PROFESSOR: Welcome back. Today, we will do two problems involving many oscillators, or at least several oscillators, coupled to each other.

Now, you will not be surprised, from past experience, that in the situation with several oscillators, we are going to end up with many equations and a lot of grindy, tough solving of equations. Already, with a single harmonic oscillator, for example, with damping, the equations have been pretty horrendous. Now, we considered a couple of oscillators with all the details, with damping, driven, et cetera, and it really becomes complicated.

Now, what we are trying to do by doing problems is to get a feel, an understanding what goes on in a given situation, so one wants to focus on new features and try to simplify the mathematics as much as possible. And so in the problems are we doing now, I am intentionally making them even more ideal than before.

Consider the first problem. Here is a highly idealized system which has 3 degrees of freedom. What we have is we have a single mass, 2m, two identical springs attached to two masses, each of mass m. This is completely idealized.

We are assuming there's no friction, that these springs are ideal springs. In other words, they're massless, obey Hooke's laws with constant k. We are assuming the system is constrained, so it can only move in one direction, it can't move up and down or in or out from the board. So as I say, it's an idealized situation.

And the question we want to answer is, for such a system, what are the normal mode frequencies? And you know from lectures given by Professor Walter Lewin that when you have coupled oscillators, in this case, 3, all right, one finds that there are very special oscillations, which we call normal mode oscillations, in which every

part of the system is oscillating with the same frequency and phase.

So first, we want to find out for this system, what are those three normal mode frequencies? And secondly, suppose I take this system, place these masses at some arbitrary positions, some maybe even moving, and let go, how would I predict where each mass would be at some later time? So those are the two questions I want to answer today for this problem.

Now, in general, coupled oscillator problems are quite difficult. But there are situations in which one can use logic alone, well, logic and our understanding of coupled oscillators, to do most of the solution by hand-waving and not do any calculations.

In this first example I'm doing in this problem, since the system has a lot of symmetry, I know from experience that under those conditions it might be possible to solve this, as I say, by logic alone and our understanding of coupled oscillators.

So the issue is, can I guess what will be the motion in the normal modes? And the answer is, actually, if you stop and think for a second, at least the first two modes you can figure out rather easily. So here I'm showing you schematically what would be one of the modes of oscillation, one of the normal modes.

Imagine that I first hold this mass fixed and I displace these symmetrically, one to the right, one to the left, and I let go. What you will find is, by symmetry, this spring will always be stretched by the same amount as this is compressed, and so the force on this mass will be 0.

This spring will be pushing this mass exactly by the same amount as this one will be pulling it and vice versa. So there will never be any net force on this mass, and so they'll stay put. And these two will now behave as a simple harmonic oscillator consisting of a spring attached to a fixed wall and the mass m.

So that's what this one will be doing, and that's what this will be doing. It will be out of phase by 180 degrees. Or what one normally says, it'll be in phase but the amplitude will be minus the other one. This, by now, you've had enough experience

and I've done it here before, I can just write what will be that frequency of oscillation here, as we've seen many times.

And, by the way, throughout my solving the problems here, I do not differentiate frequency and angular frequency, that you can see which I'm talking about is whether I'm using omega or F. Omega is the angular frequency, and there's just a fact of 2 pi between the two. But I'll use, interchangeably, I'll call this frequency. So this frequency of one of the normal modes is just k/m. So we found one normal mode frequency.

Let's look for another one. We stare at this and we see, sure, there is another one. Another one is where I take, simply, these two masses and simultaneously pull them out, in this mass, away and let go. Effectively, these two will look like that. It's as if I had a mass of 2m attached to a mass of 2m by two springs, each of spring constant k.

And as I pull this back, this will oscillate like this. Everything, both this mass and this mass, will be oscillating with the same frequency, same phase, although there's this minus sign. I can always replace the 180-degree phase shift by saying the amplitude is minus. So those are identical statements. So you have these two masses oscillating like that. OK?

