22 The z-Transform

Solutions to Recommended Problems

S22.1

(a) The z-transform $H(z)$ can be written as

$$H(z) = \frac{z}{z - \frac{1}{2}}$$

Setting the numerator equal to zero to obtain the zeros, we find a zero at $z = 0$. Setting the denominator equal to zero to get the poles, we find a pole at $z = \frac{1}{2}$. The pole-zero pattern is shown in Figure S22.1.

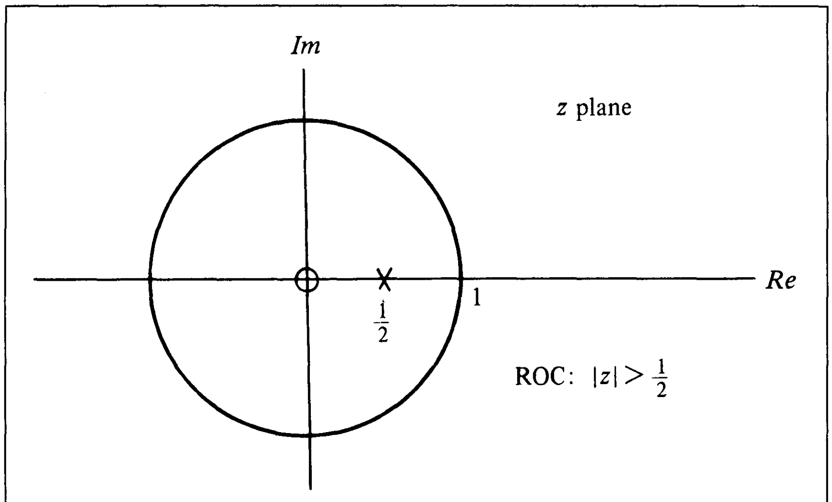

Figure S22.1

(b) Since $H(z)$ is the eigenvalue of the input z^n and the system is linear, the output is given by

$$\begin{aligned} y[n] &= \frac{1}{1 - \frac{1}{2}(\frac{4}{3})}\left(\frac{3}{4}\right)^n + 3\left[\frac{1}{1 - \frac{1}{2}(\frac{1}{2})}\right](2)^n \\ &= 3(\tfrac{3}{4})^n + 4(2)^n \end{aligned}$$

S22.2

(a) To see if $x[n]$ is absolutely summable, we form the sum

$$S_N = \sum_{n=0}^{N-1} |x[n]| = \sum_{n=0}^{N-1} 2^n = \frac{1 - 2^N}{1 - 2}$$

Since $\lim_{N\to\infty} S_N$ diverges, $x[n]$ is *not* absolutely summable.

(b) Since $x[n]$ is not absolutely summable, the Fourier transform of $x[n]$ does not converge.

(c) $$S_N = \sum_{n=0}^{N-1} \left(\frac{2}{r}\right)^n = \frac{1 - \left(\frac{2}{r}\right)^N}{1 - \left(\frac{2}{r}\right)}$$

$\lim_{N\to\infty} S_N$ is finite for $|r| > 2$. Therefore, the Fourier transform of $r^{-n}x[n]$ converges for $|r| > 2$.

(d) $$X(z) = \sum_{n=0}^{\infty} 2^n z^{-n} = \sum_{n=0}^{\infty} (2z^{-1})^n$$
$$= \frac{1}{1 - 2z^{-1}} \qquad \text{for } |2z^{-1}| < 1$$

Therefore, the ROC is $|z| > 2$.

(e) $$X_1(e^{j\Omega}) = \frac{1}{1 - \frac{2}{3}e^{-j\Omega}}$$

Therefore, $x_1[n] = (\frac{2}{3})^n u[n]$.

S22.3

(a) Since $x[n]$ is right-sided, the ROC is given by $|z| > \alpha$. Since the ROC cannot include poles, for this case the ROC is given by $|z| > 2$.

(b) The statement implies that the ROC includes the unit circle $|z| = 1$. Since the ROC is a connected region and bounded by poles, the ROC must be

$$\tfrac{2}{3} < |z| < 2$$

(c) For this situation there are three possibilities:

(i) $|z| < \frac{1}{3}$

(ii) $\frac{1}{3} < |z| < \frac{2}{3}$

(iii) $|z| > 2$

(d) This statement implies that the ROC is given by $|z| < \frac{1}{3}$.

