(\beta_{1+2+3+4}+\eta)\Gamma(\bar{\beta})}{\Gamma(\beta+\eta)}$$

$$\times C_4[\beta_{1-12}, \beta_{12}, \beta_{2-12-23}, \beta_{23}, \beta_{3-23-34}, \beta_{34}, \beta_{4-34-41}, \beta_{41}; \eta_1, \eta_2, \eta_3, \eta_4].$$

2g. BAYES ESTIMATORS - MULTIPLE OVERLAP TERMS.

In this subsection we complete the presentation of the results for the Bayes estimators (see Sec. 2d) by giving the Bayes estimators with uniform prior for the second powers of covariance and chi-squared. Since the complete description of these results is *quite* lengthy, we present them in recipe form; at this point the reader should be able to make the needed substitutions.

5) Covariance $C(\mathbf{p}) = \Sigma_{ij} p_{ij} (X_i - \mu_x)(Y_i - \mu_y)$.

Theorem 26. If $\mathrm{Re}(\nu_{ij}) > 0 \ \forall ij$ then

$$E[C^2(\mathbf{p}) \mid \mathbf{n}] = \Sigma_{ijkl} E[p_{ij}p_{kl} \mid \mathbf{n}] X_i Y_j X_k Y_l - 2\Sigma_{ijkl} E[p_{ij}p_{k\cdot}\, p_{\cdot l} \mid \mathbf{n}] X_i Y_j X_k Y_l$$
$$+ \Sigma_{ijkl} E[p_{i\cdot}\, p_{\cdot j} p_{k\cdot}\, p_{\cdot l} \mid \mathbf{n}] X_i Y_j X_k Y_l,$$

where $E[p_{ij}p_{kl} \mid \mathbf{n}]$ is found by applying Thm. 12, $E[p_{ij}p_{k\cdot}\, p_{\cdot l} \mid \mathbf{n}]$ is found by applying Thm. 14a, and $E[p_{i\cdot}\, p_{\cdot j} p_{k\cdot}\, p_{\cdot l} \mid \mathbf{n}]$ is either a single, double, or quadruple overlap term found by applying Thms. 14a, 24, and 25: if $i = k, j = l$, then apply Thm. 14a; if $i = k, j \neq l$ or $i \neq k, j = l$, then apply Thm. 24; if $i \neq k, j \neq l$ then apply Thm. 25.

6) Chi-Squared $\chi^2(\mathbf{p}) = \Sigma_{ij} \frac{(p_{ij} - p_{i\cdot} p_{\cdot j})^2}{p_{i\cdot} p_{\cdot j}}$.

Theorem 27. If $\mathrm{Re}(\nu_{ij}) > 0 \; \forall ij$, $\mathrm{Re}(\nu_{i\cdot}) > -1$ and $\mathrm{Re}(\nu_{\cdot j}) > -1$ then

$$E[(\chi^2(\mathbf{p}))^2 \mid \mathbf{n}] = \Sigma_{ijkl} E\left[\frac{p_{ij}^2 p_{kl}^2}{p_{i\cdot} p_{\cdot j} p_{k\cdot} p_{\cdot l}} \mid \mathbf{n}\right] - 2E[\chi^2(\mathbf{p}) \mid \mathbf{n}] - 1,$$

where $E[\chi^2(\mathbf{p}) \mid \mathbf{n}]$ is given in Thm. 21 and where $E\left[\frac{p_{ij}^2 p_{kl}^2}{p_{i\cdot} p_{\cdot j} p_{k\cdot} p_{\cdot l}} \mid \mathbf{n}\right]$ is either a single, double, or quadruple overlap term found by applying Thms. 14a, 24, and 25: if $i = k, j = l$, then apply Thm. 14a; if $i = k, j \neq l$ or $i \neq k, j = l$, then apply Thm. 24; if $i \neq k, j \neq l$ then apply Thm. 25.

3. EXTENSION OF THE CLASS OF CALCULABLE PRIORS

The calculations of this paper and [1] are done under the assumption of a uniform prior $P(\mathbf{p})$. However they also apply essentially unchanged when certain other priors are used. Here we briefly discuss applying the calculations of these papers to cases where the prior is not uniform. As specific examples of how the calculations of these papers are modified when the prior is not uniform, we consider priors of the form $P(\mathbf{p}) \propto \Delta(\mathbf{p}) \Pi_{i=1}^m p_i^{r_i}$ (the Dirichlet priors are the subset of these with all r_i equal) and the entropic prior $P(\mathbf{p}) \propto \Delta(\mathbf{p}) \; e^{\alpha S(\mathbf{p})}$.

Define $\bar{P}(\mathbf{p})$ implicitly through $P(\mathbf{p}) = \Delta(p)\bar{P}(\mathbf{p})$. ($\bar{P}(\mathbf{p})$ is the non-delta function part of $P(\mathbf{p})$). The uniform prior has a constant $\bar{P}(\mathbf{p})$. Even if $P(\mathbf{p})$ is not uniform, it is often the case that $I[F(\mathbf{p})\bar{P}(\mathbf{p}), \mathbf{n}] = \int d\mathbf{p} F(\mathbf{p}) P(\mathbf{p}) \Pi_{i=1}^m p_i^{n_i}$ and $I[\bar{P}(\mathbf{p}), \mathbf{n}] = \int d\mathbf{p} P(\mathbf{p}) \Pi_{i=1}^m p_i^{n_i}$ are of the form of integrals evaluated in these papers. (In this section all integrals are over $\mathbf{p}$'s with nonnegative com-

ponents). In these cases we can evaluate the Bayes estimator for $F(\mathbf{p})$ with prior $P(\mathbf{p})$ since $E[F(\mathbf{p}) \mid \mathbf{n}] = I[F(\mathbf{p})\bar{P}(\mathbf{p}), \mathbf{n}] / I[\bar{P}(\mathbf{p}), \mathbf{n}]$.

