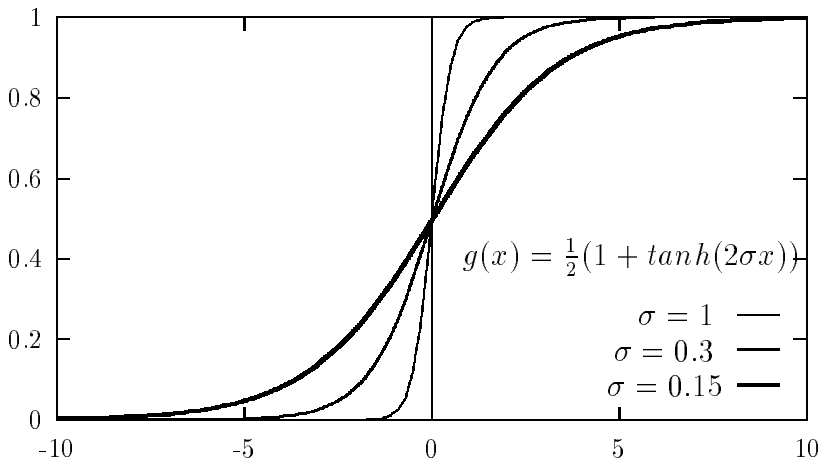
1
0.8
0.6
0.4
0.2
0
-10
-5
0
5
10
$g(x) = \frac{1}{2}(1 + tanh(2\sigma x))$
$\sigma = 1$
$\sigma = 0.3$
$\sigma = 0.15$

Figure 2:

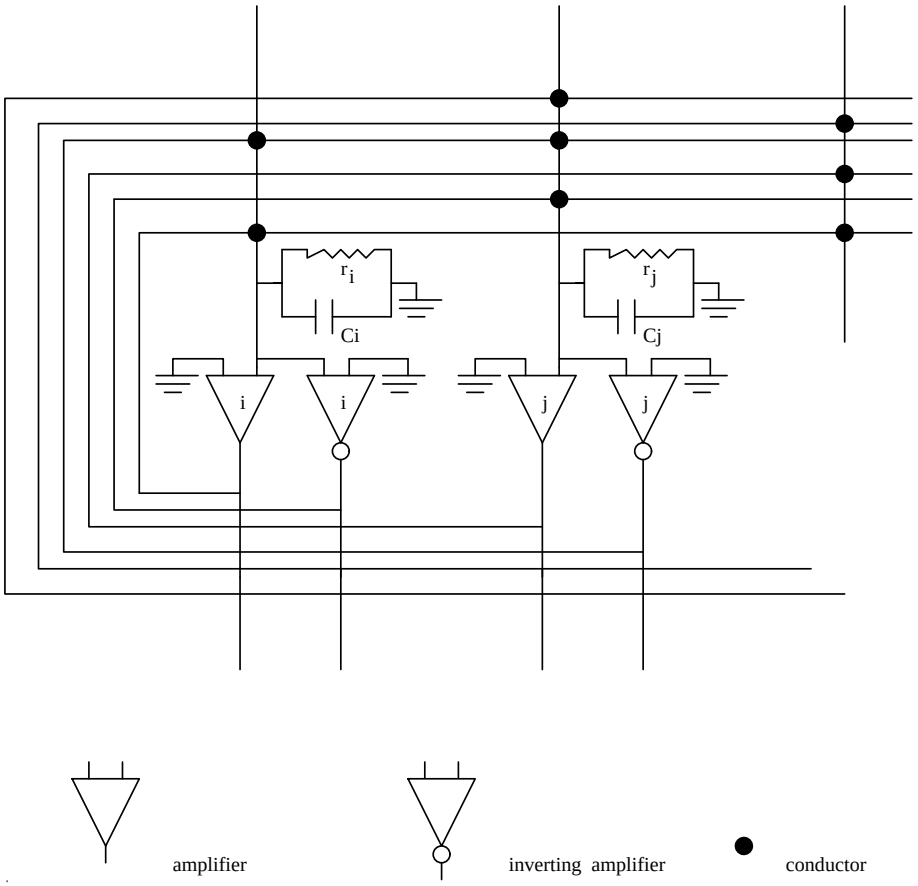

Figure 1:

Figures

Figure 1: Hopfield neural network circuit
Figure 2: The activation functions with different gain coefficients

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Let $\mu_1,\ \mu_2,\ \cdots \mu_N$ be the eigenvalues of the matrix (T_{ij}) such that $\mu_N \le \cdots \le \mu_2 \le \mu_1$. So $\mu_1 = \mu_{max}$. There exists an orthogonal matrix S such that $S^\tau TS = diag(\mu_1, \cdots, \mu_N)$. Particularly, choose $V = \overline{V} + SY$, where $Y = (y_1, \cdots, y_N)^\tau$ with $y_1 \neq 0$ and $y_2 = \cdots = y_N = 0$ in (2), by the assumption **(A2)** in Theorem 5 we have

$$E(u) \le (-\frac{\mu_{max}}{2} + \frac{1}{\lambda} + \varepsilon)\|V - \overline{V}\|^2 < 0.$$

So in any neighborhood of $\overline{u} = 0$, the function $E(u)$ can take negative value.

Set $G_i(u) = \sum_j T_{ij}(g_j(u_j) - \frac{1}{2}) - \frac{u_i}{R_i}$ and $X_i = tanh(\lambda_i u_i)$.
Then $G_i(u) = \frac{1}{2}\sum_j T_{ij}X_j - \frac{1}{\lambda}th^{-1}(X_i) = \sum_j(\frac{1}{2}T_{ij} - \frac{1}{\lambda}\delta_{ij})X_j + o(\|X\|)$.

Let $a_{ij} = \frac{1}{2}T_{ij} - \frac{1}{\lambda}\delta_{ij}$ and $A = (a_{ij})$. Then $det(A) \neq 0$ since $\lambda \neq \frac{2}{\mu_i}$, $i = 1, \cdots, N$, by the assumption **(A2)**.

So $G(u) = AX + o(\|X\|)$ and $\frac{\|G(u)\|}{\|X\|} \ge \frac{\|AX\|}{\|X\|} \ge \inf_{\|X\|=1}\|AX\| > 0$.

Set $\varepsilon = \inf_{\|X\|=1}\|AX\|$. Then there is a neighborhood $U(\overline{u})$ of $\overline{u} = 0$ on which $\|G(u)\| > 0$.

So $\overline{u} = 0$ is an unstable equilibrium, since $D_{(1)}E(u) = -\sum_i \frac{1}{C_i}g_i'(u_i)(G_i(u))^2 < 0$ on $U(\overline{u})$, $E(\overline{u}) = 0$ and $E(u)$ can take negative value in any neighborhood of $\overline{u} = 0$. Q.E.D.

Appendix 4 *Proof of Lemma 3.*

Suppose the system (33) has a nonconstant periodic solution $\psi_t(x_0)$.
Let $\mathbf{P} = \{all\ period\ of\ \psi_t(x_0)\}$ and $\tau = \inf_{T\in\mathbf{P}} T$. Then, there exists a sequence of $\tau_n \in \mathbf{P}$ such that $\tau_n \downarrow \tau$.

If $\tau > 0$, then τ is the smallest period of $\psi_t(x_0)$.

Let $\psi_{t_1}(x_0) = \inf\{\psi_t(x_0),\ 0 \le t \le \tau\}$ and $\psi_{t_2}(x_0) = \sup\{\psi_t(x_0),\ 0 \le t \le \tau\}$.

Since $\psi_t(x_0)$ is not a constant solution, $\psi_{t_1}(x_0) < \psi_{t_2}(x_0)$. Without loss of generality, we assume $t_1 < t_2$. So $0 \le t_1 < t_2 \le \tau \le t_1 + \tau$ and $\psi_{t_1+\tau}(x_0) = \psi_{t_1}(x_0) < \psi_{t_2}(x_0)$. By the continuity of $\psi_t(x_0)$, there exist $t'\ and\ t''$ such that $t_1 < t' < t_2 < t'' < t_1 + \tau$ and $\psi_{t'}(x_0) = \psi_{t''}(x_0)$. Then for any t, $\psi_{t-t'}(\psi_{t'}(x_0)) = \psi_{t-t'}(\psi_{t''}(x_0))$, i.e. $\psi_t(x_0) = \psi_{t+t''-t'}(x_0)$. Therefore $t'' - t'$ is a period of $\psi_t(x_0)$ but it is less than τ. This is a contradiction.

If $\tau = 0$, then $\tau_n \downarrow 0$. There exist $t_1 < t_2$ such that $\psi_{t_1}(x_0) \neq \psi_{t_2}(x_0)$ since $\psi_t(x_0)$ is not a constant solution.

