17 Interpolation

Solutions to Recommended Problems

S17.1

It is more convenient to solve this problem in the time domain than in the frequency domain. Since $x_p(t) = x(t)p(t)$ and $p(t)$ is an impulse train, $x_p(t)$ is a sampled version of $x(t)$, as shown in Figure S17.1-1.

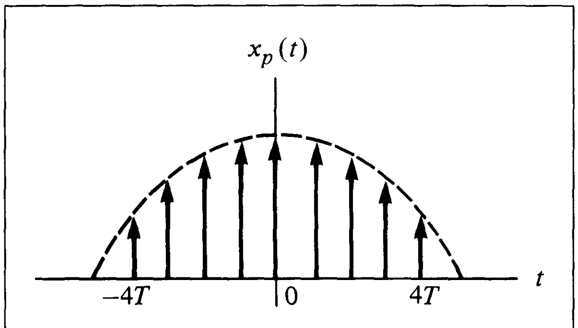

Figure S17.1-1

$$x_p(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t - nT)$$

Since $y(t) = x_p(t) * h(t)$ and $x_p(t)$ is impulsive, the convolution carried out in the time domain is as shown in Figure S17.1-2.

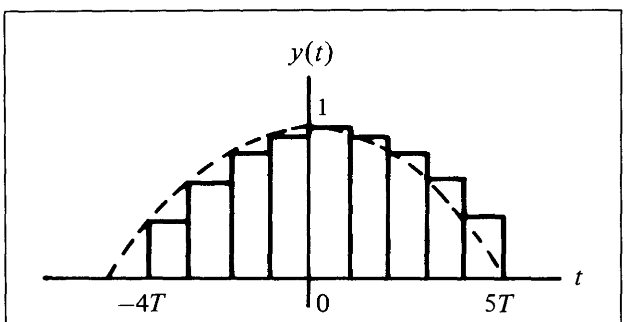

Figure S17.1-2

Here we have that $x(t)$ is sampled by a rectangular pulse train as opposed to an impulse train.

Since $h(t) * (1/T)h(t)$, shown in Figure S17.1-3, is wider than the sampling period T, the resultant $w(t)$ is not a triangularly sampled version of $x(t)$. $w(t)$ consists of the superposition of waveforms shown in Figure S17.1-4.

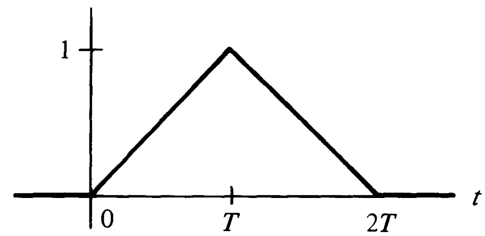

Figure S17.1-3

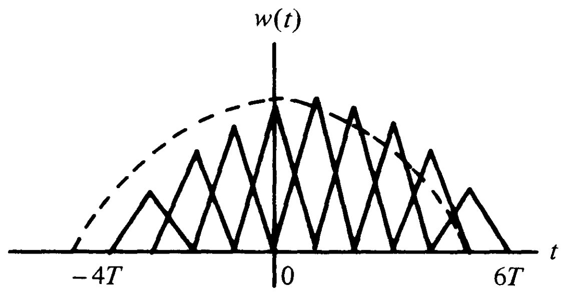

Figure S17.1-4

We note that this superposition is actually a linear interpolation between the samples of $x(t)$. For example, Figure S17.1-5 convolved with Figure S17.1-6 equals Figure S17.1-7.