For example, the Bayes estimator for $F(\mathbf{p})$ with prior $P(\mathbf{p})$ can often be evaluated using the calculations of these papers when $P(\mathbf{p}) \propto \Delta(\mathbf{p}) \Pi_{i=1}^{m} f_i(p_i)$. In particular consider priors of the form $P(\mathbf{p}) \propto \Delta(\mathbf{p}) \Pi_{i=1}^{m} p_i^{r_i}$ with $Re(r_i) > -1$, $i = 1, \ldots, m$. (Dirichlet priors have $r_i = r$ for all i.) When $P(\mathbf{p})$ is of this form the Bayes estimator for $F(\mathbf{p})$ with prior $P(\mathbf{p})$ is given by

$$E[F(\mathbf{p}) \mid \mathbf{n}] = \frac{I[F(\mathbf{p}), \mathbf{n} + \mathbf{r}]}{I[1, \mathbf{n} + \mathbf{r}]}. \tag{15}$$

As another example, consider the class of priors with $\bar{P}(\mathbf{p})$ represented by a taylor series converging everywhere in the domain of $\mathbf{p}$. Using this taylor series representation expand both $I[F(\mathbf{p})\bar{P}(\mathbf{p}), \mathbf{n}]$ and $I[\bar{P}(\mathbf{p}), \mathbf{n}]$ into infinite sums of integrals. If all of the integrals are of a form evaluated in this paper the we can find $E[F(\mathbf{p}) \mid \mathbf{n}]$ for the prior $P(\mathbf{p})$.

For example, consider the entropic prior $P(\mathbf{p}) \propto \Delta(\mathbf{p})\ e^{\alpha S(\mathbf{p})}$. In some applications the entropy $S(\mathbf{p})$ is taken to be $S(\mathbf{p}) \equiv -\Sigma_{i=1}^{m} p_i \log(p_i)$, where α is some constant, while in image processing applications $S(\mathbf{p})$ is often defined as $S(\mathbf{p}) \equiv \Sigma_{i=1}^{m} \left((p_i - m_i) - p_i \log\left(\frac{p_i}{m_i}\right)\right)$, where $\mathbf{m}$ is known as the "model" [11]. In either case, $\bar{P}(\mathbf{p})$ may be expanded in the series

$$\bar{P}(\mathbf{p}) = \Sigma_{i=0}^{\infty} \frac{\alpha^i S^i(\mathbf{p})}{i!}, \tag{16}$$

Whenever the products $S^i(\mathbf{p})F(\mathbf{p})$, $i = 0, 1, \ldots$, are of the form of some function integrated in these papers then closed form results (up to series truncations) for the Bayes estimator for $F(\mathbf{p})$ with an entropic prior are available. An application of these ideas appears in [12], where they are used to calculate the normalization constant of the entropic prior

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APPENDICES

A. HYPERGEOMETRIC FUNCTION NOTATION

Here we present the notation for the hypergeometric functions used in this paper. Paralleling Lebedev [13], let $\mathbf{a}$ and $\mathbf{b}$ be vectors of dimensions p and q respectively. Define $(\lambda)_k \equiv \Gamma(\lambda + k)/\Gamma(\lambda)$. Define the single summation hypergeometrics ${}_pF_q$ by

$$ {}_pF_q[\mathbf{a};\mathbf{b};\tau] \equiv \Sigma_{i=0}^{\infty} \left(\frac{\Pi_{\alpha=1}^{p} (a_\alpha)_i}{\Pi_{\beta=1}^{q} (b_\beta)_i} \right) \frac{\tau^i}{i!}. \tag{A.1} $$

An example of a single summation hypergeometric is ${}_1F_1[\alpha;\beta;\tau]$, which has the integral representation for $\beta > \alpha > 0$ [13]

$$ {}_1F_1[\alpha;\beta;\tau] = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} \int_0^1 dx\ e^{\tau x}\ x^{\alpha-1} (1-x)^{\beta-\alpha-1}. \tag{A.2} $$

Now, given vectors $\mathbf{a}^1$, $\mathbf{a}^2$, $\mathbf{a}^{12}$, $\mathbf{b}^1$, $\mathbf{b}^2$, $\mathbf{b}^{12}$ of dimensions p_1, p_2, p_{12}, q_1, q_2, q_{12} respectively, define the double summation hypergeometrics

$$ {}_{p_1, p_2, p_{12}}F_{q_1, q_2, q_{12}}[\mathbf{a}^1, \mathbf{a}^2, \mathbf{a}^{12}; \mathbf{b}^1, \mathbf{b}^2, \mathbf{b}^{12}; \tau_1, \tau_2] \equiv $$

$$ \Sigma_{i=0}^{\infty} \Sigma_{j=0}^{\infty} \frac{(\Pi_{\alpha_1=1}^{p_1} (a_{\alpha_1}^1)_i)\ (\Pi_{\alpha_2=1}^{p_2} (a_{\alpha_2}^2)_j)\ (\Pi_{\alpha_{12}=1}^{p_{12}} (a_{\alpha_{12}}^{12})_{i+j})}{(\Pi_{\beta_1=1}^{q_1} (b_{\beta_1}^1)_i)\ (\Pi_{\beta_2=1}^{q_2} (b_{\beta_2}^2)_j)\ (\Pi_{\beta_{12}=1}^{q_{12}} (b_{\beta_{12}}^{12})_{i+j})} \frac{\tau_1^i}{i!} \frac{\tau_2^j}{j!} \tag{A.3} $$

In writing arguments of F's, vectors will be denoted by a list of the elements, e.g., $\mathbf{c} = (c_1, \ldots, c_k)$. However, when listing the components of a 1-dimensional vector the parentheses will be dropped. Further, when any of the p or q subscripts are zero (which corresponds to an empty argument for that position), the empty vector argument of the hypergeometric will simply be omitted from the list of arguments.