Let $c = |\psi_{t_1}(x_0) - \psi_{t_2}(x_0)|$. Then

$$|\psi_{t_1+k\tau_n}(x_0) - \psi_{t_2}(x_0)| = c. \tag{36}$$

By the continuity of $\psi_t(x_0)$, there exists a $\delta > 0$, such that $|\psi_s(x_0) - \psi_{t_2}(x_0)| < c$ when $|t_2 - s| < \delta$. Since $\tau_n \downarrow 0$, for sufficiently large integer n $0 < \tau_n < \delta$. Let $t_1 + k\tau_n \le t_2 < t_1 + (k+1)\tau_n$ for some integer k. Then $|t_1 + k\tau_n - t_2| < \tau_n < \delta$, and $|\psi_{t_1+k\tau_n}(x_0) - \psi_{t_2}(x_0)| < c$. This is contradictory to (36).
Q.E.D.

3.2.3, 3.4.1, and 3.4.3, Chapter 3). The degrees [4] of the two maps $x - F$ and $x - x_0$ are both one. So if the map F has more than one fixed point it must have at least three fixed points. One of them can be found along a path of the homotopy H starting from the point x_0. Among the other fixed points, there must exist two fixed points y_1 and y_2 such that the two points $(y_1, 1)$ and $(y_2, 1)$ are on the same path of the homotopy, but the signs of the two values $det(H'_x(y_1, 1))$ and $det(H'_x(y_2, 1))$ are opposite, i.e.

$$det(H'_x(y_1, 1))det(H'_x(y_2, 1)) < 0.$$

Therefore, from the BDEs(see PP.26 in Ref. [4]) of the homotopy H:

$$\frac{dt}{dr} = (-1)^{N+1} det(H'_x(x, t)),$$

there must exists a point $(\overline{y}, \overline{t})$ on the path connecting $(y_1, 1)$ and $(y_2, 1)$ such that $det(H'_x(\overline{y}, \overline{t})) = 0$. This is contrary to the conclusion that the matrix $H'_x(x, t)$ is of full rank for $0 \leq t \leq 1$ and $x \in D$. So the equilibrium of the system (1) must be unique. Q.E.D.

Appendix 2 *Proof of Theorem 2:*

Theorem 1 is proved in Appendix 1. To prove Theorem 2, all proofs for Theorem 1 are needed except for the three inequalities in (34) where we want to prove that
$Y^\tau B(x, t)Y < 0$ for all $Y \in R^N \setminus \{0\}$.

Since the derivative of each activation function is positive and bounded, we have $Y^\tau B(x, t)Y = Y^\tau TY - \sum_i \frac{1}{tR_i g'_i(x_i)} y_i^2 \leq Y^\tau TY - \sum_i \frac{1}{R_i \sigma_i} y_i^2$.
The quadratic form $Y^\tau TY$ in the above is equal to
$\frac{1}{2}\sum_{ij}(T_{ij} + T_{ji})y_i y_j$
$\leq \sum_i T_{ii} y_i^2 + \frac{1}{2}\sum_{i,j:i\neq j} |T_{ij} + T_{ji}||y_i y_j|$
$\leq \sum_i T_{ii} y_i^2 + \frac{1}{4}\sum_{i,j:i\neq j} |T_{ij} + T_{ji}|(y_i^2 + y_j^2)$
$= \sum_i (T_{ii} + \frac{1}{2}\sum_{j:i\neq j} |T_{ij} + T_{ji}|)y_i^2$.
Therefore, under the condition (7), we know
$Y^\tau B(x, t)Y \leq \sum_i (T_{ii} + \frac{1}{2}\sum_{j:i\neq j} |T_{ij} + T_{ji}| - \frac{1}{R_i \sigma_i})y_i^2 < 0$ for all non-zero Y.
Q.E.D.

Appendix 3 *Proof of Theorem 5:*

Here we take the modified energy function $E(u)$ defined by (2) as the Lyapunov function to analyze the instability of the system (1) when the connection matrix is symmetric. It is easily seen that $\overline{u} = 0$ is an equilibrium of the system (1) if the condition **(A1)** in Theorem 5 holds. Since $g_i^{-1}(v) = \frac{R_i}{2\lambda} ln \frac{v}{1-v}$, we have the following

$$\lim_{V_i \to \overline{V_i}} \frac{\frac{1}{R_i}\int_{\overline{V_i}}^{V_i} (g_i^{-1}(v) - g_i^{-1}(\overline{V}_i))dv}{(V_i - \overline{V_i})^2} = (\frac{1}{\overline{V_i}} + \frac{1}{1 - \overline{V_i}})\frac{1}{4\lambda}, \tag{35}$$

where $V_i = g_i(u_i),\ i = 1, \cdots, N$, and $V = (V_1, \cdots, V_N)^\tau$. When $\overline{u} = 0$, $\overline{V_i} = g_i(\overline{u_i}) = \frac{1}{2}$, the above limit is $\frac{1}{\lambda}$. Choose an $\varepsilon > 0$ such that $\frac{1}{\lambda} + \varepsilon < \frac{\mu_{max}}{2}$. Then by (35), there is a neighborhood of $\overline{V} = (\overline{V_1}, \cdots, \overline{V_N})^\tau$ on which

$$\sum_i \frac{1}{R_i} \int_{\overline{V_i}}^{V_i} (g_i^{-1}(v) - g_i^{-1}(\overline{V}_i))dv \leq (\frac{1}{\lambda} + \varepsilon)\|V - \overline{V}\|^2.$$

For the closed chain neural network, some conditions have been found to make it oscillatory. These results show the importance of the gain factor in determining the dynamics of neural networks. When the gain factor is sufficiently small, the neural network has an exponentially stable equilibrium. However, if the gain factor is sufficiently large, then this neural network can oscillate periodically. The modeling approach given in this paper supports the conclusion in Ref. [16]. Some forms of learning can be realized by adjustments of postsynaptic response properties rather than synaptic strengths. To regulate the postsynaptic response in the neural circuit, we can adjust the gain factors of the amplifiers to improve the performance of the neural network.

In this paper, we use the same assumptions to assure the uniqueness and exponential stability. The uniqueness is useful in determining the global minimum point of the energy function in [11] when the neural networks are used to solve some optimal problems by the correspondence between the equilibria and the minimum points.

6 Appendix

Appendix 1 *Proof of Theorem 1:*

Let $F_i(u) = R_i(\sum_j T_{ij} g_j(u_j) + I_i), \quad i = 1, \ldots, N$, and $F(u) = (F_1(u), \ldots, F_N(u))^\tau$.

The equation (3) can be rewritten as $\overline{u} = F(\overline{u})$ where $\overline{u}$ is an equilibrium or a fixed point of the system (1). Since the functions $\{g_i(u)\}$ are bounded, we can choose constants $m_i > 0$ such that $|F_i(u)| < m_i, \quad i = 1, \ldots, N$.
Let

$$D = \{(x_1, \ldots, x_N) \in R^N : |x_i| \leq m_i,\ i = 1, \ldots, N\}, \quad \text{and}$$
$$D^0 = \{(x_1, \ldots, x_N) \in R^N : |x_i| < m_i,\ i = 1, \ldots, N\}.$$

It is easy to see the map $F : D \to D$ is a continuous map. By Brouwer's fixed point theorem [4], F has at least one fixed point or equilibrium.

Choosing any point $x_0 \in D^0$, we define a homotopy function
$H(x,t) = (1-t)(x - x_0) + t(x - F(x))$, for $x \in D$ and $0 \leq t \leq 1$. If $H(x_1, t_1) = 0$, then $x_1 = (1-t_1)x_0 + t_1 F(x_1) \in D^0$ since both x_0 and $F(x_1)$ are in the open convex set D^0. Therefore, $H : D \times [0,1] \to R^N$ is boundary-free for $0 \leq t \leq 1$.