S22.4

(a) (i) $$X_1(z) = \sum_{n=-\infty}^{\infty} x_1[n]z^{-n} = \sum_{n=0}^{\infty} (\tfrac{1}{2}z^{-1})^n$$
$$= \frac{1}{1 - \frac{1}{2}z^{-1}},$$

with an ROC of $\left|\frac{1}{2z}\right| < 1$, or $|z| > \frac{1}{2}$.

(ii) $$X_2(z) = \sum_{n=-\infty}^{-1} (\tfrac{1}{2})^n z^{-n}$$

Letting $n = -m$, we have

$$X_2(z) = -\sum_{m=1}^{\infty} (\tfrac{1}{2})^{-m} z^m$$
$$= -\sum_{m=1}^{\infty} (2z)^m = -\frac{2z}{1 - 2z}$$
$$= \frac{1}{1 - \frac{1}{2}z^{-1}},$$

with an ROC of $|2z| < 1$, or $|z| < \frac{1}{2}$.

(b) (i)

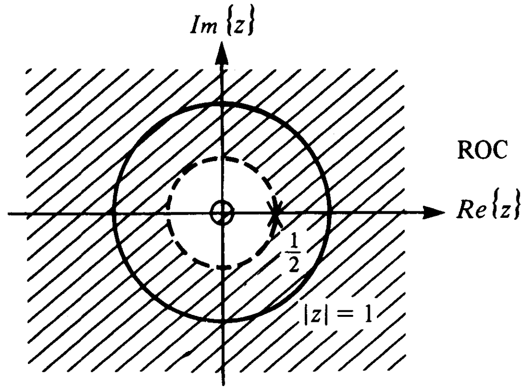

Figure S22.4-1

(ii)

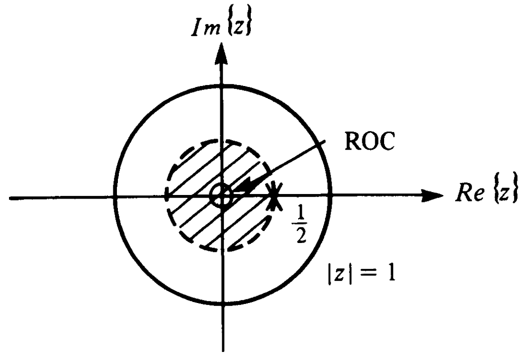

Figure S22.4-2

(c) (i) $X_3(z) = 2\sum_{n=0}^{\infty} z^{-n} = 2\left(\frac{1}{1 - z^{-1}}\right) = \frac{2z}{z-1}$. The ROC is $|z| > 1$, as shown in Figure S22.4-3.

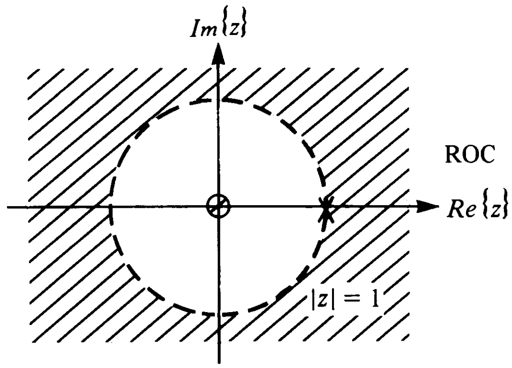

Figure S22.4-3

(ii)

$$X_4(z) = -\sum_{n=-\infty}^{-1} 2^n z^{-n} = -\sum_{n=1}^{\infty} 2^{-n} z^n$$

$$= -\sum_{n=1}^{\infty}\left(\frac{z}{2}\right)^n = -\frac{z/2}{1 - (z/2)} = \frac{z}{z-2},$$

with an ROC of $|z/2| < 1$, or $|z| < 2$, shown in Figure S22.4-4.