By the way, I've said that in the normal mode I want everything to move with the same frequency. Some of you may argue, hold on. Did I cheat on you? Here, this mass is not oscillating. So am I contradicting myself? And the answer is no. It's a diabolical answer.

But I'll say, no, this is oscillating with this frequency, but with 0 amplitude. And you can't tell me that I'm wrong. It's not moving because the amplitude is 0, but it is oscillating with the same frequency.

So now, what is the frequency of this oscillation? Again, I can do it in my head. I can imagine that the middle of this spring is not moving. And so this mass is a half a length of the spring and a mass 2 attached to it oscillating, and I can calculate the

frequency of oscillation of that.

Or if you prefer, I can do the following. I can move this mass out a distance dx and this one dx to the other side and calculate what will be the restoring force. Well, now each one has a spring constant k. There is two of them, so that's 2k. Although this mass has been moved by dx, this by dx, the springs have been extended by twice dx, so it's 2 dx. So that's the total restoring force by Hooke's law, which is equal minus 4k dx. All right?

And so this will behave like a single mass, like this system, a single mass with a single spring where the spring has a spring constant 4k. The mass is 2m. And so the angular frequency, or frequency of this mode is 2k/m. So we've found two of them.

Now, we know from our studies of coupled oscillators that for a system of 3 degrees of freedom, so it would be three masses, there are three normal modes. What is the third normal mode? And I intentionally did not write it down here because I wanted you for a second to think. Well, how else can this system move such that every mass is moving in phase with the same frequency? All right?

And I tell you, when I first saw it, it took me a long time to figure out, and most people don't find it. And yet the answer is extremely simple. This answer is there is one more normal mode. It is one of almost infinite amplitude, but 0 frequency. Just the opposite to here, where this was 0 amplitude, OK, and the angular frequency doesn't even matter. Here it has very large amplitude, infinite in fact, but 0 frequency.

What does such motion look like at any instant of time? It's moving with the same velocity. So if I take and if I consider the motion where each one of these is moving uniformly to the right with a constant velocity, that is a normal mode with 0 frequency. So the last one is omega C is equal to 0. OK? So we have now found the three normal modes. OK.

All right. Now that I've got a piece of chalk, let's continue. So we've answered the

first part of the question, what are the normal mode frequencies of the system? And I found you those three normal modes.

Now, the next question we want to answer is, if, at any given instant of time, I know the positions and the velocities of the three masses, can I predict what will happen at the end? That's equivalent of saying, can I write equations for the positions of the masses as a function of time?

For that I need to define for myself a coordinate system. So I'll say those are those three masses. All the motion is along the x direction, so there's just one variable, one variable for each mass, x.

So the position of the first mass I'll call x1, position of the second mass x2, and of the third is x3. And the origin of coordinates I will take to be through the center of each mass at a time when these are in equilibrium. In other words, the spring is unstretched, et cetera. OK.

So I can now, using the information we've obtained, write down the description of each mass in each mode. So in mode A, I know that the x1 is 0 all the time. It's not moving, OK?

Now, in a normal mode, that's what we mean by a normal mode, each mass is moving with the same frequency and the same phase, and it's oscillating. So there will be some sinusoidal function of the mode frequency t plus this phase times some arbitrary amplitude.

The other one, the last one, will also be oscillating in the normal mode, so everything will be the same. It will be cosine of omega A t plus phi A with some other amplitude.

But from our analysis here, we know that these two are not both arbitrary. Because we said that in the normal mode, it was the case where the 2m mass was stationary, but these were going like this. That was the first mode. So whenever this one was going to the right a distance A, this one was going to the left a distance A.

So the overall normalization is arbitrary. It can be anything. But these are not independent of the other, because we discovered that in that mode, this was the frequency, and the amplitudes were opposite to each other. So this describes the situation for this system if it's oscillating in the first normal mode. It's the most general description of that.

Next, I will describe what it will do in the second normal mode. And now I can go much quicker. This time, it'll have the second normal mode frequency, different phase. But these have to be the same because all of these masses are moving in the same normal mode, same normal frequency. These amplitudes, overall, it's arbitrary. I can make it anything. It'll satisfy the equations.