Denote $H'_x(x,t) = (\frac{\partial H_i(x,t)}{\partial x_j})_{N \times N}$ the derivative of the homotopy function, and $I = (\delta_{ij})$ the identity matrix. Then $H'_x(x,t) = (\delta_{ij} - tR_i T_{ij} g'_j(x_j))_{N \times N}$.
Let $B(x,t) = (T_{ij}) - diag(\frac{1}{tR_1 g'_1(x_1)}, \ldots, \frac{1}{tR_N g'_N(x_N)})$. For $0 < t \leq 1$, the matrix $H'_x(x,t)$ can be changed to $B(x,t)$ by a succession of elementary row and column operations since $R_i > 0$, and $g'_i(x) > 0, i = 1, \ldots, N$. So $B(x,t)$ and $H'_x(x,t)$ have the same rank. For any $Y = (y_1, \ldots, y_N)^\tau \in R^N \setminus \{0\}$, noticing $Y^\tau T Y = Y^\tau (\frac{T + T^\tau}{2}) Y$, we have

$$Y^\tau B(x,t) Y \leq \mu_{max} \|Y\|^2 - \sum_i \frac{1}{tR_i g'_i(x_i)} y_i^2 \leq \sum_i (\mu_{max} - \frac{1}{R_i \sigma_i}) y_i^2 < 0. \tag{34}$$

So the linear equation $B(x,t)Y = 0$ has the zero solution only. Hence $det(B(x,t)) \neq 0$, and $det(H'_x(x,t)) \neq 0$ for $0 < t \leq 1$. In addition, $H'_x(x,0) = I$. Therefore $H'_x(x,t)$ is of full rank, for $0 \leq t \leq 1$. The following proofs are based on the results in Ref. [4] (Theorem

Theorem 8 *Let $N \geq 3$. In addition to the conditions* **(A5)** *and* **(A6)**, *assume*

(A8) $C_i = C,\ R_i = R,\ i = 1, \ldots, N$ *and* $\frac{1}{R} < \frac{1}{2}(-T_{1N}T_{NN-1}\cdots T_{21}\lambda_1\lambda_2\cdots\lambda_N)^{\frac{1}{N}}cos(\frac{\pi}{N})$.

Then the system (1) has a nonconstant periodic solution.

Proof:
Set $\delta = (-T_{1N}T_{NN-1}\cdots T_{21}\lambda_1\lambda_2\cdots\lambda_N)$ and the matrix $A = (\frac{\partial \overline{g}_i}{\partial u_j}(0))_{N\times N}$. The eigenvalues of A can be found by solving the equation $det(A - ZI) = (Z + \frac{1}{RC})^N + \frac{\delta}{(2C)^N} = 0$. They are $Z_k = -\frac{1}{RC} + \frac{1}{2C}\delta^{\frac{1}{N}}exp(\frac{2k+1}{N}\pi i),\ \ k = 0, \cdots, N-1$. The N complex numbers $(Z_k)_{k=0,N-1}$ are distributed on the circle $|Z + \frac{1}{RC}| = \frac{1}{2C}\delta^{\frac{1}{N}}$ in the complex plane. So the matrix A has no repeated eigenvalues. By condition **(A8)**, we know $Re(Z_0) = -\frac{1}{RC} + \frac{1}{2C}\delta^{\frac{1}{N}}cos(\frac{\pi}{N}) > 0$. Therefore the condition **(A7)** in Theorem 7 holds.
Q.E.D.

For the closed chain neural network, we can prove that the μ_{max} is always positive. In fact, if we set $f(X) = X^\tau AX$ for $X \in R^N$ and choose $x_4 = \cdots = x_N = 0$ and $x_2 = 1$, then $f(X) = T_{21}x_1 + T_{32}x_3$ which takes both positive and negative values. Because of this and $f(X) \leq \mu_{max}\|X\|^2$, μ_{max} must be greater than zero.

When $\lambda_i = \frac{\lambda}{R}$, we showed in Section 2.4 that if $0 < \lambda < \frac{2}{\mu_{max}}$, than the system (1) contains a unique equilibrium which is exponentially stable. On the other hand, in this case the condition **(A8)** becomes

(A9) $$\frac{2}{(-T_{1N}T_{NN-1}\cdots T_{21})^{\frac{1}{N}}cos(\frac{\pi}{N})} < \lambda.$$

Therefore, under the conditions **(A5)**, **(A6)** and **(A9)**, the system (1) has a nonconstant periodic solution.

With the two constraints **(A5)** and **(A6)** on the connection matrix, one may have two different neural dynamics due to different constraints on the gain factor. One is stable if the gain factor is small enough such that $0 < \lambda < \frac{2}{\mu_{max}}$ is true, and the other one is oscillatory if the gain factor is large enough such that **(A9)** holds.

Note that the constraint **(A9)** gives no upper bound but a lower bound for the gain factor to guarantee the smooth oscillation in the closed chain network.

5 Discussion and Conclusions

In this paper, we present some results about the exponential stability, the almost sure exponential stability and the oscillation of asymmetric neural networks. The activation functions are more general than the ones in [10], and not restricted to some particular functions. The assumptions for assuring the stability and uniqueness of the equilibrium are more concise than the ones given in Refs.[2], [14] and [15]. For any synaptic connection matrix (T_{ij}), we can adjust the gain factors of the amplifiers or the resistances in the neural circuit to make the neural network possess a unique equilibrium which is exponentially stable. As a corollary to this result, we can set up a one-to-one and smooth map between input currents and the equilibria of the neural network.

On the other hand, if $u_1 < 0$ *and* $u_N < 0$, then

$$g_N(u_N) < \frac{1}{2} \quad and \quad \overline{g}_1(u_N, u_1) > \frac{1}{C_1}(\frac{1}{2}T_{1N} - \frac{u_1}{R_1} + I_1) = -\frac{1}{C_1}\frac{u_1}{R_1} > 0.$$

By Lemma 2, the closed chain neural network has a nonconstant periodic solution. Q.E.D.

In Theorem 7, we assume $N \geq 3$. We shall see at least three neurons are needed to make a closed chain network periodically oscillating.

When $N = 2$, the equations for the closed chain network are

$$\frac{du_1}{dt} = \frac{1}{C_1}(T_{12}g_2(u_2) - \frac{u_1}{R_1} + I_1) \quad and \quad \frac{du_2}{dt} = \frac{1}{C_2}(T_{21}g_1(u_1) - \frac{u_2}{R_2} + I_2). \tag{31}$$

Denote $P = \frac{1}{C_1}(T_{12}g_2(u_2) - \frac{u_1}{R_1} + I_1)$ and $Q = \frac{1}{C_2}(T_{21}g_1(u_1) - \frac{u_2}{R_2} + I_2)$.

Since $\frac{\partial P}{\partial u_1} + \frac{\partial Q}{\partial u_2} = -\frac{1}{R_1} - \frac{1}{R_2} < 0$, by Bendixson's criterion [6], the system (31) has no nonconstant periodic solution.

When $N = 1$, the system (1) becomes

$$C\frac{du}{dt} = T_{11}g(u) - \frac{u}{R} + I. \tag{32}$$

By the following Lemma, this equation also has no nonconstant periodic solution, i.e. one isolated formal neuron cannot oscillate.

Lemma 3 *If a one-dimensional equation*

$$\frac{dx}{dt} = f(x), \quad x(0) = x_0 \in R^1, \quad t \geq 0, \tag{33}$$

has a unique solution for each $x_0 \in R^1$, then this equation has no nonconstant periodic solution.

Proof: see Appendix 4 for mathematical proof.

Intuitively, if the equation (33) has a nonconstant periodic solution, then there will be two time points t_1 and t_2 such that

$$x(t_1) = x(t_2) \quad but \quad \left.\frac{dx}{dt}\right|_{t=t_1} \neq \left.\frac{dx}{dt}\right|_{t=t_2} .$$

This is contradictory to the fact that

$$\left.\frac{dx}{dt}\right|_{t=t_1} = f(x(t_1)) = f(x(t_2)) = \left.\frac{dx}{dt}\right|_{t=t_2} .$$

Theorem 7 indicates that a closed chain of three or more neurons can form a neural oscillator under the conditions **(A5)-(A7)**. However, it is very difficult to examine the condition **(A7)**. In the next Theorem, some more straight forward and concrete conditions other than the ones in Ref.[1] are provided to guarantee that the closed chain neural network has a nonconstant periodic solution.

Lemma 2 *(See [1]on pp.86)*
Let the Jacobian matrix $E = (\frac{\partial g_i}{\partial y_j}(0))_{N\times N}$ *have at least one eigenvalue with positive real part but no repeated eigenvalues. Then the system (28) has a nonconstant periodic solution.*

For the closed chain neural network, the system (1) can be rewritten as the following:

$$\frac{du_1}{dt} = \overline{g}_1(u_N, u_1), \quad \frac{du_j}{dt} = \overline{g}_j(u_{j-1}, u_j), \quad 2 \le j \le N, \tag{29}$$

where $\overline{g}_1(u_N, u_1) = \frac{1}{C_1}(T_{1N}g_N(u_N) - \frac{u_1}{R_1} + I_1)$, and
$\overline{g}_i(u_{i-1}, u_i) = \frac{1}{C_i}(T_{ii-1}g_{i-1}(u_{i-1}) - \frac{u_i}{R_i} + I_i), \quad i = 2, \cdots, N.$

In this section, we assume $g_i(x) = \frac{1}{2}(1 + tanh(\lambda_i x))$, $i = 1, \cdots, N$. By using the Lemma 2, we have Theorem 7 below which tells us that the closed chain of three or more neurons can build a neural oscillator.