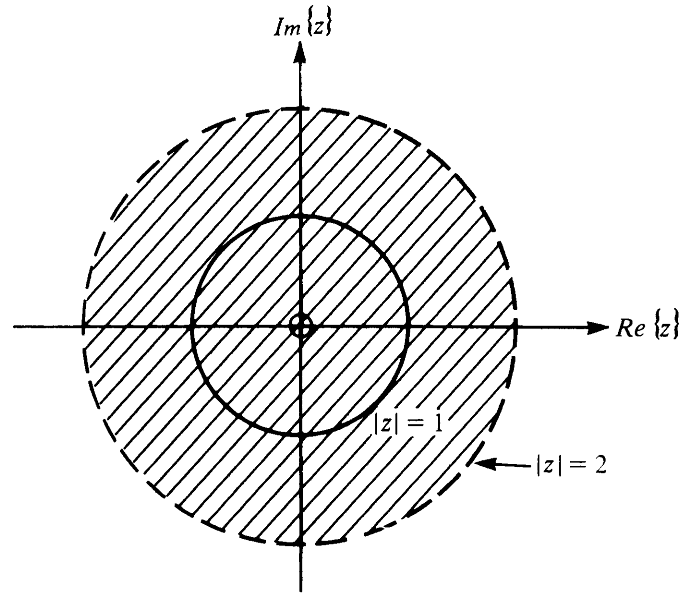

Figure S22.4-4

(d) For the Fourier transform to converge, the ROC of the z-transform must include the unit circle. Therefore, for $x_1[n]$ and $x_4[n]$, the corresponding Fourier transforms converge.

S22.5

Consider the pole-zero plot of $H(z)$ given in Figure S22.5-1, where $H(a/2) = 1$.

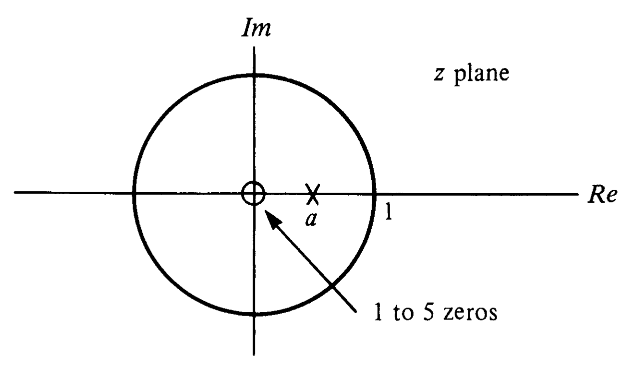

Figure S22.5-1

(a) When $H(z) = z/(z - a)$, i.e., the number of zeros is 1, we have

$$H(e^{j\Omega}) = \frac{\cos \Omega + j \sin \Omega}{(\cos \Omega - a) + j \sin \Omega}$$

Therefore,

$$|H(e^{j\Omega})| = \frac{1}{1 + a^2 - 2a \cos \Omega},$$

and we can plot $|H(e^{j\Omega})|$ as in Figure S22.5-2.

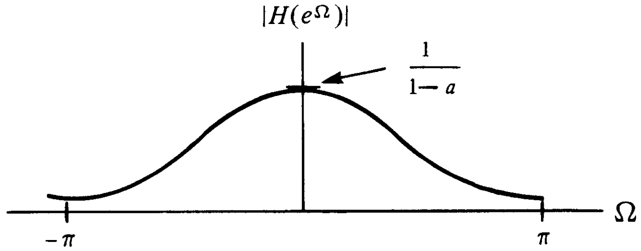

Figure S22.5-2

When $H(z) = z^2/(z - a)$, i.e., the number of zeros is 2, we have

$$H(e^{j\Omega}) = \frac{\cos 2\Omega + j \sin 2\Omega}{(\cos \Omega - a) + j \sin \Omega}$$

Therefore,

$$|H(e^{j\Omega})| = \frac{1}{1 + a^2 - 2a \cos \Omega}$$

Hence, we see that the magnitude of $H(e^{j\Omega})$ does not change as the number of zeros increases.

(b) For one zero at $z = 0$, we have

$$H(z) = \frac{z}{z - a},$$
$$H(e^{j\Omega}) = \frac{e^{j\Omega}}{e^{j\Omega} - a}$$

We can calculate the phase of $H(e^{j\Omega})$ by $[\Omega - \sphericalangle \text{(denominator)}]$. For two zeros at 0, the phase of $H(e^{j\Omega})$ is $[2\Omega - \sphericalangle \text{(denominator)}]$. Hence, the phase changes by a linear factor with the number of zeros.

(c) The region of the z plane where $|H(z)| = 1$ is indicated in Figure S22.5-3.