But I know that in the second normal mode, this one is moving in that direction and those two in the opposite direction with the same magnitude. So these two have to have the same size and magnitude, but this has to be opposite. So this describes the motion in the second mode.

And finally, in the third normal mode, we said that it was 0 frequency. All right? And all that was happening, the three masses were moving with uniform velocity. So the positions will be some constant plus the velocity of that system. So this describes the system in the third mode, and these are the other two.

Now, I know that, in general, each one of these masses can move in a superposition of the normal modes. So the most general expression for x1 of t has to be of the form which is the sum of its possible motion in each of the modes.

So x1, so this is the mass 2m, will have possible motion in the first mode, which is 0. All right? So I'll write 0 plus B times cosine omega B t plus phi B, all right, plus C plus v. So that will be the most general description of the motion of the big mass, the 2m one. The B is arbitrary, this phase is arbitrary, this is arbitrary, and v. Those quantities will be determined by the initial conditions.

Well, how about x2? Well, I know that if x1 is 0, x2 is A cosine omega A t plus phi A. When the x1 is this, then x2 is minus B cosine omega B t plus phi B, OK, and plus C

plus v. OK?

And finally, x3 of t, again now just following the same pattern, this is minus A cosine omega A t plus phi A. This one is minus B cosine omega B t plus phi B plus C plus v.

And I'm almost home. I have described the motion of each one. For each particle, it is oscillating in a superposition of its normal modes. And the amplitudes are arbitrary within the constraint that the ratios between them is determined by the constraints of the system.

So now the question is, can I predict the future? Well, in order to predict the future, say, where will particle 3 be at time t? I need to know the value of these arbitrary constants. Omega is not an arbitrary constant. We found it. Omega A is here. OK? This is arbitrary, this is arbitrary, this, and that.

So there are six arbitrary constants. How do I find them? They are determined by the initial conditions. And you know, fortunately, or else it would mean we've made a mistake, we have three conditions. At the beginning, someone has to tell me where each mass was, the location, and with what velocity is it moving at that time.

So I can write these three equations at, say, with t equals 0, at t equals 0, and equate this to the position of x1 of t equals 0, wherever it is. When the t is 0, I make it equal to the position of x2 at t equals 0. When t is 0, this tells me has to be equal to where x3 is. That's three equations.

I can differentiate each with respect to time to get the velocities, and likewise, get three more equations. So I'm going to end up with six algebraic equations. And you, as well as I, know that if we have six algebraic equations, I can solve for six unknowns. And that will give me those six unknowns.

So once I've used the initial conditions, I can then find these arbitrary constants. They're no longer arbitrary for a specific situation. And I can predict the future, what will happen in this case. OK? So that's the way one would solve a problem when you are lucky enough that there is sufficient symmetry in the original situation so you can guess the normal modes.

The next problem I'll do, I will do one where you cannot guess the normal modes. What do you do under those circumstances? And you won't be surprised that it's going to be mathematically much, much harder, but conceptually no harder.

So now we'll move to the next problem. Here it is. But in order to be able to do this problem, I need more board space. So we're going to erase the boards and continue from there. OK. So now let's go and do the second problem. And it's going to be a problem with no symmetry, so we can't guess the answer like we did before.

And I'll tell you what I decided to do. You remember Professor Walter Lewin in his lectures discussed a double pendulum. In other words, a situation where you had a string, a mass, another string, and mass. He gave you the answer, but he didn't prove it. He said it's hard and lengthy, et cetera. And I thought that would be a perfect example to do here from first principles.

So let me just describe in detail the problem we're trying to solve. What we have is, again, an idealized situation. We have a simple pendulum with a string of length L, attached to it a mass m. Attached to this mass at one end is another string of length L to another mass m. We'll again idealize the situation.

So the assumptions we are making, that this string and this string is massless, which, by the way, is equivalent to saying that it's taut and straight all the time. You can think about that for yourself. Next, these masses are point masses. And we're assuming no friction.