Theorem 7 *If* $N \ge 3$ *and the following conditions hold:*

(A5) *all elements of* (T_{ij}) *are zero except* $T_{1N} < 0$ *and* $T_{ii-1} > 0, i = 2, \ldots, N$*;*

(A6) $\frac{T_{1N}}{2} + I_1 = 0$*;*

(A7) *and the Jacobian matrix* $A = (\frac{\partial \overline{g}_i}{\partial u_j}(0))_{N\times N}$ *has at least one eigenvalue with positive real part, but no repeated eigenvalues.*

Then the system (1) has a nonconstant periodic solution.

Proof:
By **(A6)**, we know $\overline{g}_i(0,0) = 0, \quad i = 1, \cdots, N.$
Because of **(A5)**, we have $\frac{\partial \overline{g}_1}{\partial u_N} = \frac{1}{C_1}T_{1N}g'_N(u_N) < 0$ and $\frac{\partial \overline{g}_i}{\partial u_{i-1}} = \frac{1}{C_i}T_{ii-1}g'_{i-1}(u_{i-1}) > 0.$

Moreover, $\frac{\partial \overline{g}_i}{\partial u_i} = -\frac{1}{R_i} < 0, \quad i = 1, \cdots, N.$

We can choose N positive constants $u_i^* > 0, \quad i = 1, \cdots, N$ such that for any u_N

$$\overline{g}_1(u_N, -u_1^*) > 0 \ \ and \ \ \overline{g}_j(-u_{j-1}^*, -u_j^*) > 0, \quad j = 2, \cdots, N. \tag{30}$$

In fact, since $\overline{g}_1(u_N, -u_1^*) = \frac{1}{C_1}(T_{1N}g_N(u_N) + \frac{u_1^*}{R_1} + I_1)$
$\ge \frac{1}{C_1}(T_{1N} + \frac{u_1^*}{R_1} + I_1) = \frac{1}{C_1}(\frac{1}{2}T_{1N} + \frac{u_1^*}{R_1})$ for any u_N,
if we choose $u_1^* > -\frac{1}{2}T_{1N}R_1 > 0$, then $\overline{g}_1(u_N, -u_1^*) > 0$ for any u_N.

In addition, $\overline{g}_2(-u_1^*, -u_2^*) = \frac{1}{C_2}(T_{21}g_1(-u_1^*) + \frac{u_2^*}{R_2} + I_2) \ge \frac{1}{C_2}(\frac{u_2^*}{R_2} + I_2)$ since $T_{21} > 0$.
If we choose $u_2^* > max\{-I_2R_2, 0\}$, then $\overline{g}_2(-u_1^*, -u_2^*) > 0$.

Similarly, we can choose $u_3^* > 0, \cdots, u_N^* > 0$ such that the inequalities (30) hold.

When $u_1 > 0 \ \ and \ \ u_N > 0$, we have $g_N(u_N) > g_N(0) = \frac{1}{2}$ and

$$\overline{g}_1(u_N, u_1) = \frac{1}{C_1}(T_{1N}g_N(u_N) - \frac{u_1}{R_1} + I_1) < \frac{1}{C_1}(\frac{1}{2}T_{1N} - \frac{u_1}{R_1} + I_1) = -\frac{1}{C_1}\frac{u_1}{R_1} < 0$$

since $T_{1N} < 0$ and $\frac{1}{2}T_{1N} + I_1 = 0$ by **(A5)** and **(A6)**.

were found in the proof of Theorem 3:
$D_{(16)}V(x) \le -k\|x\|^2$, for $\|x\| < \delta$, and $k_2\|x\|^2 \le V(x) \le k_3\|x\|^2$ for $x \in R^N$.
In addition, $\sum_{ij} |\frac{\partial^2 V}{\partial x_i \partial x_j}| \le b$, since $\frac{\partial^2 V}{\partial x_i \partial x_j} = \delta_{ij} C_i \alpha_i$, where $b > 0$ is a constant.

Assume

$$|\sigma_{ij}(x) - \sigma_{ij}(y)| \le \beta\|x - y\|, \quad i = 1, \cdots, N, \ j = 1, \cdots, L, \tag{27}$$

where the constant β satisfies $0 < \beta < \frac{k}{bL}$. Then $\mathcal{L}V \le (-k + bL\beta)\|x\|^2$.

By the Lemma 1, we have the following Theorem 6.

Theorem 6 *Assume the condition (27) and the conditions in Theorem 3 hold, then the solution of the system (23) is almost surely exponentially stable.*

Theorem 6 tells us that under the conditions in Theorem 3 each equilibrium of the system (1) is almost surely exponentially stable in the stochastic environment described by (23), if the diffusion coefficient matrix satisfies the condition (27).

4 Neural oscillator

Here the neural oscillator is a special neural network formed by a closed chain of neurons [1] with the connections T_{ij} being zero except $T_{i,i-1} > 0$ for $i = 2, \cdots, N$ and $T_{1N} < 0$.

The neural network model in Ref.[1], which is described by a set of nonlinear integral equations, is different from the model in this paper. When the neural network is a closed chain of neurons, the equation (1) is a special case of the following equation which is studied in Ref.[8]:

$$\frac{dY_1}{dt} = g_1(Y_N, Y_1), \quad \frac{dY_j}{dt} = g_j(Y_{j-1}, Y_j), \quad 2 \le j \le N, \tag{28}$$

where $N \ge 3$.

The two-variable functions $g_j(\cdot, \cdot), j = 1, \cdots, N$, in the above equations (28) satisfy the following conditions:

(A3) $g_j(0, 0) = 0, \quad j = 1, \cdots, N$;

(A4) **(a)** there are constants $y_1^*, \cdots, y_N^* > 0$ such that on the set
$Q = \{y = (y_1, \cdots, y_N) \in R^N : y_j > y_j^*, \ j = 1, \cdots, N \}$,
$\frac{\partial g_j}{\partial y_{j-1}} > 0, \quad 2 \le j \le N$, $\frac{\partial g_1}{\partial y_N} < 0$, $\frac{\partial g_j}{\partial y_j} < 0, \quad 2 \le j \le N$,

(b) $g_j(-y_{j-1}^*, -y_j^*) \ge 0, \quad g_1(y_N, -y_1^*) > 0$, for $y_N \ge -y_N^*$,

(c) $g_1(y_N, y_1) < 0$ for $y_1 > 0$ and $y_N > 0$,

(d) $g_1(y_N, y_1) > 0$ for $y_1 < 0$ and $y_N < 0$,

(e) and $\frac{\partial g_1}{\partial y_1}$ is bounded above on the set Q.

For the system (28), Hastings et al (1977) got the following result.

Let $X^{s,x}(t)$ be the solution of (S.D.E) given the initial condition $X^{s,x}(s) = x$. Then $X^{s,x}(t)$ is a Markov process . Its infinitesimal differential generator $\mathcal{L}$ can be represented by

$$\mathcal{L}V = \frac{\partial V}{\partial t} + \sum_i b_i(t,x)\frac{\partial V}{\partial x_i} + \frac{1}{2}\sum_{ij} a_{ij}(t,x)\frac{\partial^2 V}{\partial x_i \partial x_j}, \tag{24}$$

where $a_{ij}(t,x) = \sum_k \sigma_{ik}(t,x)\sigma_{jk}(t,x)$ and for any $V = V(t,x) \in C^{1,2}$.

To discuss the almost sure exponential stability of the neural network we need the following Lemma.

Lemma 1 *Assume the equation (S.D.E.) has a zero solution $\overline{x} = 0$ and there is a function $V(t,x) \in C^{1,2}$ and a constant $K_1 > 0$ such that the set $G = \{x \in R^N \ : \ \mathcal{L}V(x) \leq -K_1\|x\|^2 \ \}$ contains point 0 and has a nonempty interior, i.e. the set $G - \partial G$ is not empty and $K_2\|x\|^2 \leq V(x) \leq K_3\|x\|^2$. Then the solution of (S.D.E.) is almost surely exponentially stable, i.e. for each $x \in G - \partial G$,*

$$\|X^{s,x}(t)\| \leq b_1(s,x,\omega)e^{-b_2 t},$$

where $b_2 > 0$ is a constant and $b_1(s,x,\omega) \geq 0$ is a finite function which does not depend on t.

Proof:
Let $(X^{s,x}(t))$ be the solution of (S.D.E.) and $\tau = \inf\{t \geq s \ : \ X^{s,x}(t) \text{ not in } G\}$ be the first exit time of the process $(X^{s,x}(t))$ from the set G.

Set $W(x,t) = V(x,t)e^{\frac{K_1}{K_3}t}$. Then $\mathcal{L}W \leq 0$. By Ito's formula, $W(X^{s,x}(t \wedge \tau), t)$ is a positive supmartingale. Therefore, with probability one the $\lim_{t\to\infty} W(X^{s,x}(t \wedge \tau), t)$ exists for each $s \geq 0$ and $x \in G$.