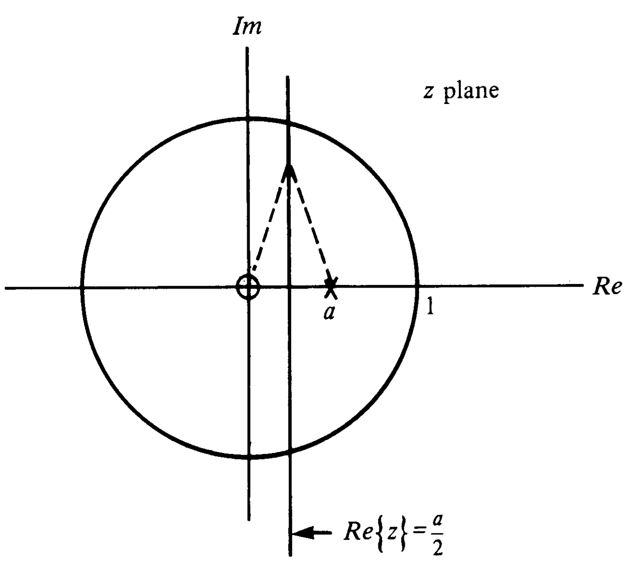

Figure S22.5-3

S22.6

(a) $(\frac{1}{3})^n u[n] \xleftrightarrow{Z} \sum_{n=0}^{\infty} (\frac{1}{3})^n z^{-n}$

$$= \sum_{n=0}^{\infty} (3z)^{-n} = \frac{1}{1 - \frac{1}{3}z^{-1}} = \frac{z}{z - \frac{1}{3}}$$

Therefore, there is a zero at $z = 0$ and a pole at $z = \frac{1}{3}$, and the ROC is

$$\left|\frac{1}{3z}\right| < 1 \quad \text{or} \quad |z| > \frac{1}{3},$$

as shown in Figure S22.6-1.

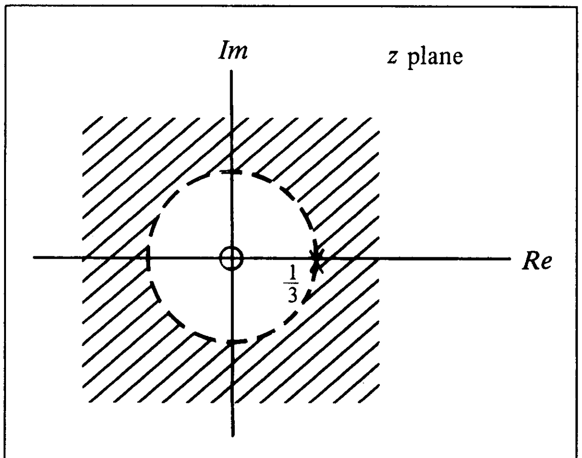

Figure S22.6-1

(b) $\delta[n + 1] \xleftrightarrow{Z} \sum_{n=-\infty}^{\infty} \delta[n + 1]z^{-n} = z,$

with the ROC comprising the entire z-plane, as shown in Figure S22.6-2.

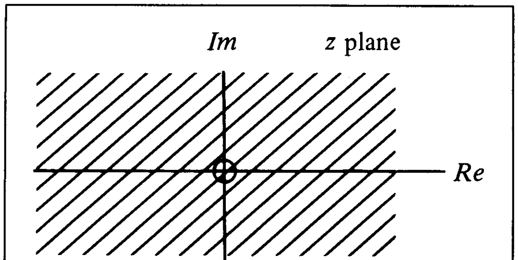

Figure S22.6-2

S22.7

(a) Using long division, we have

$$X(z) = 1 - \tfrac{1}{2}z^{-1} + \tfrac{1}{4}z^{-2} - \tfrac{1}{8}z^{-3} + \cdots$$

We recognize that

$$x[n] = (-\tfrac{1}{2})^n u[n]$$

(b) Proceeding as we did in part (a), we have

$$x[n] = (-\tfrac{1}{2})^n u[n]$$

(c)
$$\begin{aligned} X(z) &= \frac{z-a}{1-az} \\ &= -\frac{1}{a} + \frac{\left(\frac{1-a^2}{a}\right)}{1-az} \\ &= -\frac{1}{a} - \frac{\left(\frac{1-a^2}{a^2}\right)z^{-1}}{1-a^{-1}z^{-1}} \end{aligned}$$

Therefore,

$$x[n] = -\frac{1}{a}\,\delta[n] - \frac{(1-a^2)}{a^{n+1}}\,u[n-1]$$

Solutions to Optional Problems

S22.8

(a)
$$\begin{aligned} \mathcal{F}\{r^{-n}x[n]\} &= \mathcal{F}\{r^{-n}u[n]\} \\ &= \sum_{n=0}^{\infty} r^{-n}e^{-j\Omega n} \\ &= \sum_{n=0}^{\infty} (re^{+j\Omega})^{-n} \end{aligned}$$

For the sum to converge, we must have

$$\left|\frac{1}{re^{+j\Omega}}\right| < 1$$

Thus, $|r| > 1$.