Furthermore, as we've done in the past, we'll assume that all displacements when this is oscillating, at all times the angle that the string makes with the vertical is always such that we can ignore the difference between a sine theta, theta, or [INAUDIBLE] can make the assumption that cosine theta is 0.

By the way, the fact that we'll make this assumption is equivalent to saying that we'll ignore vertical motion of these masses. The motion is sufficiently small that vertically the masses are not moving. We can assume the acceleration vertically is 0, et cetera. OK? This is in the vertical plane. Gravity is down. OK?

And what we are trying to derive for this, predict what are the normal mode frequencies of this. And once we do that, of course, we can use the same kind of technique as we did before. Once we've managed to find the normal modes and the frequencies, we can always write the most general expression. And then using the boundary conditions, initial conditions, predict what this will do as a function of time.

OK, so let's get going. OK. Now, we are not guessing. We are not using logic. We are following the kind of prescription I've told you in the past. This is our description of the situation. And all the new words and ordinary language, we must now translate it into mathematics. We've got to describe this problem in terms of mathematical equations.

So step one is we redraw this and define some coordinate system. So we will say that the angle this first string makes with respect to the vertical is theta 1. This angle the second string makes with respect to the vertical is theta 2. We will take the vertical through this pivot here as our origin of coordinate x. x equals 0 here, is defined here.

From the point of your motion of the masses, it's a one-dimensional problem. The masses only move along the x-axis. So we'll define this distance as x1, and we'll define this distance as x2.

Now we have to use the laws of physics, Newtonian mechanics, to derive the equations of motion for the two masses. So I draw a force diagram separately for the two masses.

So the mass is there, what forces act on that mass? Well, I have to look at the mass and see what's attached to it. There is a string here attached to it. It's taut, so there will be a tension in it. So along this string, there will be tension T1, which will exert a force T1 along this string.

This string is attached to that mass. It has a tension, so it's pulling on this mass. That tension I call T2, so there will be a force along that direction T2. This sits in a

gravitational field. Gravity acts on this mass. There is a force downwards due to gravity, Fg.

The sum of these forces, by Newton's laws, must equal to the mass times the acceleration. That acceleration, in this approximation that is not moving up, is horizontally, and it's the second derivative of x1. Similarly, for the second mass, again, I'll go now faster. There is this mass m, and this tension exerts a force here of T2 and the gravity on it Fg.

The only subtlety is, why did I say this force is equal to that force? Think about it for a second. Why should the two forces be equal? Why is the tension at both ends of the string equal?

And the answer is, actually, a subtle one. It's equal because we made the assumption that the string has no mass, and there can be no net force on an object of 0 mass. Because if there was, that object would disappear, would have an infinite acceleration.

So if you treat the string as a mass, there cannot be net force on it. And so the force of each of these masses must be pulling on this string with exactly the same force equal and opposite. OK? And we're using the third law to equate those forces. So the net result is this T2 is the same as that, but in opposite direction, and this is the mass of that. OK. So these are the force diagrams.

Using now Newton's laws of motion, I can translate this into equations of motion. So let me consider the horizontal motion of each mass separately. First this mass, and so its mass times acceleration is equal to the horizontal force of T2. And remember that this angle here is theta 2. And we see that the force due to T2, and there's a T2 sine theta 2. And the force due to T1 is minus T1 sine theta 1 because it's in the opposite direction.

For this, the only horizontal component is the force horizontal component of T2. And x2 double dot is equal to minus T2 sine theta 2. OK, that's horizontal motion.

Applying Newton's laws of motion to the vertical motion, we said that, because

cosine theta is 0 or approximately 0, the masses are not moving up and down. There's no acceleration of the masses. So the vertical acceleration of the first mass is 0 must be equal to the vertical component of T1, which is T1 cosine theta 1, minus the vertical component of this, which is minus T2 cosine theta 2 minus mg.