Hence $\sup_t W(X^{s,x}(t \wedge \tau), t) = A_{sx}(\omega) < \infty$, i.e. $V(X^{s,x}(t \wedge \tau), t) \leq A_{sx}(\omega)e^{-\frac{K_1}{K_3}t}$.

So

$$\|X^{s,x}(t \wedge \tau)\|^2 \leq \frac{1}{K_2}A_{sx}(\omega)e^{-\frac{K_1}{K_3}t}, \tag{25}$$

and

$$\|X^{s,x}(\tau)\| = \lim_{t\to\infty} \|X^{s,x}(t \wedge \tau)\| = 0. \tag{26}$$

The stopping time τ must be ∞, otherwise the process $\|X^{s,x}(\tau)\| > 0$ which is a contradiction of (26). So $\tau = \infty$, and from (25) we have $\|X^{s,x}(t)\|^2 \leq \frac{1}{K_2}A_{sx}(\omega)e^{-\frac{K_1}{K_3}t}$.
Q.E.D.

In the proof of Lemma 1, Ito's formula [3] is applied to the stochastic system (23) in which the noise term is based on the Brownian Motion. The essential result of Lemma 1 is invariant for a noise term based on a continuous local martingale, which is a more general stochastic model than the Brownian Motion, as long as the infinitesimal differential generator $\mathcal{L}$ is modified and Ito's formula for semimartingale in the martingale theory [3] is applied accordingly.

Applying Lemma 1 to the stochastic system (23), we can choose the function $V(x)$ defined in the proof of Theorem 3 as the Lyapunov function to analyze the stochastic stability of the system (23). Under the conditions of Theorem 3, the following inequalities

3 Stochastic Stability

In previous sections, the dynamic system is described by the following type of ordinary differential equations:

$$dX_i/dt = b_i(t, X), \quad i = 1, \cdots, N, \quad t \geq t_0, \quad X(t_0) = X_0 \in R^N. \qquad (O.D.E.)$$

When the neural network is disturbed by some noise, we should use the following stochastic differential equation to represent the system:

$$dX_i = b_i(t, X)dt + \sum_k \sigma_{ik}(t, X)dW_k(t), \quad i = 1, \cdots, N, \quad t \geq t_0, \quad X(t_0) = X_0 \in R^N, \qquad (S.D.E.)$$

where $(W(t),\ t \geq 0)$ is an L-dimensional Brownian Motion.

Let $\overline{u}$ and $(u(t))$ be the solution of system (3) and (1) respectively.

Under the assumptions of Theorem 1, $u(t) \to \overline{u}$ exponentially as $t \to \infty$ as long as the initial state $u(0)$ is close enough to $\overline{u}$. The equations (16) can be rewritten as the following:

$$\frac{dx_i}{dt} = \frac{1}{C_i}\sum_j T_{ij}(g_j(x_j + \overline{u}_j) - g_j(\overline{u}_j)) - \frac{x_i}{R_i C_i}, \quad i = 1, \cdots, N, \qquad (22)$$

where $x(t) = u(t) - \overline{u}$. The convergence $u(t) \to \overline{u}$ is equivalent to the convergence $x(t) \to 0$.

When the above system (22) is disturbed by a noise term with a diffusion coefficient matrix $\sigma(x + \overline{u}) - \sigma(\overline{u})$, it becomes

$$\begin{aligned} dx_i = &(\frac{1}{C_i}\sum_j T_{ij}(g_j(x_j + \overline{u}_j) - g_j(\overline{u}_j)) - \frac{x_i}{R_i C_i})dt \\ &+ \sum_k (\sigma_{ik}(x + \overline{u}) - \sigma_{ik}(\overline{u}))dW_k(t), \quad i = 1, \cdots, N, \end{aligned} \qquad (23)$$

where $(W(t),\ \ t \geq 0)$ is an L-dimensional Brownian Motion.

Under the effect of the noise, the trajectory $(x(t))$ of the system (23) becomes a stochastic process. There are various kinds of convergence concepts to describe limiting behaviors of stochastic processes[7]. The almost sure convergence is the most useful because it is closer to the real situation during computation than other forms of convergence. Let $(x(t))$ be a stochastic process in a n-dimensional real space R^n and $\overline{x}$ be a random variable or a constant in R^n. We say that the stochastic process $(x(t))$ is almost surely convergent to $\overline{x}$ if $Prob(\ lim_{t\to\infty}\ x(t) = \overline{x}\ ) = 1$. In particular, if the stochastic process $(x(t))$ converges to $\overline{x}$ exponentially with probability one, we say that $(x(t))$ is almost surely exponentially convergent to $\overline{x}$. If the trajectory $(x(t))$ of the system (23) is almost surely exponentially convergent to 0, we say that the system (23) has almost sure exponential stability.

To analyze the almost sure exponential stability of the system (23), we need the following notation.

Let $C^{1,2}$ denote all functions $V = V(t, x) = V(t, x_1, \cdots, x_N)$ which are twice continuously differentiable with respect to each x_i and once continuously differentiable with respect to t.

Corollary 2 in Ref.[15](pp.237). This result is useful for analyzing a lot of asymmetric networks, but it is not as good as the result in this paper for anti-symmetric networks. Consider the following example. Assume in the system (1), $N = 2$, $R_i = R$ and $\sigma_i = \sigma$ such that $\frac{1}{R\sigma} > 2$, and the connection matrix $T = (T_{ij}) = \begin{pmatrix} a & -2 \\ 2 & a \end{pmatrix}$, where $\frac{1}{R\sigma} - 2 < a < \frac{1}{R\sigma}$. By Corollary 2, this network has a unique equilibrium which is exponentially stable since $T_{ii} = a < \frac{1}{R\sigma}$. But the test matrix for this network is not an M-matrix since
$det(T) = (\frac{1}{R\sigma} - a)^2 - 4 < 0$. Therefore the result in Ref.[15] mentioned above does not apply to this network.

2.5 Energy function and instability

In some earlier works, the proofs of stability were based on proving that the energy function given below was decreasing,

$$\mathcal{E}(u) = -\frac{1}{2}\sum_{ij} T_{ij}V_iV_j + \sum_i \frac{1}{R_i}\int_{\frac{1}{2}}^{V_i} g_i^{-1}(v)dv, \ V_i = g_i(u_i), \ i = 1, \cdots, N.$$

Here in this section, by giving a instability result, we show that those proofs based only on the decreasing property of the energy function are not complete. In order to prove stability, some additional conditions, like (15) or (17), should be met otherwise the system can be unstable even if the energy function is decreasing.

The function $\mathcal{E}(u)$ is not a Lyapunov function unless it is modified as (2). The modified energy function $E(u)$ can be taken as a Lyapunov function to analyze the stability of the system (1) when the connection matrix is symmetric. It can also be used in Theorem 5 below to get some conditions under which the system (1) is not stable.

Theorem 5 *Assume* $g_i(x) = \frac{1}{2}(1 + tanh(\lambda_i x))$, *where* $\lambda_i = \frac{\lambda}{R_i}$, $i = 1, \cdots, N$, *and the conditions below hold, then the origin* $\overline{u} = (0, \cdots, 0)$ *is an unstable equilibrium of the system (1):*

(A1) $\sum_j T_{ij} + 2I_i = 0, \quad i = 1, \cdots, N,$

(A2) $\mu_{max} > 0$, $\lambda > \frac{2}{\mu_{max}}$ *and* $\lambda \neq \frac{2}{\mu_i}$, $i = 1, \cdots, N$, *where* μ_i, $i = 1, \cdots, N$ *are the eigenvalues of* (T_{ij}) *and* $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N$.

Proof: see Appendix 3.

Combining the results in this section and Section 2.4, we see that by adjusting the gain factor λ we can change the dynamic behavior of the neural network from stable to unstable. In the next two sections, we shall see more effects of the gain factor on the stochastic stability and oscillation of the neural network.

Corollary 1 *Assume that $g_i'(x) > 0$ and the condition (19) holds. Then all equilibria of the system (1) are exponentially stable.*

Note that when one uses the condition (18), one needs to know $g_i'(\overline{u_i})$, the derivative of the activation function at the equilibrium. This is inconvenient since one needs to know the equilibrium in advance. On the other hand, the condition (19) only involves the maximum of the derivatives within the region defined by the network parameters.

When the condition (5) holds, the condition (15) and (18) are implied by (6) and (7) respectively. Therefore, we have the following corollary.