B. TRANSFORMS

In App. B.1. we discuss the T transform. In App. B.2. we discuss the Z transform.

B.1. THE T TRANSFORM

In order to calculate averages of the form $E[(\rho^{\eta})(\log(\rho^{\eta}))\mid \mathbf{n}]$ (with $\rho \equiv \Sigma_i p_{ij}$) and similar averages, the identity

$$\Gamma(-\eta) = \int_0^{\infty} u^{-\eta-1} e^{-u}\, du,\ \mathrm{Re}(-\eta) > 0 \tag{B.1}$$

for the gamma function is needed. (Here $-\eta$ has been used instead of η in order to simplify the following.) Make the change of variables $u = \rho t$. With $\rho > 0$, independent of t, we find

$$\rho^{\eta} = \frac{1}{\Gamma(-\eta)} \int_0^{\infty} t^{-\eta-1} e^{-\rho t}\, dt,\ \mathrm{Re}(-\eta) > 0,\ \rho > 0. \tag{B.2}$$

Define the operator T^{-1} by $T^{-1}[F(\cdot)](\eta) \equiv \frac{1}{\Gamma(-\eta)} \int_0^{\infty} t^{-\eta-1} F(t)\, dt,\ \mathrm{Re}(-\eta) > 0$, and define the transform T by $T[T^{-1}[F(\cdot)]] = F(\cdot)$. As defined, the transform T is closely related to the Mellin transform [14] (it is an inverse-Mellin transform) and we rely on this similarity to establish the conditions for the existence of the transform and its inverse. Of interest in this work is the following: For ρ independent of t, the functions ρ^{η} and $e^{-\rho t}$ form a transform pair. That is,

$$T[\rho^{\eta}](t) = e^{-\rho t} \text{ and } T^{-1}[e^{-\rho t}](\eta) = \rho^{\eta}. \tag{B.3}$$

As an example of the use of the transform T, when finding $E[\rho^{\eta} \mid \mathbf{n}]$, where ρ is a sum of a subset of the p_i's and $\mathrm{Re}(\eta) < 0$, the T transform will be taken of $E[\rho^{\eta} \mid \mathbf{n}]$ with respect to η, which substitutes $e^{-\rho t}$ for ρ^{η}:

$$T[E[\rho^{\eta} \mid \mathbf{n}]](t) = T[\int d\mathbf{p}\ P(\mathbf{p} \mid \mathbf{n})\ \rho^{\eta}] = \int d\mathbf{p}\ P(\mathbf{p} \mid \mathbf{n})\ T[\rho^{\eta}] = \int d\mathbf{p}\ P(\mathbf{p} \mid \mathbf{n})\ e^{-\rho t}. \tag{B.4}$$

See App. C.1 for the justification of the commutation of the integrals. After the transform has been done, often the last integral may be computed in closed form (see Sec. 2) and then the T^{-1} transform may be applied.

B.2. THE Z TRANSFORM

Let $f(\mathbf{n})$ be any function that factors as $f(\mathbf{n}) = \Pi_{i=1}^{m} f_i(n_i)$. For such functions, the Z transform $Z[f](z) \equiv \Sigma_{n=0}^{\infty} f(n) z^n$ is useful in simplifying calculations involving sums $\Sigma_{\mathbf{n}} f(\mathbf{n})$, where the summation extends over all $\mathbf{n}$ having non-negative integer components and $\Sigma_i n_i = N$. Define the discrete convolution product of two functions g and h by $(g \otimes h)(n) \equiv \Sigma_{i=0}^{n} g(i) h(n-i)$. (Note that $\otimes$ is both commutative and associative, so that the order that the convolutions are taken in is irrelevant, justifying the use of the above notation when several functions are involved.)

The Z transform convolution theorem may be thought of as a discretized form of the Laplace convolution theorem (see Thm. 2).

Theorem B.1: If $F(N) \equiv \Sigma_{\mathbf{n}} f(\mathbf{n})$ and $f(\mathbf{n}) = \Pi_{i=1}^{m} f_i(n_i)$ then $F(N) = (\otimes_{i=1}^{m} f_i)(N)$ and $Z[F](z) = \Pi_{i=1}^{m} Z[f_i](z)$, for all z such that $Z[f_i](z)$, $i = 1, \ldots, m$, converges.

Proof: For $m = 2$ we have $F(N) = (f_1 \otimes f_2)(N)$ and the Z transforms of f_1 and f_2 are given by $Z[f_i](z) = \Sigma_{n=0}^{\infty} f_i(n) z^n$, $i = 1, 2$, respectively. For z within the radii of convergence of both of these power series, we have (after collecting terms having the same power of z)

$$Z[f_1](z) \times Z[f_2](z) = \Sigma_{n=0}^{\infty} z^n \Sigma_{i=0}^{n} f_1(i) f_2(n-i),$$

The right-hand side is immediately seen to be $Z[F](z)$. The result for arbitrary m follows by induction. QED.