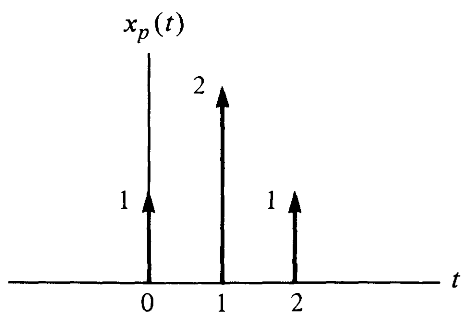

Figure S17.1-5

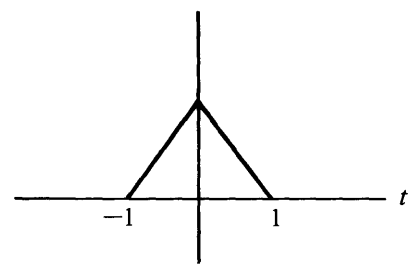

Figure S17.1-6

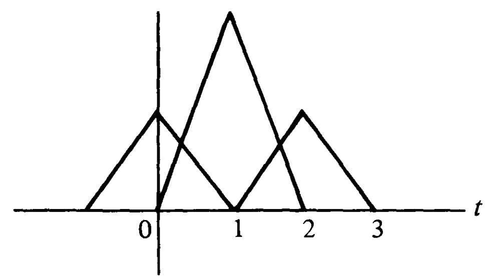

Figure S17.1-7

Now adding the shifted and scaled triangles yields Figure S17.1-8, which we see is the linear interpolation between samples of $x_p(t)$.

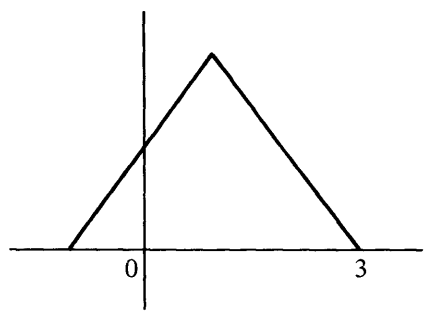

Figure S17.1-8

Now since $(1/T)h(t) * h(t)$ is Figure S17.1-9, we expect that $w(t)$ is the linear interpolation of $x_p(t)$ shifted right by T, shown in Figure S17.1-10.

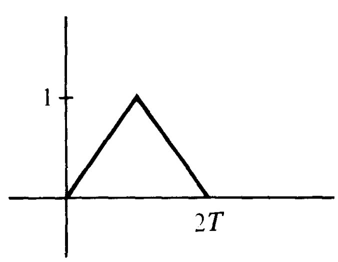

Figure S17.1-9

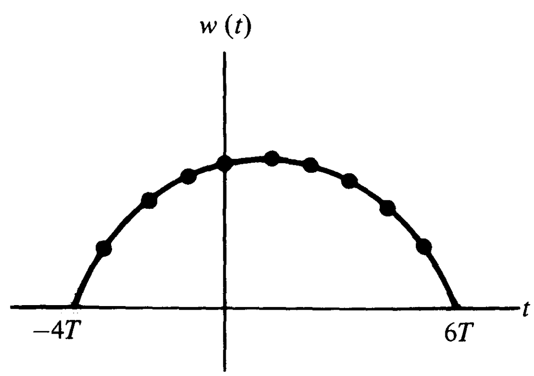

Figure S17.1-10

S17.2

$y(t)$ in all cases is the superposition of two signals.

(a)

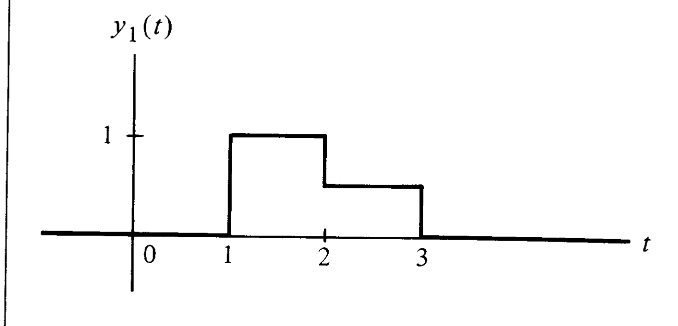

Figure S17.2-1

(b)