(b) (i)

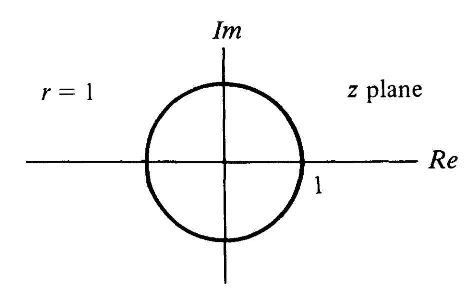

Figure S22.8-1

(ii)

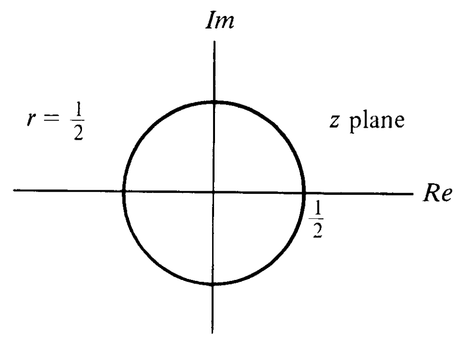

Figure S22.8-2

(iii)

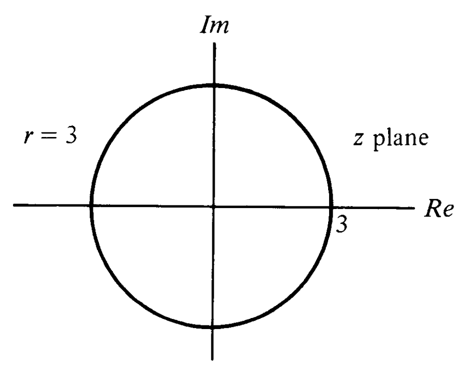

Figure S22.8-3

(c)

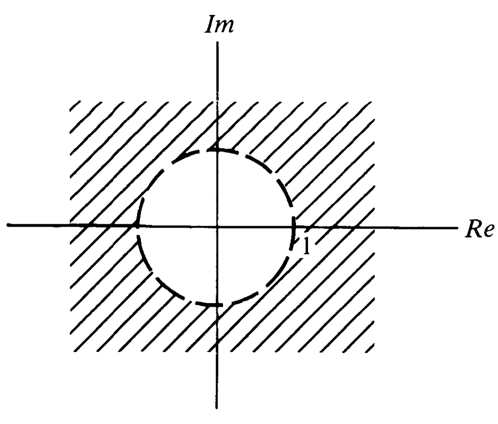

Figure S22.8-4

S22.9

(a) The inverse transform of

$$\frac{1}{1 - \frac{1}{3}e^{-j\Omega}}$$

is $(\frac{1}{3})^n u[n]$. But from the given relation, we have

$$2^{-n}x[n] = (\tfrac{1}{3})^n u[n],$$

or

$$x[n] = (\tfrac{2}{3})^n u[n]$$

(b)
$$\begin{aligned}\frac{3 + 2z^{-1}}{2 + 3z^{-1} + z^{-2}} &= \frac{3 + 2z^{-1}}{(2 + z^{-1})(1 + z^{-1})} \\ &= \frac{1}{2 + z^{-1}} + \frac{1}{1 + z^{-1}}, \\ X(z) &= \frac{\frac{1}{2}}{1 + \frac{1}{2}z^{-1}} + \frac{1}{1 + z^{-1}} \\ &= \tfrac{1}{2}(-\tfrac{1}{2})^n u[n] + (-1)^n u[n]\end{aligned}$$

S22.10

(a) $H(z) = \dfrac{A(z^{-1} - a)}{(1 - az^{-1})}$, with A a constant

Therefore,

$$\begin{aligned}H(e^{j\Omega}) &= \frac{A(e^{-j\Omega} - a)}{1 - ae^{-j\Omega}} \\ |H(e^{j\Omega})|^2 &= \frac{A^2(e^{-j\Omega} - a)(e^{j\Omega} - a)}{(1 - ae^{-j\Omega})(1 - ae^{j\Omega})} = A^2,\end{aligned}$$

and thus,

$$|H(e^{j\Omega})| = |A|$$

(b) (i) $|v_1|^2 = 1 + a^2 - 2a \cos \Omega$

(ii)
$$\begin{aligned}|v_2|^2 &= 1 + \frac{1}{a^2} - \frac{2}{a}\cos \Omega \\ &= \frac{1}{a^2}(a^2 + 1 - 2a \cos \Omega) \\ &= \frac{1}{a^2}|v_1|^2\end{aligned}$$