Note that due to the uniqueness of power series representations, inverses of Z transforms exist on the nonnegative integers.

C. COMMUTING LINEAR OPERATORS

In App. C.1. we discuss the interchange of integrals. In App. C.2. we discuss the interchange of derivatives and integrals.

C.1. COMMUTING TWO INTEGRALS

Interchanging the integrals appearing in these papers as the **p** integral and the T transform integral is possible due to Fubini's theorem [15], which justifies the interchange of uncoupled integrations (region of integration of either integral does not depend on the other integral's parameters) when the double integral exists.

C.2. COMMUTING INTEGRALS AND DERIVATIVES

Consider differentiating the integral $\int F(x, t)\, dx$ with respect to t. Theorem C.1 generalizes Thm. 9.42 of [10] and establishes conditions general enough to allow the commutation of the derivative and integral for the functions $F(x, t)$ appearing in this paper. Define $D_2F(x, t)$ to be the partial derivative of F with respect to its second argument, evaluated at (x, t).

Theorem C.1: If (1) $F(x, t)$ and $D_2F(x, t)$ are defined for $(x, t) \in \Delta_x \times \Delta_t$,

where $\Delta_x = (0, \infty)$, and where Δ_t is convex,

(2) $\int_0^\infty F(x, t)dx$ exists $\forall t \in \Delta_t$,

(3) $\forall \varepsilon > 0$ and $b > 0$, $\exists$ f (x) with $f(x) > 0$ for $x \in \Delta_x$, and $\delta > 0$, ∋

$\int_b^\infty f(x)dx < \varepsilon$ and

$\forall x \geq b, \forall s, t \in \Delta_t, |t - s| < \delta \Rightarrow |D_2F(x, t) - D_2F(x, s)| < f(x)$

then $D_2\int_0^\infty F(x, t)dx = \int_0^\infty D_2F(x, t)dx$ on $\Delta_x \times \Delta_t$.

Proof: Let $\phi(s, t) \equiv \frac{F(x, t) - F(x, s)}{t - s}$ for $s \neq t$. By (1) and the mean value theorem, $\forall t > s$ with $t, s \in \Delta_t$, $\exists u(s, t) \in [s, t]$ ∋ $\phi(s, t) = D_2F(x, u(s, t))$. Using this and (3) we have that for any $\varepsilon > 0$, $\forall b > 0$, $\exists \delta > 0$ and a nowhere-negative (in Δ_x) f (x) obeying $\int_b^\infty f(x)dx < \varepsilon$ such that if $|t - s| < \delta$ and $x \geq b$, then $|\phi(s, t) - D_2F(x, t)| = |D_2F(x, u(s, t)) - D_2F(x, t)| < f(x)$. From this and (2) it follows that for all $b > 0$, $\exists \delta > 0$, and a nowhere-negative (in Δ_x) f (x) obeying $\int_b^\infty f(x)\,dx < \varepsilon$ such that if $|t - s| < \delta$, then $\left|\int_b^\infty \phi(s, t)\,dx - \int_b^\infty D_2F(x, t)\,dx\right| \leq \int_b^\infty f(x)\,dx < \varepsilon$.

Taking the limit $s \to t$, noting that $\lim_{s \to t}\int_b^\infty \phi(s, t)\,dx = D_2\int_b^\infty F(x, t)dx$, and finally taking $\varepsilon \to 0$ with $b = \varepsilon$, we arrive at the desired result. QED.

The functions F(x, t) of interest in this paper have the form $F(x, t) = x^t \log(x)^m e^{-cx}$ with

$\mathrm{Re}(t) > -1$ and $c > 0$. For these functions it may be shown that the conditions of Thm. C.1 hold.

D. EXPANDING ETA'S DOMAIN

To apply the T transform, the assumption that all $\eta_i < 0$ had to be made. Here we present a simple theorem that expands the region of validity of the various expressions derived in this paper to the region where any of the η_i may be non-negative. We present the theorem for the single subset sum case only, although the multiple non-overlapping subset case and the contained overlap case may be handled in an almost identical manner.

Theorem D.1: If $\mathrm{Re}(\eta + \beta_1) > 0$, $\mathrm{Re}(\eta) \geq 0$ and $\mathrm{Re}(n_i) > -1$, $i = 1, \ldots, m$ then

$$I[\rho^{\eta}, \mathbf{n}] = \frac{\gamma_{\mathbf{n}}}{\Gamma(\beta_1)} \frac{\Gamma(\eta + \beta_1)}{\Gamma(\eta + \beta)}.$$

Proof: Note that $\eta > 0$ implies that there is an integer $q > 0$ and an $\bar{\eta} < 0$ such that $\eta = \bar{\eta} + q$. Thus $I[\rho^{\eta}, \mathbf{n}]$ may be rewritten as

$$I[\rho^{\eta}, \mathbf{n}] = \Sigma_{i \in \sigma} I[p_i \rho^{\bar{\eta} + q - 1}, \mathbf{n}] = \Sigma_{i \in \sigma} I[\rho^{\bar{\eta} + q - 1}, \mathbf{n} + \mathbf{e}_i],$$

where $[\mathbf{e}_i]_j = \delta_{ij}$ and δ is the Kronecker delta function. Iterate this operation q times (removing one power from ρ and summing with an increased count vector each time) to find

$$I[\rho^{\eta}, \mathbf{n}] = \Sigma_{i_1 \in \sigma} \ldots \Sigma_{i_q \in \sigma} I[\rho^{\bar{\eta}}, \mathbf{n} + \mathbf{e}_{i_1} + \ldots + \mathbf{e}_{i_q}].$$