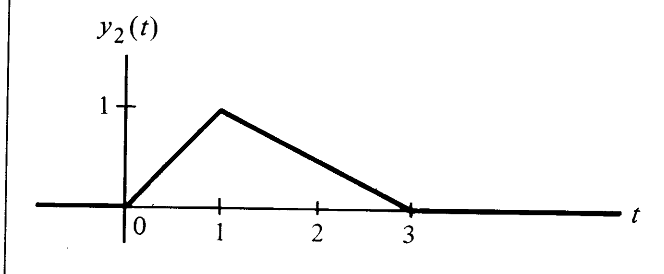

Figure S17.2-2

(c)

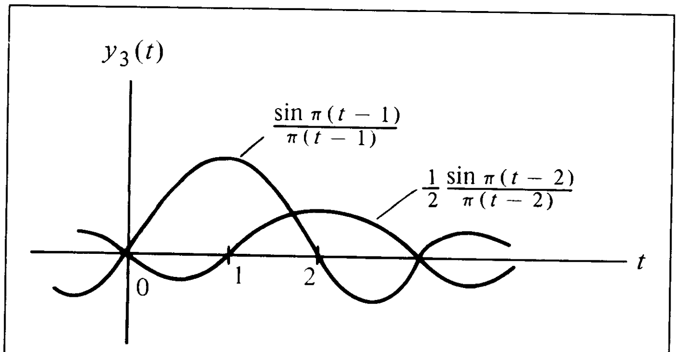

Figure S17.2-3

S17.3

(a)

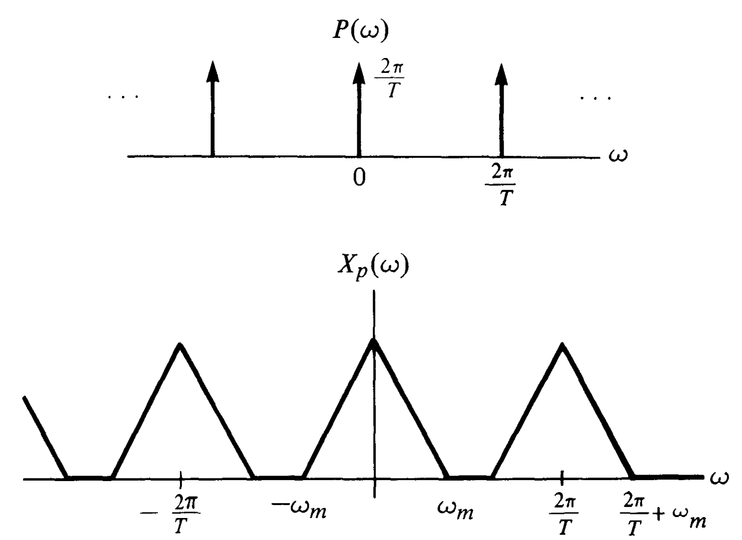

Figure S17.3-1

We have

$$\frac{2\pi}{T} - \omega_m > \omega_m,$$
$$\frac{2\pi}{T} > 2\omega_m,$$
$$\omega_s > 2\omega_m,$$

where ω_s is the sampling frequency. To ensure no aliasing we require that $T < \pi/\omega_m$.

(b) To recover $x(t)$ from $x_p(t)$, we must interpolate. We have previously shown that by lowpass filtering, the spectrum is recovered, assuming that sampling has been performed at a sufficiently high rate. The interpolation may be done in many different ways, however, depending on the cutoff frequency we choose for the lowpass filter. For example, any of the filters $H_1(\omega)$, $H_2(\omega)$, and $H_3(\omega)$ in Figures S17.3-3 to S17.3-5 may be used to interpolate $x_p(t)$, whose Fourier transform is shown in Figure S17.3-2, to yield $x(t)$.