Corollary 2 *Under the conditions of Theorem 1 or Theorem 2, and the condition (14), the system (1) has a unique equilibrium which is exponentially stable.*

In the above theorems and corollaries, the I/O functions of the amplifiers can be any function as long as they satisfy the conditions in Theorem 1. In particular, we can take $g_i(x) = \frac{1}{2}(1 + tanh(\lambda_i x))$, where $\lambda_i = \frac{\lambda}{R_i}$ and tanh(x) is the hyperbolic tangent function. λ is called a gain factor of the amplifier. In this case, $\sigma_i = \frac{\lambda}{2R_i}$ and the conditions of Theorem 1 hold under the following assumptions which do not involve the resistance R_i:

$$\mu_{max} \leq 0 \quad or \quad 0 < \lambda < \frac{2}{\mu_{max}}. \tag{20}$$

The gain factor λ plays an important role in determining the dynamic behavior of the neural network. For any synaptic connection matrix (T_{ij}), if the inequality $\mu_{max} \leq 0$ does not hold, the other inequality in (20) can be satisfied by a sufficiently small gain factor.

The inequalities in (20) put some constraints on the gain factor to guarantee the uniqueness and the exponential stability of the equilibrium. It is shown in Figure 2 that the activation functions are nearly linear when the gain factor is close to zero. Theoretically, those constraints give no lower bound for the gain factor as long as it is positive, and give an upper bound only when $\mu_{max} > 0$.

By Corollary 2, under the conditions of Theorem 1 or Theorem 2, we can set up a map: $f: \ R^N \to R^N$. For each $I \in R^N$, define $f(I) = \overline{u}$ to be the unique and exponentially stable equilibrium of the system (1) corresponding to the input currents $I = (I_i)$.

Let $u^1 = f(I^1)$ and $u^2 = f(I^2)$ be two equilibria of the system (1). If $u^1 = u^2$, by (3), we know that $I^1 = I^2$, i.e. the map f is a one-to-one map.

The map f is the inverse map of the map defined by (21) below

$$I_i = -\sum_j T_{ij} g_j(\overline{u}_j) + \frac{\overline{u}_i}{R_i}, \quad i = 1, \ldots N. \tag{21}$$

If $\{g_i(u)\}_{i=1,N}$ is a set of smooth functions, so is the map f.

To apply the stability result in [15] to the system (1), we define the following test matrix $D = (d_{ij})$:

$$d_{ij} = \begin{cases} \frac{1}{R_i \sigma_i} - T_{ii}, & i = j \\ -|T_{ij}|, & i \neq j \end{cases}$$

Assume that the hypothesis (5) is true and the matrix D is an M-matrix, i.e. all successive principal minors of D are positive, then the system (1) is asymptotically stable by the

By applying Gerschgorin's circle theorem, we can get the following sufficient condition for (15)

$$T_{ii} + \frac{1}{2}\sum_{i\neq j} |T_{ij} + T_{ji}| < min_i\{\frac{1}{R_i g_i'(\overline{u_i})}\}, \quad i = 1, \cdots, N. \tag{17}$$

As explained earlier, a condition of form (17) related to the elements of the connection matrix is more desirable from a computational viewpoint for a large network than one of the form (15) which requires calculation of the maximum eigenvalue.

Without using Gerschgorin's circle theorem, we can get another constraint (18) below which is less restrictive than the constraint (17) in the sense that the constraint (18) is easier to be met than (17):

$$T_{ii} + \frac{1}{2}\sum_{i\neq j} |T_{ij} + T_{ji}| < \frac{1}{R_i g_i'(\overline{u_i})}, \quad i = 1, \cdots, N. \tag{18}$$

Similar to the proof of Theorem 3, we can prove that under the above constraint on the connection matrix, the equilibrium $\overline{u}$ is exponentially stable. We choose the same Lyapunov function as the proof of Theorem 3 but estimate the derivative of the function in a different way. Let

$$a_{ij} = \begin{cases} |T_{ij} + T_{ji}|, & \text{if } i \neq j, \\ 0, & \text{if } i = j. \end{cases}$$

Since (a_{ij}) is a symmetric matrix, we have the equality $\frac{1}{2}\sum_{ij} a_{ij}(y_i^2 + y_j^2) = \sum_{ij} a_{ij} y_i^2$. This is applied to estimate the quadratic form

$\sum_{ij} y_i T_{ij} y_j = \frac{1}{2}\sum_{ij} y_i (T_{ij} + T_{ji}) y_j$

$\leq \sum_i T_{ii} y_i^2 + \frac{1}{2}\sum_{ij} a_{ij}|y_i||y_j| \leq \sum_i T_{ii} y_i^2 + \frac{1}{4}\sum_{ij} a_{ij}(y_i^2 + y_j^2)$

$= \sum_i T_{ii} y_i^2 + \frac{1}{2}\sum_{i\neq j} |T_{ij} + T_{ji}| y_i^2 = \sum_i (T_{ii} + \frac{1}{2}\sum_{j\neq i} |T_{ij} + T_{ji}|) y_i^2$.

Therefore

$D_{(16)}V(x) \leq -\sum_i \frac{\alpha_i x_i^2}{R_i} + \sum_{ij} \alpha_i x_i T_{ij} \alpha_j x_j + b_0 \|x\|_1 \|x\|^2$

$\leq \sum_i (-\frac{1}{\alpha_i R_i} + T_{ii} + \frac{1}{2}\sum_{j\neq i} |T_{ij} + T_{ji}| + b_1 \|x\|_1) y_i^2$

where $y_i = \alpha_i x_i$, $\alpha_i = g_i'(\overline{u}_i)$ and b_0 and b_1 are positive constants.

Hence, we have the following theorem

Theorem 4 *Let $\overline{u}$ be an equilibrium of the system (1). Assume that $g_i'(\overline{u_i}) > 0$ and the condition (18) holds. Then the equilibrium $\overline{u}$ is exponentially stable.*

If $\overline{u}$ is an equilibrium of the system (1), $\overline{u}_i = R_i(\sum_j T_{ij} g_j(\overline{u_j}))$, then $|\overline{u}_i| \leq R_i(\sum_j |T_{ij}| + |I_i|)$, $\quad i = 1, \cdots, N$, i.e. all equilibria of the system (1) are located in a bounded region.

Let $\delta_i = \max\{g_i'(x_i) : |x_i| \leq R_i(\sum_j |T_{ij}| + |I_i|)\}$, $\quad i = 1, \cdots, N$. Then the following condition is a sufficient condition for the condition (18):

$$T_{ii} + \frac{1}{2}\sum_{i\neq j} |T_{ij} + T_{ji}| < \frac{1}{R_i \delta_i}, \quad i = 1, \cdots, N. \tag{19}$$

Usually δ_i is smaller than σ_i. If $\delta_i = \sigma_i$ for all i, then (19) becomes (7).

2.4 Stability of equilibria

In section 2.2, we derived results for the existence and uniqueness of the equilibria. Here, we determine the conditions under which the equilibria of the system (1) are exponentially stable.

In this section, we assume that $\{g_j(x)\}$ are twice differentiable functions with bounded second derivative, i.e. there is a constant b such that

$$|g_j''(x)| \leq b, \ j = 1, \ldots, N, \quad for\ all\ x \in R^N. \tag{14}$$

Theorem 3 *Let $\overline{u}$ be an equilibrium of the system (1). Assume that $g_i'(\overline{u_i}) > 0$ and the following condition*

$$\mu_{max} g_i'(\overline{u_i}) < \frac{1}{R_i}, \ i = 1, \cdots, N, \tag{15}$$

holds. Then the equilibrium $\overline{u}$ is exponentially stable.

Proof:
By Theorem 1, the system (1) has a unique fixed point $\overline{u}$ which satisfies the equation (3). Set $x_i = u_i - \overline{u}_i$. The set of equations (1) become

$$C_i \frac{dx_i}{dt} = \sum_j T_{ij}(g_j(x_j + \overline{u}_j) - g_j(\overline{u}_j)) - \frac{x_i}{R_i}, \quad i = 1, \cdots, N. \tag{16}$$

Note

$$g_j(x_j + \overline{u}_j) - g_j(\overline{u}_j) = g_j'(\overline{u}_j)x_j + \frac{1}{2} g_j''(\xi_j) x_j^2,$$

where ξ_j is a real number between $x_j + \overline{u}_j$ and $\overline{u}_j$.

Let $\alpha_i = g_i'(\overline{u}_i)$, $V(x) = \frac{1}{2}\sum_i C_i \alpha_i x_i^2$, and $Y = (y_1, \cdots, y_N)^\tau = (\alpha_1 x_1, \cdots, \alpha_N x_N)^\tau$. Then
$D_{(16)}V(x) = \sum_i \alpha_i x_i (\sum_j T_{ij}(g_j(x_j + \overline{u}_j) - g_j(\overline{u}_j)) - \frac{x_i}{R_i})$
$= \sum_i \alpha_i x_i (-\frac{x_i}{R_i} + \sum_j T_{ij}(g_j'(\overline{u}_j)x_j + \frac{1}{2} g_j''(\xi_j)x_j^2))$
$\leq -\sum_i \frac{\alpha_i x_i^2}{R_i} + \sum_{ij} \alpha_i x_i T_{ij} \alpha_j x_j + b_0 \|x\|_1 \|x\|^2$
$\leq -\sum_i \frac{\alpha_i x_i^2}{R_i} + \mu_{max}\|Y\|^2 + b_0 \|x\|_1 \|x\|^2$
$\leq \sum_i (-\frac{1}{\alpha_i R_i} + \mu_{max} + b_1 \|x\|_1)\alpha_i^2 x_i^2$, where $b_0 > 0$ and $b_1 > 0$ are constants.