Simplify this to yield

$$I[\rho^{\eta}, \mathbf{n}] = \Sigma_{\mathbf{q}} \binom{q}{\mathbf{q}} I[\rho^{\bar{\eta}}, \mathbf{n} + \mathbf{q}],$$

where the vector $\mathbf{q}$ has nonnegative integer components summing to q with $q_i = 0$ for $i \notin \sigma$. The

symbol $\binom{q}{\mathbf{q}} \equiv \frac{\Gamma(q+1)}{\Pi_{i=1}^{m}\Gamma(q_i+1)}$ is the usual multinomial coefficient. Since $\bar{\eta} < 0$, evaluate the integral $I[\rho^{\bar{\eta}}, \mathbf{n}+\mathbf{q}]$ using Thm. 12 with $k = 1$ (noting that β_1 and β increase by q due to $\mathbf{q}$ being added to $\mathbf{n}$) to find $I[\rho^{\eta}, \mathbf{n}] = \frac{\gamma_{\mathbf{n}}\Gamma(\nu_1+\bar{\eta}+q)}{\Gamma(\nu_1)\Gamma(\nu+\bar{\eta}+q)}\Sigma_{\mathbf{q}}\binom{q}{\mathbf{q}}\gamma_{\mathbf{n}+\mathbf{q}}$(*). Now, we put $\Sigma_{\mathbf{q}}\binom{q}{\mathbf{q}}\gamma_{\mathbf{n}+\mathbf{q}}$ into closed form by noting that it is the discrete convolution product of the functions of $\Gamma(n_i+q_i+1)/\Gamma(q_i+1)$ of q_i given by

$$\Sigma_{\mathbf{q}}\binom{q}{\mathbf{q}}\gamma_{\mathbf{n}+\mathbf{q}} = \Gamma(q+1)\,[\otimes_{i=1}^{m}\Gamma(n_i+q_i+1)/\Gamma(q_i+1)]\,(q).$$

Apply the Z transform convolution theorem (see App. B.2.) to find

$$\Sigma_{\mathbf{q}}\binom{q}{\mathbf{q}}\gamma_{\mathbf{n}} = \Gamma(q+1)Z^{-1}\,[\Pi_{i=1}^{m}Z\,[\Gamma(n_i+q_i+1)/\Gamma(q_i+1)]\,]\,.$$

Note that $Z\left[\frac{\Gamma(n_i+q_i+1)}{\Gamma(q_i+1)}\right](z) = \Gamma(n_i+1)\,(1-z)^{-(n_i+1)}$ for $|z| < 1$ and substitute for the Z transforms to find $\Sigma_{\mathbf{q}}\binom{q}{\mathbf{q}}\gamma_{\mathbf{n}+\mathbf{q}} = \gamma_{\mathbf{n}}\frac{\Gamma(\beta_1+q)}{\Gamma(\beta_1)}$. Substituting this result in (*) and simplifying leads to the desired result. QED.

We resort to analytic continuation in the non-contained overlap case.

E. EXISTENCE CONDITIONS

Here we present an example of a calculation for determining the conditions of existence of the various integrals $I[\cdot|\cdot]$ appearing in these papers. For the integrands of these papers, existence of these integrals depends upon the behavior of the singularities appearing at the edges of the region of integration.

Consider the single pair overlap intergral $I[\rho_1^{\eta_1}\rho_2^{\eta_2}, \mathbf{n}]$, where ρ_1, ρ_2, σ_1, and σ_2 are as in the

definitions for Thm. 14a, with the minor change that σ_{1+2} contains all m indices, which may be made without loss of generality. We show that the conditions for existence of this integral are $\mathrm{Re}(\nu_i) > 0$, $i = 1, \ldots, m$, $\mathrm{Re}(\beta_1 + \eta_1) > 0$ and $\mathrm{Re}(\beta_2 + \eta_2) > 0$. Write the integral as

$$I[\rho_1^{\eta_1}\rho_2^{\eta_2}, \mathbf{n}] = \int d\mathbf{p}\ \Delta(\mathbf{p})\ \Theta(\mathbf{p})\ \rho_1^{\eta_1}\rho_2^{\eta_2}\ \Pi_{i=1}^{m} p_i^{n_i}. \tag{E.1}$$

The first condition, $\mathrm{Re}(\nu_i) > 0$, $i = 1, \ldots, m$, follows immediately from the fact that $\int dp_i\ p_i^{n_i}$ exists iff $\mathrm{Re}(\nu_i) > 0$ and the fact that any p_i may independently be near zero for this particular overlap case. Now, either ρ_1 or ρ_2 may also be near zero. We consider the first case, in which ρ_1 is near zero, and use symmetry to supply the result for the second case. Letting $x \equiv \Sigma_{i \in \sigma_1} p_i$ and $y \equiv \Sigma_{i \in \sigma_{12}} p_i$, rewite Eq. (E.1) in a form that isolates ρ_1 as