All right. Similarly for the second mass, it's 0 is equal to this. 0 must equal to the vertical component of T1 minus the vertical component of T2 minus the force of gravity down. And similarly for the second mass, we know that vertically the acceleration is 0. That must be equal to the vertical component of T2 minus the gravitational force pulling down.

Now I can make use of what we made the assumption-- oops, I see there is an error. If you were in this room, I'm sure you would have corrected me. For very small angles, the sine of an angle is approximately equal to the angle. But for small angles, the cosine is 1. This is an error, and I apologize for that.

But that's the approximation we are making. And it is this approximation which is equivalent to saying that we can ignore the vertical motion. All right. So with the assumption that the cosines are all 1, I can, from these two equations, I clearly derive that T2 must equal to mg, and T1 equals twice mg.

Actually, it makes sense. Imagine this is hanging completely vertically. Obviously, this string, the upper part of the string, is supporting not only this mass but also that. It's supporting 2m masses, while this string is only supporting 1. So that's consistent with what we see, that the top string has twice the tension and the lower one half the tension.

Having determined T1 and T2, we can now go back into our equation, replace the T1 and T2. We also can replace-- we know what sine theta 1 and sine theta 2 are. We can replace those. And we end up-- the two equations of motions are written here. All right? So here is the equation of motion for x1. And the second one, the equation for motion of x2, is written here. OK?

So these are the equations of motion, all right, which we have to solve. Now, to

simplify the algebra, let me define the quantity g/l by omega squared over 2. And you'll recognize this. This is not [INAUDIBLE]. I'm using that terminology. This is the frequency of oscillation of a mass on a string of length l. All right?

So here are our two differential equations, the two equations of motion, one for x1 and one for x2. When we had one mass, one harmonic oscillator, we had a single second-order differential equation. We now have two masses, so we have two second-order differential equations. These are it.

They are coupled differential equations. You see, the derivatives of x1 is related both to x1 and x2. The second derivative of x2 is related both to x1 and to x2. So these are two coupled differential equations. OK?

This is the end of step 1. What we succeeded in, we've translated the physical situation into mathematics. These two second-order differential equations describe exactly that idealized situation we had.

So if I now want to answer the question, what will be the motion of those masses, I have to go into the world of mathematics. I have to solve these equations. And life is not as easy as it was for a single second-order differential equation. It's more complicated.

But here I will use my general knowledge of what happens when you have coupled oscillators. I know that the general solution of coupled oscillations is a superposition of normal modes. Here I have 2 degrees of freedom, two second-order differential equations. There will be two normal modes. If I succeed in finding them, I will have found the most general solution to this problem, because it will be the sum of the two normal modes.

So I will now do it by trial. In a normal mode, we know that x1 will be oscillating with a single frequency omega, some phase phi, some amplitude A1. This will be a solution to these equations. If this is a solution to those equations, the other mass must also be oscillating in the same normal mode, meaning with the same frequency and same phase. It'll have some arbitrary amplitude.

So I don't know what A1 is, and I don't know what omega is. I don't know what phi is. But I know that the solutions to these equations must be of this form. OK? So let me now try to find these various constants.

Well, if this and that satisfies these equations-- and it has to because that's what it's saying, these are the solutions of those equations-- then I can take these, x1 and x2, plug it into this equation. In other words, calculate the second derivative of x1, et cetera, calculate second, and I'll end up with two equations. OK?

So I won't bother to go in great detail here. But just to start off with, x1 double dot, I have to differentiate A1 cosine omega t twice. So I get minus omega squared A1 times cosine omega t plus phi. If I take the next term, I get this, and the third term that.

And in each case, if you notice, since it'll be multiplied by the same cosine function, I've just canceled in my head the cosine function there. All right? Just saving time. Similarly, next equation, I end up with this.

So if my guessed functions, and I'm returning to this, if these two satisfy those equations, then here these algebraic equations must be satisfied. But you'll notice something interesting. If I take the first of these equations, it boils down to A1 times this quantity plus A2 to [? then ?] this quantity equal to 0. And the second equation boils down to this.