By the condition $\mu_{max} < \frac{1}{R_i \alpha_i}$, there exists a $\delta > 0$ such that $-\frac{1}{\alpha_i R_i} + \mu_{max} + b_1 \delta < 0$, for $\|x\| < \delta$, and $i = 1, \ldots, N$.

Hence, $DV_{(16)}(x) \leq -k\|x\|^2$, for $\|x\| < \delta$, where the constant $k > 0$.

On the other hand, it is obvious that there exist positive constants k_2 and k_3 such that $k_2\|x\|^2 \leq V(x) \leq k_3\|x\|^2$.

By Theorem 9.10 in Ref. [17] (chapter 5), the zero solution of the system (16) is exponentially stable, i.e. there exist positive constants d_1 and $d_2 > 0$ such that

$$|u_i(t) - \overline{u}_i| = |x_i(t)| \leq d_1 |u_i(0) - \overline{u}_i| e^{-d_2(t - t_0)}, \quad when \ \ |u_i(0) - \overline{u}_i| < \delta,$$

for $t \geq t_0$.
Q.E.D.

Applying the result in Ref. [12], we know that if $|g_i'(x)| \leq \beta$ for all $x \in R$ and $i = 1, \cdots, N$, and

$$\beta \|T\| < 1 \tag{9}$$

then the system (8) is a contractive network and has a unique equilibrium, which is globally asymptotically stable.

The global asymptotical stability result is also given in Ref. [9] for a very general network whose dynamical equation has the form:

$$\frac{dx_i}{dt} = F_i(x_1, \cdots, x_n, I_1, \cdots, I_n). \tag{10}$$

Let $< x, y >$ denote the inner product of vectors x,y, and $DF(y)$ the Jacobian matrix of the function F. Hirsch's result tells us that the system (10) is globally asymptotically stable if there is a constant $\mu > 0$ such that

$$< DF(y)x, x > \leq -\mu \|x\|^2 \quad for \quad all \quad x, y \in R^n. \tag{11}$$

This result implies the uniqueness of the equilibrium because of the global stability. But it is difficult to verify the assumption (11). One sufficient condition for it is given in Ref. [9] for additive networks. According to Hirsch's result, the system (1) is globally asymptotically stable provided $|g_i'| \leq \beta$ and the condition (12) below holds:

$$\beta(T_{ii} + \frac{1}{2}\sum_{i \neq j}\{|T_{ij}| + |T_{ji}|\}) < \frac{1}{R_i}, \quad i = 1, \cdots, N, \tag{12}$$

Now we can compare our result with those of Kelly and Hirsch. Our assumption (6) is less restrictive than Kelly's assumption (9) when μ_{max} is not positive. For example, for $T = (T_{ij}) = \begin{pmatrix} -2 & 2 \\ 0 & -2 \end{pmatrix}$, we have $\mu_{max} \leq 0$, and $\|T\| = \sqrt{6 + 2\sqrt{5}}$. By Kelly's result, the system (8) has a unique equilibrium when $\beta < \frac{1}{\sqrt{6+2\sqrt{5}}}$, where β is a common gain coefficient for all activation functions. But according to Theorem 1, this restriction is not necessary because the assumption (6) is true for any gain coefficients for this example.

Hirsch's assumption (12) is much stronger than our assumption (7). For instance, for an anti-symmetric matrix, where $T_{ij} = -T_{ji}$, the assumption (7) is

$$T_{ii} < \frac{1}{R_i \sigma_i}, \quad i = 1, \cdots, N. \tag{13}$$

When T_{ii} is zero or negative the above assumption becomes true for any gain coefficients and resistances.

We can apply the uniqueness results in Section 2.2 to analyze the Master/Slave model in Ref.[13]. Let (T_{ijkl}) be the connection matrix of the Master net in Ref.[13]. We can prove that the largest eigenvalue of the matrix (T_{ijkl}) is not positive. So the Master net has a unique equilibrium which is exponentially stable. From this equilibrium, we can find the best connection matrix for the slave net (see Ref.[21]).

and

$$\mu_{max} < \frac{1}{R_i \sigma_i} \tag{6}$$

where $\sigma_i > 0$, $i = 1, \ldots, N$, *are gains of the activation functions, and* R_i , $i = 1, \ldots, N$, *are resistances in equation (1).*

Then the system (1) has a unique equilibrium.

Proof: see Appendix 1.

The novelty of our approach is to apply Brouwer's fixed point theorem [4] and the homotopy method [4] to get the existence and uniqueness of the equilibrium on the whole state space. It should be noted that the assumptions on the activation functions in Theorem 1 are very general. The following typical sigmoid functions satisfy these assumptions: $g_i(x) = \frac{1}{2}(1 + tanh(2\sigma_i(x - \theta_i))$, with parameters σ_i and threshold θ_i, $i = 1, \cdots N$. Both $g_i'(x)$ and $g_i''(x)$ are bounded and the gain of $g_i(x)$ is σ_i. Note that σ_i represents the gain of the amplifier at the threshold.

Figure 2 shows the effect of the gain. The larger the the gain the steeper the function becomes at the threshold $\theta = 0$.

Theorem 1 says that if the condition $\mu_{max} \leq 0$ is true then we have the existence and uniqueness of the equilibrium otherwise, to reach the same conclusion, we need the constraint $\mu_{max} < \frac{1}{R_i \sigma_i}$ which can be satisfied for sufficiently small resistances R_i or sufficiently small gain coefficients σ_i, $i = 1, \ldots, N$. Note the condition $\mu_{max} \leq 0$ is included in the inequality (6).

If one is to apply the condition (6) directly, one needs to compute μ_{max}. But the computation of μ_{max} could be expensive for a large network. It is more convenient to use the Theorem 2 below which puts the constraints directly on the elements of the connection matrix.

Theorem 2 *Under the assumption (5) and the condition below*

$$T_{ii} + \frac{1}{2} \sum_{i \neq j} |T_{ij} + T_{ji}| < \frac{1}{R_i \sigma_i}, \quad i = 1, \cdots, N, \tag{7}$$

the system (1) has a unique equilibrium.

Proof: see Appendix 2.

Note the condition (7) cannot be obtained by Gerschgorin's circle theorem if the numbers $\{R_i \sigma_i\}$ are not the same for all values of i. In general, neither the condition (6) includes (7) nor (7) includes (6). When all numbers $\{R_i \sigma_i\}$ are equal for all values of i, the condition (7) is the sufficient condition for (6).

Each of the conditions (6) and (7) is for ensuring the uniqueness of the equilibrium. By Brouwer's fixed point theorem , there always exists at least one equilibrium even when any of these two conditions is not satisfied (see Appendix 2).

2.3 Comparison with previous results

Kelly(1990) analyzed the following system which is a particular form of (1), namely:

$$\mu \frac{du_i}{dt} = \sum_j T_{ij} g_j(u_j) - u_i + I_i, \quad i = 1, \ldots N. \tag{8}$$

Section 2.5 and Section 3. A neural oscillator formed by a closed chain of neurons is introduced in Section 4, and both its stability and oscillation are studied.

2 Stability Analysis

2.1 Basic notation and assumptions

The following conventions are used in this paper:

X^τ	transpose of a matrix X,
$\|x\|_1 = \sum_{i=1}^n \|x_i\|$	1-norm of a vector $x \in R^N$,
$\|x\| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$	2-norm of $x \in R^N$,
$\|A\| = \max_{x \neq 0} \frac{\|Ax\|}{\|x\|}$	norm of a matrix A which is also the largest eigenvalue of the matrix $A^\tau A$,
$det(A)$	determinant of a matrix A
μ_{max}	the largest eigenvalue of the symmetric matrix $\frac{T+T^\tau}{2}$ where $T = (T_{ij})$ is the connection matrix
$\sigma_i = \max_x g_i'(x)$	gain of the i-th amplifier.

Given a differential equation labeled with (S)

$$\frac{dx}{dt} = (F_1(x), \cdots, F_N(x))^\tau \qquad (S)$$

and a differentiable function $V : R^N \to R$, define $DV_{(S)} = \sum_{i=1}^N \frac{\partial V}{\partial x_i} F_i(x)$ which is the derivative of the function V along trajectories of the system (S).

In this paper, the connection matrix $T = (T_{ij})$ is real and may not be symmetric. The I/O or activation functions $\{g_i(u)\}$ of the neurons are differentiable, bounded and increasing functions.