S22.11

In all the parts of this problem, draw the vectors from the poles or zeros to the unit circle. Then estimate the frequency response from the magnitudes of these vectors, as was done in the lecture. The following rough association can be made:

(a) $\longleftrightarrow$ (i)
(b) $\longleftrightarrow$ (ii)
(c) $\longleftrightarrow$ (iv)
(d) $\longleftrightarrow$ (iii)
(e) $\longleftrightarrow$ (iv)

S22.12

(a) $x[n] = (\frac{1}{2})^n[u[n] - u[n-10]]$. Therefore,

$$X(z) = \sum_{n=0}^{9} (\tfrac{1}{2})^n z^{-n}$$
$$= \sum_{n=0}^{9} (2z)^{-n} = \frac{1-(2z)^{-10}}{1-(2z)^{-1}}$$
$$= \frac{z^{10} - (\frac{1}{2})^{10}}{z^9(z-\frac{1}{2})}, \qquad |z| > 0,$$

shown in Figure S22.12-1. The Fourier transform exists.

(b) $x[n] = (\frac{1}{2})^{|n|} = (\frac{1}{2})^n u[n] + (\frac{1}{2})^{-n}u[-n-1]$. But

$$\left(\frac{1}{2}\right)^n u[n] \overset{Z}{\longleftrightarrow} \frac{z}{z-\frac{1}{2}}, \qquad |z| > \frac{1}{2},$$

and

$$\left(\frac{1}{2}\right)^{-n} u[-n-1] \overset{Z}{\longleftrightarrow} \frac{-z}{z-2}, \qquad |z| < 2$$

Summing the two z-transforms, we have

$$X(z) = \frac{-\frac{3}{2}z}{(z-\frac{1}{2})(z-2)}, \qquad \frac{1}{2} < |z| < 2$$

(See Figure S22.12-2.) The Fourier transform exists.

(c) $x[n] = 7(\frac{1}{3})^n \cos\left(\frac{2\pi n}{6} + \frac{\pi}{4}\right) u[n]$

Therefore,

$$X(z) = \sum_{n=0}^{\infty} 7\left(\frac{1}{3}\right)^n \cos\left[\frac{2\pi n}{6} + \frac{\pi}{4}\right] z^{-n}$$
$$= \frac{7}{2}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n [e^{j[(2\pi/6)n + (\pi/4)]} + e^{-j[(2\pi/6)n+(\pi/4)]}]z^{-n}$$
$$= \frac{7}{2}\left[e^{j\pi/4}\sum_{n=0}^{\infty}\left(\frac{1}{3}e^{j(2\pi/6)}z^{-1}\right)^n + e^{-j\pi/4}\sum_{n=0}^{\infty}\left(\frac{1}{3}e^{-j(2\pi/6)}z^{-1}\right)^n\right]$$
$$= \frac{7}{2}\left[\frac{e^{j\pi/4}}{1-\frac{1}{3}e^{j(2\pi/6)}z^{-1}} + \frac{e^{-j\pi/4}}{1-\frac{1}{3}e^{-j(2\pi/6)}z^{-1}}\right]$$
$$= \frac{7z}{2}\left[\frac{e^{j\pi/4}}{z-\frac{1}{3}e^{j(2\pi/6)}} + \frac{e^{-j\pi/4}}{z-\frac{1}{3}e^{-j(2\pi/6)}}\right]$$
$$= \frac{7z}{2}\left[\frac{2z\cos\left(\frac{\pi}{4}\right) - \frac{2}{3}\cos\left(\frac{2\pi}{6}-\frac{\pi}{4}\right)}{(z-\frac{1}{3}e^{j(2\pi/6)})(z-\frac{1}{3}e^{-j(2\pi/6)})}\right], \qquad \text{where } |z| > \frac{1}{3}$$

The pole-zero plot and ROC are shown in Figure S22.12-3. Clearly, the Fourier transform exists.