$$\begin{aligned} I[\rho_1^{\eta_1}\rho_2^{\eta_2}, \mathbf{n}] = \int_0^1 dx\, x^{\eta_1} \int_0^x dy\, (1-x+y)^{\eta_2} \Big(\int d\mathbf{p}_{1-12}\, \delta(\Sigma_{1-12} p_i - (x-y))\, \Pi_{1-12} p_i^{n_i} \\ \times \int d\mathbf{p}_{12}\, \delta(\Sigma_{12} p_i - y)\, \Pi_{12} p_i^{n_i} \times \int d\mathbf{p}_{2-12}\, \delta(\Sigma_{2-12} p_i - (1-x+y))\, \Pi_{2-12} p_i^{n_i}\Big), \end{aligned} \tag{E.2}$$

where here the subscript notation indicates the sets of indices involved, e.g. $1-12$ indicates $i \in \sigma_{1-12}$. Each of the three integrals over $\mathbf{p}$ in Eq. (E.2) may be done in closed form. Do these integrals using Thm. 9a and induction to find

$$\begin{aligned} I[\rho_1^{\eta_1}\rho_2^{\eta_2}, \mathbf{n}] = \frac{\gamma_{\mathbf{n}}}{\Gamma(\beta_{1-12})\Gamma(\beta_{12})\Gamma(\beta_{2-12})} \\ \times \int_0^1 dx\, x^{\eta_1} \int_0^x dy\, (x-y)^{\beta_{1-12}-1}\, y^{\beta_{12}-1}\, (1-x+y)^{\eta_2+\beta_{2-12}-1}. \end{aligned} \tag{E.3}$$

Apply the binomial theorem in (E.3) to expand two of the three factors in the integrand, $(x-y)^{\beta_{1-12}-1}$ and $(1-x+y)^{\eta_2+\beta_{2-12}-1}$, in series. Using these series, note that each term in the

series for $(x-y)^{\beta_{1-12}-1}$ will contribute the same power of x after the integration over y, while the terms of the series for $(1-x+y)^{\eta_2+\beta_{2-12}-1}$ contribute increasing powers of x after the integration over y. Note also that if the lowest-power-of-x term is integrable over x in a region containing 0, then all terms are. Thus, the worst case occurs with the constant term from the binomial series for $(1-x+y)^{\eta_2+\beta_{2-12}-1}$. After integration over y with this constant term, and considering the small x region of integration, we are left with the integral over x given by

$$C \int_0^{\underline{x}} dx \; x^{\eta_1+\beta_1-1}, \tag{E.4}$$

where $0 < \underline{x} < 1$, and C is a constant. This integral exists for $\mathrm{Re}(\beta_1+\eta_1) > 0$. This, symmetry, and the first condition (given by $\mathrm{Re}(\nu_1) > 0$, $i = 1, \ldots, m$) establish the result. The method for more complicated overlap structures is also indicated by this discussion.

The discussion above is of interest in another way: it provides a general method for finding multiple overlap integrals without the use of transform theory.

F. DERIVATIVES OF OVERLAP CONVOLUTIONS - POLES

In this appendix we find derivatives with respect to η of expressions such as $\Gamma(k-\eta)/\Gamma(-\eta)$, where $k \in \{0, 1, \ldots\}$, and η may be any number within the constraints of existence. We consider the various cases that arise when some combination of poles occurs and demonstrate the various simplified expressions for the derivatives in these cases.

When η is not an integer, there are no poles in either $\Gamma(k-\eta)$ or $\Gamma(-\eta)$ and the usual derivative expressions hold. When η is an integer and $\eta < 0$, the usual expressions also hold since $k \geq 0$ and therefore $k-\eta > 0$. The case where the usual expressions hold will be denoted as case 0.

When η is an integer and $\eta \geq 0$, there are two cases are of interest. The first, case 1, occurs when $\eta \geq 0$ and $k-\eta > 0$, so that there are poles in the denominator of $\Gamma(k-\eta)/\Gamma(-\eta)$ only. The

second, case 2 occurs when $\eta \geq 0$ and $k - \eta \leq 0$, so that there are poles in both the numerator and the denominator.

In order to find expressions for the derivatives in cases 1 and 2 we use the following facts. 1) The only singularities of the gamma function $\Gamma(x)$ are simple poles at $x = -n$, $n = 0, 1, \ldots$ with residues $(-1)^n/n!$ respectively. 2) $\Delta\Phi^{(n)}(k - \eta, -\eta) = (-1)^{n-1}\Gamma(n)\Sigma_{i=1}^{k} \frac{1}{(k - i - \eta)^n}$ whenever the expressions exist. (The identity $\Gamma(k - \eta)/\Gamma(-\eta) = \Pi_{i=1}^{k}(k - i - \eta)$ may be used in deriving this.) 3) $\Gamma(k - \eta)/\Gamma(-\eta)$ is the representation (away from the poles in the gamma functions) of an everywhere-analytic function (note that $k \geq 0$ is still assumed). Using these facts, the expressions for cases 1 and 2 are found by substituting $\xi = \eta + \varepsilon$ for η (now restricted by the conditions of cases 1 and 2 to be a nonnegative integer) in the corresponding case 0 expressions and taking the $\varepsilon = 0$ limit.