If this is true, then this must be true. It's A1 divided by A2 has to be equal to that. If this is true, then A1 divided by A2 has got to be equal to this. That means that this has to be equal to that, or else I have an inconsistency.

So what I have found is that, in general, my guess is not a good one. In general, these equations would not satisfy my second-order differential equations. There is only under very special circumstances that they do satisfy it, and that is if this is equal to that. In general, they will not be equal.

So under what conditions will these be equal? Well, let me force them. Let me say this is equal to that and see what it means for omega. I originally took omega to be

an unknown quantity. What I am now seeing, that those functions would work only for specific values of omega. So I'm going to solve this equation for omega and see for what values of omega do I get a consistent solution.

And this is a hard grind, unfortunately. If I take this, multiply it out, I get a quadratic equation in omega squared. If this quadratic equation is satisfied, then all these are self-consistent.

All right? You know how to solve a quadratic equation, and this is a quadratic equation in omega squared. Omega square will have to be equal to minus this plus or minus the square root of this, 4 times this times that, divided by 2a, right? This is the standard formula for the solution of a quadratic equation. And there's a plus or minus.

If I calculate this out, I get that omega squared has to be equal to 2 plus or minus the square root of 2 times omega 0 squared. All right? Now, omega 0 squared, earlier on, I defined to be g/l. So omega squared has to be this.

So let's stop for a second and think and review what we've done. We found that our physical situation can be described by these two coupled differential equations. We looked for solutions, which are normal modes where both x1 and x2 is oscillating with same frequency and phase.

And we found that it was possible to find solution of this form to these equations if and only if omega squared has one of two possible values. I've rewritten them here. One is 2 plus root 2 times g/l, and the other's 2 minus root 2 g/l. With these values, these two equations satisfies that.

So what we found is the two normal modes. Since this system consists of the coupled oscillators, in other words, two oscillators coupled with each other, we know there are two normal modes. So these are the normal mode frequencies in this situation.

And by the way, I'm delighted to see that, if you remember Professor Walter Lewin's

lectures, he actually quoted these numbers. So he was right. He did not make a mistake, as we've just seen. OK? These are the two normal mode frequencies.

Now, how about if we want to predict everything, the amplitude, et cetera? We've got to be careful here with the logic we've applied. For any one of these normal frequencies, say this one, earlier on we found that A1 to A2 is given by this equation or by this equation. But for the given values of omega, we found these are now equivalent.

And so if you know what omega is, say omega A, one of the normal modes, this we know. It's g/l. And this is known. So A1/A2, once we fix omega A, is fixed. So these are not two independent amplitudes.

So if I go back to here when we said let's guess a solution, and this was an arbitrary number and this was an arbitrary number, now we see that in a normal mode the ratio between them is fixed. It depends on the normal mode frequency. But the [? overall ?] normalization is still arbitrary. I could make this 10 times this and 10 times that. [? It would still ?] work.

So back to here, we found one of the normal mode's frequencies. Using the equation above, it determines the value of A1 to A2. Taking the other normal mode frequency determines the ratio of those two but not the overall amplitude.

Not to confuse the two, I will call this B1 and B2. This is the amplitude of the first mass, and this is of the other one. The ratio is fixed, determined by the value of omega B. But the overall amplitude is not.

So now we have ended up in the same situation we were in our three masses, where, if you remember, I started by guessing the normal modes and guessing the relative amplitudes. And with that information alone and the initial conditions, I could predict what will happen. I could now repeat that here. But to save time, I think you can do that for yourself.

For each, you can write now a single equation saying what a given mass will do as the sum of two normal modes, where you know the frequency of them and the

amplitudes. And you have to remember that for the two masses, the ratio of the amplitudes in each normal mode have to be fixed given by what we've done here.

And therefore, you can get the single equation for each mass, one for one mass, one for the other. Which will tell you what they will do in the future if you know what they're doing initially, for example, the initial conditions, that you can solve to get the final predicted position of each mass.

So that's as much as I was hoping to do today on coupled oscillators. And we'll continue next time on situations with many, many more degrees of freedom. Thank you.