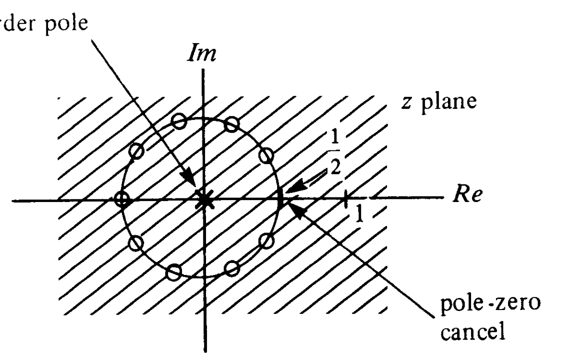

Figure S22.12-1

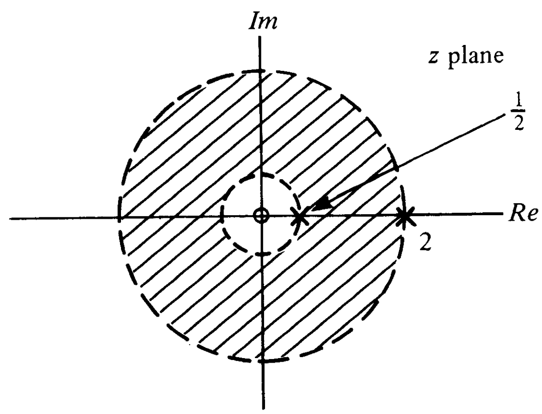

Figure S22.12-2

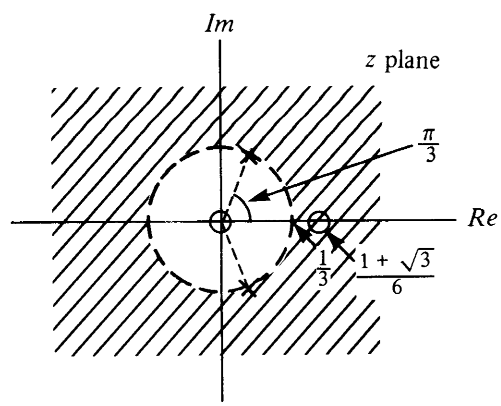

Figure S22.12-3

(d) $X(z) = \sum_{n=0}^{9} z^{-n} = \frac{1 - z^{10}}{1 - z^{-1}} = \frac{z^{10} - 1}{z^9(z - 1)}$

The ROC is all z except $z = 0$, shown in Figure S22.12-4. The Fourier transform exists.

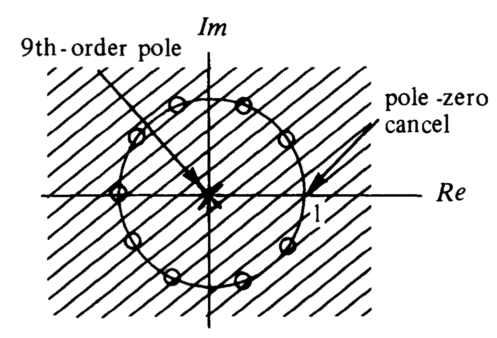

Figure S22.12-4

S22.13

(a) From

$$\log(1 - w) = -\sum_{i=1}^{\infty} \frac{w^i}{i}, \qquad |w| < 1,$$

we find

$$\log(1 - 2z) = -\sum_{i=1}^{\infty} \frac{(2z)^i}{i}, \qquad |2z| < 1$$

$$= -\sum_{i=1}^{\infty} \frac{2^i}{i} z^i, \qquad |z| < \frac{1}{2},$$

$$x[n] = \begin{cases} \dfrac{2^{-n}}{n}, & n < 0, \\ 0, & n \geq 0 \end{cases}$$

(b) We solve this similarly to the way we solved part (a).

$$\log\left(1 - \frac{1}{2}z^{-1}\right) = -\sum_{i=1}^{\infty} \frac{(\frac{1}{2}z^{-1})^i}{i}, \qquad \left|\frac{1}{2}z^{-1}\right| < 1$$

$$= -\sum_{i=1}^{\infty} \frac{(\frac{1}{2})^i}{i} z^{-i}, \qquad \frac{1}{2} < |z|,$$

$$x[n] = \begin{cases} -\dfrac{1}{n}\left(\dfrac{1}{2}\right)^n, & n > 0, \\ 0, & n \leq 0 \end{cases}$$