Case 0: η non-integer, or η an integer with $\eta < 0$ and $k - \eta > 0$. There are no poles in the numerator or denominator. The first derivative is given by

$$\partial_\eta^1\left[\frac{\Gamma(k - \eta)}{\Gamma(-\eta)}\right] = -\frac{\Gamma(k - \eta)}{\Gamma(-\eta)}\Delta\Phi^{(1)}(k - \eta, -\eta). \qquad \text{(F.1)}$$

The r^{th} derivative may be found by iteration, using Eq. (F.1) and the recursion relation

$$\partial_\eta^1\Phi^{(n)}(k - \eta) = -\Phi^{(n+1)}(k - \eta). \qquad \text{(F.2)}$$

For example, taking the derivative of Eq. (F.1), applying Eq. (F.2) with Eq. (F.1) in the process, yields the second derivative

$$\partial_\eta^2\left[\frac{\Gamma(k - \eta)}{\Gamma(-\eta)}\right] = \frac{\Gamma(k - \eta)}{\Gamma(-\eta)}\left[\Delta\Phi^{(1)}(k - \eta, -\eta)^2 + \Delta\Phi^{(2)}(k - \eta, -\eta)\right]. \qquad \text{(F.3)}$$

Case 1: η an integer, $\eta \geq 0$ and $k - \eta > 0$. The denominator contains a pole.

The zeroth derivative is 0. Taking the appropriate limit in Eq. (F.1) gives us the first

derivative

$$(-1)^{\eta+1}\eta!\Gamma(k-\eta), \tag{F.4}$$

while taking the appropriate limit in Eq. (F.3) yields the second derivative

$$(-1)^{\eta}2\eta!\Gamma(k-\eta)\Sigma_{i=1,\,i\neq k-\eta}^{k}(k-i-\eta)^{-1}. \tag{F.5}$$

Case 2: η integer, $\eta \geq 0$ and $k-\eta \leq 0$: Both the numerator and denominator contain poles. The zeroth derivative is simply $\frac{(-1)^k\eta!}{(\eta-k)!}$. Taking the appropriate limit in Eq. (F.1) gives us the first derivative

$$(-1)^{k+1}\frac{\eta!}{(\eta-k)!}\Delta\Phi^{(1)}(k-\eta,-\eta), \tag{F.6}$$

while taking the appropriate limit in Eq. (F.3) yields the second derivative

$$(-1)^{k}\frac{\eta!}{(\eta-k)!}\left[\Delta\Phi^{(1)}(k-\eta,-\eta)^2+\Delta\Phi^{(2)}(k-\eta,-\eta)\right]. \tag{F.7}$$

G. MULTIPLE OVERLAP RESULTS

Here we define the general *Convolution Form* (CF) notation and demonstrate the results stated without proof in Sec. 2e. Let $\mathbf{k} = (k_1, \ldots, k_n)$ be a vector of nonnegative summation indices, (with n the number of such indices) and let $C(\cdot)$ be any function of an n-vector of nonnegative integers. Further, let $t^{\mathbf{k}} \equiv \Pi_{i=1}^{n} t_i^{k_i}$, $t_{\mathbf{a}} \equiv \Sigma_{i=1}^{n} a_i t_i$, where $\mathbf{a} = (a_1, \ldots, a_n)$ with components in the reals, and let $\alpha_{\mathbf{k}}$ be a complex number indexed by $\mathbf{k}$.

Define the CF symbol $(\mathbf{a};\mathbf{b})$ to be the set of expressions of the form

$$\exp(-\tau t_{\mathbf{a}})\,\Sigma_{\mathbf{k}} C(\mathbf{k})\tau^{\alpha_{\mathbf{k}}-1}t^{\mathbf{k}}, \tag{G.1}$$

where τ is known as the *convolution variable* and where the *occurrence* indicator vector $\mathbf{b}$ has components in $\{0, 1\}$ and $b_i = 1$ if and only if t_i *occurs* in the summation $\Sigma_{\mathbf{k}} C(\mathbf{k}) \tau^{\alpha_k - 1} t^{\mathbf{k}}$, i.e. $b_i = 1$ iff $\exists \mathbf{k} \ni C(\mathbf{k}) \neq 0$ and $k_i > 0$. All the aspects of $C(\mathbf{k})$ that are relevant to the analysis of Sec. 2e are expressed by the vector $\mathbf{b}$. A particular member of the CF $(\mathbf{a};\mathbf{b})$ will be written as $(\mathbf{a};\mathbf{b})_u$, i.e. u indexes the CF $(\mathbf{a};\mathbf{b})$.

In order to represent convolutions of CF expressions, let $[(\mathbf{a};\mathbf{b}), (\mathbf{c};\mathbf{d})]$ be the set of all expressions that result from the convolution of two CF members from $(\mathbf{a};\mathbf{b})$ and $(\mathbf{c};\mathbf{d})$ respectively, i.e.

$$[(\mathbf{a};\mathbf{b}), (\mathbf{c};\mathbf{d})] \equiv \{\int_0^\tau (\mathbf{a};\mathbf{b})_u(x) \ (\mathbf{c};\mathbf{d})_w(\tau - x) \ dx\}, \tag{G.2}$$

where by $(\mathbf{a};\mathbf{b})_u(x)$ we mean $(\mathbf{a};\mathbf{b})_u$ with its convolution variable evaluated at x, and where u and w range over all indices for the CF's $(\mathbf{a};\mathbf{b})$ and $(\mathbf{c};\mathbf{d})$ respectively.