2.2 Uniqueness of equilibrium

A point $\overline{u}$ in R^N is called an equilibrium of the system (1) if this point $\overline{u}$ satisfies the equation (3). The knowledge of the network equilibria is very useful in characterizing the neural dynamics. Generally, a non-linear system may not have an equilibrium. If it has one, it may not be unique. The uniqueness of the network equilibrium is discussed in Refs. [14] and [19] on some specific regions in the state space. The following Theorem 1 gives the existence and uniqueness of the equilibrium on the whole state space.

Theorem 1 *Assume that the derivative of each activation function is positive and bounded,*

$$\textit{for all } x \in R^1,\ 0 < g_i'(x) \leq \sigma_i, \quad i = 1, \ldots, N \qquad (5)$$

these constraints for neural networks and get the qualitative properties including the exponential stability and the instability of the equilibria. The M-matrix concept is used to express some of the constraints in Ref. [15] for an asymptotically stable network. The linearization method for ordinary differential equations is used in Ref. [9] to find a constraint under which the network is globally asymptotically stable. In this paper, a constraint on the connection matrix is found which is weaker or less restrictive than those in Refs. [15] and [9] for an asymmetric network. For the symmetric network, our constraint is the same as those in Refs. [15] and [9]. If a neural network satisfies this constraint, it has a unique and exponentially asymptotically stable equilibrium. We shall compare these results in Section 2.3.

All neural network learning methods apply certain rules to modify the parameters in the network. Some learning methods based on Hebb's rule or the δ-rule [18] are to update the connections or synaptic strengths in the network. After training, some constraints are put on the connections to acquire the desired neural dynamics. But the learning method suggested by Midtgaard and Hounsgaard in Ref. [16] is to modify the postsynaptic response rather than synaptic strengths. The postsynaptic responses of the neurons are determined by the resistances and gains of the amplifiers. Here the gain of each amplifier is the maximum value of the derivative of the activation function. Some constraints on these parameters have been found in the present paper under which the neural network has a unique and exponentially stable equilibrium. These constraints can be satisfied by just adjusting the parameters of each neuron.

The number of equilibria of the neural network relates to its storage capacity. Some neural networks may have infinite associative memories. The network described by the differential equation (4) includes examples in which the neural network can have nondenumerably many equilibria [2].

The uniqueness of the equilibrium is discussed in Refs. [14] and [19] on each of the 2^N quadrants in the state space. According to these results, under certain conditions the maximum number of stable equilibria or the capacity of the network (1) is 2^N. In this paper, the uniqueness of the equilibrium will be analyzed on the whole state space instead of some specific regions in the state space.

If the neural network is viewed as an associative memory, the more equilibria the neural network has, the greater the storage capacity. But when we use the neural network to solve optimization problems, we want to design a neural network with less equilibria. For example, in Refs. [11] and [20], if an equilibrium is unique, it will be the global minimum point of the related energy function. We can also apply the uniqueness result to analyze the Master/Slave model and find out whether the master net can find the best connection matrix for the slave net.

To analyze the dynamic behaviors of the Hopfield graded response neural network, we apply the Lyapunov method to find some constraints on the connection matrix which itself may be asymmetric. These constraints can be satisfied through the adjustment of the gains of the amplifiers and the resistances in the neural net circuit.

The paper is organized as follows. Some basic notation and assumptions are introduced in Section 2.1. The uniqueness of the equilibrium on the whole state space is discussed in Section 2.2. Our uniqueness results are compared with some published results in Section 2.3. The stability, instability and stochastic stability are analyzed in Section 2.4,

and I_i the input current. The resistance R_i depends on the connection matrix: $\frac{1}{R_i} = \sum_j |T_{ij}| + \frac{1}{r_i}$, where r_i is the input resistance to model the cell membrane impedance. This model is widely used to simulate associative memory and solve some optimization problems [10], [11].

In this paper, the stability of the system (1) is analyzed by using the Lyapunov method. When the (T_{ij}) is symmetric, the following function, which is the modification of the energy function in Ref. [10], can be taken as the Lyapunov function for the system (1):

$$E(u) = -\frac{1}{2}\sum_{i,j} T_{ij}(V_i - \overline{V}_i)(V_j - \overline{V}_j) + \sum_i \frac{1}{R_i}\int_{\overline{V}_i}^{V_i} (g_i^{-1}(v) - g_i^{-1}(\overline{V}_i))dv, \qquad (2)$$

where $V_i = g_i(u_i)$ and $\overline{V}_i = g_i(\overline{u}_i), i = 1, \ldots N$, and $\overline{u}$ is an equilibrium of the system (1), i.e. $\overline{u}$ satisfies the following equations:

$$\sum_j T_{ij} g_j(\overline{u}_j) - \frac{\overline{u}_i}{R_i} + I_i = 0, \quad i = 1, \ldots N \qquad (3)$$

The energy function used in Ref. [10] is not a Lyapunov function in the mathematical sense unless it is modified as (2). By calculating the derivative of the function E(u) along the trajectories of the system (1), we know that the energy function E(u) is decreasing along the trajectories when the connection matrix is symmetric. But a decreasing energy function does not mean that the equilibria are stable. A instability result of the neural network will be given in Section 2.5 to explain this. More instability examples can be found in Ref. [15]. These examples suggest to us that decreasing energy alone cannot assure the stability of the equilibria. To assure the stability of a system, the Lyapunov function should have more properties than just having a decreasing energy function.

Cohen and Grossberg(1983) analyzed the following competitive dynamic system which is a more general neural network model than the system (1):

$$\frac{dx_i}{dt} = a_i(x_i)(b_i(x_i) - \sum_{j=1}^{n} C_{ij} d_j(x_j)), \quad i = 1, ..., n. \qquad (4)$$

Some of their results in Ref. [2] cannot be applied to the system (1). This was discussed in Ref. [14]. Theorem 1 in Ref. [2] (PP.820) also deals with a symmetric neural network. But the symmetry assumption is not plausible in many neural network applications. We will see that the symmetry is not essential for the stability of the neural network. Moreover, the simulation results on the Master/Slave network in Ref. [13] suggest that the asymmetric synaptic connections offer additional potential for the neural network to model general dynamic systems. It is well known that the dynamics of asymmetric neural networks are very complex. Stable, unstable, oscillatory and chaotic behaviors may appear in the dynamics of the neural network depending on the connection structures. Both stability and oscillation will be discussed in this paper. More results about complex and chaotic oscillations can be found in Ref. [2].

The key issue in designing a stable or oscillatory neural network is to control the neural dynamics by putting some constraints on the connection matrix. If a neural network is treated as a large scale dynamic system, the theories in this area can be applied to find

Exponential Stability and Oscillation of Hopfield Graded Response Neural Network

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Abstract

Both exponential and stochastic stabilities of the Hopfield neural network are analyzed. The results are especially useful for analyzing the stabilities of asymmetric neural networks. A constraint on the connection matrix has been found under which the neural network has a unique and exponentially stable equilibrium. Given any connection matrix, this constraint can be satisfied through the adjustment of the gains of the amplifiers and the resistances in the neural net circuit. A one-to-one and smooth map between input currents and the equilibria of the neural network can be set up. The above results can be applied to the Master/Slave net to prove that the master net can find the best connection matrix for the slave net. For the neural network disturbed by some noise, the stochastic stability of the network is also analyzed. A special asymmetric neural network formed by a closed chain of formal neurons is also studied for its stability and oscillation. Both stable and oscillatory dynamics are obtained in the closed chain network through the adjustment of the gains and resistances of the amplifiers.

1 Introduction

The collective computational capabilities of neural networks, such as optimization, association and oscillation, rely on the dynamic behaviors of the neural networks. The qualitative analyses of neural networks, including the the analyses of stability and oscillations, characterize the dynamics of these networks.

The Hopfield graded response neural network is based on an analog circuit which consists of some amplifiers, resistances and capacitors as shown in Figure 1 reproduced from [10].

The I/O functions of the amplifiers, which are called activation functions, are sigmoid monotonically increasing functions. In this circuit, a formal neuron is modeled by a subcircuit composed of an amplifier, a reverse amplifier, a capacitor and a resistance. The neural network is made from the formal neurons connected by resistances which model synapses. The reverse amplifiers are used to model the inhibitory synapses. The dynamics of the neural network is described by the following differential equation:

$$C_i \frac{du_i}{dt} = \sum_j T_{ij} g_j(u_j) - \frac{u_i}{R_i} + I_i, \quad i = 1, \ldots N, \tag{1}$$

$$u(t_0) = u^0 \in R^N,$$

where N is the number of neurons, $u = (u_i)$ the vector, u_i the average membrane potential in i-th neuron, (T_{ij}) are the synaptic connections among the neurons, C_i is the capacitor

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