Now we show that the result of the convolution of any two members of CF's may be written as a member of some CF. More precisely, we prove Eq. (13) of Sec. 2e,

$$[(\mathbf{a};\mathbf{b}), (\mathbf{c};\mathbf{d})] \subset (\mathbf{c} \ ; \ \mathbf{b} \vee \mathbf{d} \vee \mathrm{nz}(\mathbf{c} - \mathbf{a})). \tag{G.3}$$

To prove Eq. (G.3) start by noting that each integrated convolution in Eq. (G.2) has the representation (see App. A, Eq. (A.2))

$$e^{-\tau t_c} \Sigma_{\mathbf{j}} \Sigma_{\mathbf{k}} C(\mathbf{j}, \mathbf{k}) t^{\mathbf{j}} t^{\mathbf{k}} \tau^{\alpha_j + \alpha_k - 1} \frac{\Gamma(\alpha_j)\Gamma(\alpha_k)}{\Gamma(\alpha_j + \alpha_k)} \times {}_1F_1(\alpha_j; \alpha_j + \alpha_k; \tau t_{\mathbf{c} - \mathbf{a}}). \tag{G.4}$$

In Eq. (G.4), as before, $\mathbf{j}$ and $\mathbf{k}$ are vectors of nonnegative summation indices, $C(\cdot, \cdot)$ is a function of the vectors $\mathbf{j}$ and $\mathbf{k}$ (specifically, it is the product of the C's appearing in the two CF's $(\mathbf{a};\mathbf{b})_u(x)$ and $(\mathbf{c};\mathbf{d})_w(x)$), $t^{\mathbf{j}} \equiv \Pi_{i=1}^n t_i^{[\mathbf{j}]_i}$ (and similarly for $t^{\mathbf{k}}$), $t_{\mathbf{c}} \equiv \Sigma_{i=1}^n c_i t_i$, and $t_{\mathbf{c}-\mathbf{a}} \equiv \Sigma_{i=1}^n (c_i - a_i) t_i$. Now, expand the hypergeometric ${}_1F_1$ in the series (see App. A)

$${}_1F_1(\alpha_j; \alpha_j + \alpha_k; \tau t_{\mathbf{c}-\mathbf{a}}) = \frac{\Gamma(\alpha_j + \alpha_k)}{\Gamma(\alpha_j)\Gamma(\alpha_k)} \Sigma_{i=0}^{\infty} \frac{(\alpha_j)_i}{(\alpha_j + \alpha_k)_i} \frac{(\tau t_{\mathbf{c}-\mathbf{a}})^i}{i!}. \tag{G.5}$$

Substitute Eq. (G.5) into Eq. (G.4) and note the following four things: (1) if $b_i = 1$ then t_i occurs in Eq. (G.4), (2) if $d_i = 1$ then t_i occurs in Eq. (G.4), (3) if $c_i - a_i \neq 0$ then t_i occurs in Eq. (G.4), and (4) if none of the cases 1-3 hold then t_i does not occur in Eq. (G.4). Thus, the occurrence indicator vector for the result of the convolution Eq. (G.4) is $(\mathbf{c} \; ; \; \mathbf{b} \vee \mathbf{d} \vee \mathrm{nz}(\mathbf{c} - \mathbf{a}))$. Noting also that the constant in the exponential's argument is $t_{\mathbf{c}}$ establishes Eq. (G.3).

Finally, Eq. (14) of Sec. 2e follows immediately from the definition of the CF in Eq. (G.1) and the properties of the T transform in App. A.

FOOTNOTES

[1] An alternate proof of Thm. 12 leads to an identity. Where the inverse T transform is applied in the proof of Thm. 12, instead find the convolution $(\tau^{\beta_1 - 1} e^{-\tau t} \otimes \tau^{\bar{\beta} - 1})$ using Thm. 10.2 and express it in terms of ${}_1F_1$. Now, do the inverse transform and equate this result to the result of Thm. 12 to find *Gauss's identity*: ${}_2F_1((a, b);c;1) = \dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$.

[2] An alternate proof of Thm. 14a leads to another identity. Instead of applying Thm. 11, apply Thm. 10.1 to the first two terms of the pairwise overlap convolution $p^{\beta_{1-12} - 1} e^{-pt_1} \otimes p^{\beta_{12} - 1} e^{-p(t_1 + t_2)}$ and immediately take the inverse T_1 transform. Now, do the final convolution with $p^{\beta_{2-12} - 1} e^{-pt_2}$ and take the inverse T_2 transform. Note that there is only a single summation in the result, whereas in the result in Thm. 14a there are two summations. On the other hand, the convolution $p^{\beta_{12} - 1} e^{-p(t_1 + t_2)} \otimes p^{\beta_{2-12} - 1} e^{-pt_2}$ can be taken first, followed by a convolution with $p^{\beta_{1-12} - 1} e^{-pt_1}$, effectively interchanging indices 1 and 2. Equating these two single-sum forms gives the identity

$$
{}_3F_2[a_1, a_2, a_3; b_1, b_2; 1] = \frac{\Gamma(b_1)\Gamma(b_2)\Gamma(b_1 + b_2 - (a_1 + a_2 + a_3))}{\Gamma(a_2)\Gamma(b_1 + b_2 - (a_1 + a_2))\Gamma(b_1 + b_2 - (a_2 + a_3))}
$$

$$
\times {}_3F_2[b_1 - a_2, b_1 + b_2 - (a_1 + a_2 + a_3), b_2 - a_2; b_1 + b_2 - (a_1 + a_2), b_1 + b_2 - (a_2 + a_3); 1],
$$

while equating either of the single-sum results just described to the original result of Thm. 14a yields Gauss's identity, discussed in footnote 1.

[3] Utilizing Gauss's identity (see footnote 1 and [9], Eq. 15.1.1) provides further simplification in Thm. 15a for cases 15a.2 and 15a.4. These simplifications are due to simplifications appearing in $F^{(10)}$ and $F^{(01)}$ respectively. The choice of the form of the results presented was made considering the simplicity of the results and consistency between the results.

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