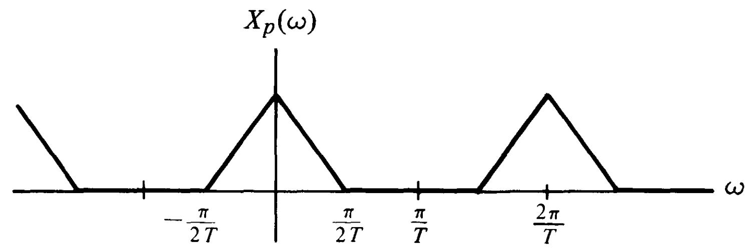

Figure S17.3-2

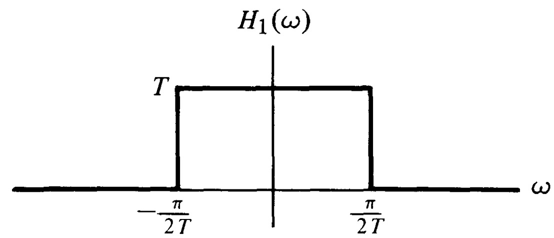

Figure S17.3-3

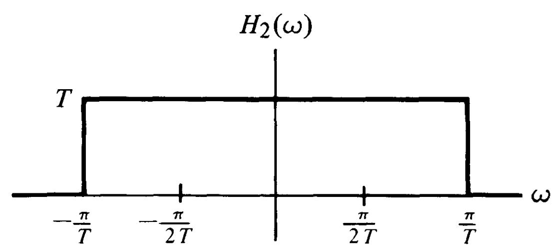

Figure S17.3-4

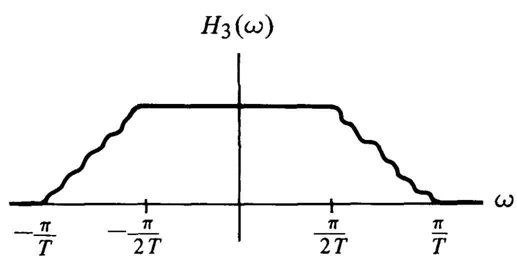

Figure S17.3-5

(c) If

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT)$$

and $H(\omega)$ is an ideal lowpass filter whose Fourier transform is shown in Figure S17.3-6, then

$$h(t) = \frac{T \sin \omega_c t}{\pi t} = \frac{T\omega_c}{\pi} \operatorname{sinc} \frac{\omega_c t}{\pi}$$

and

$$x(t) = x_p(t) * h(t) = \frac{T\omega_c}{\pi} \sum_{n=-\infty}^{\infty} x(nT) \operatorname{sinc} \frac{(t - nT)}{\pi}$$

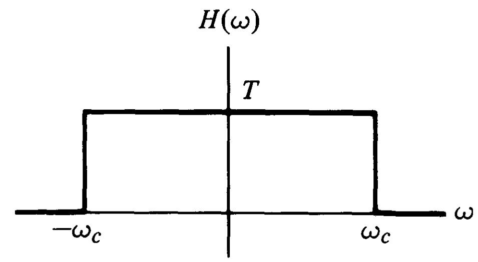

Figure S17.3-6

S17.4

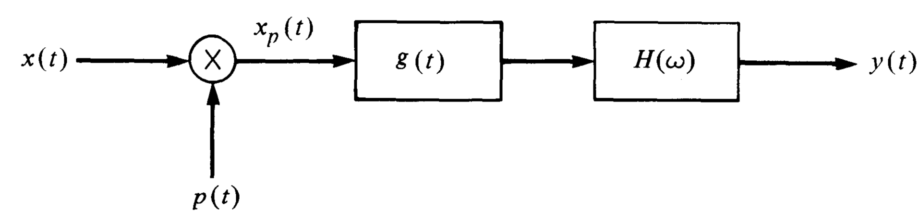

Figure S17.4

We want to choose $H(\omega)$ so that the cascade of the two filters is an ideal lowpass filter. In this example,

$$G(\omega) = 2 \sin \frac{\omega\Delta/2}{\omega}\, e^{-j\omega\Delta/2}$$

so that

$$H(\omega) = \frac{\omega}{2 \sin \omega\Delta/2}\, e^{j\omega\Delta/2}, \qquad |\omega| < \pi/\Delta,$$
$$H(\omega) = 0 \qquad \text{otherwise}$$

S17.5

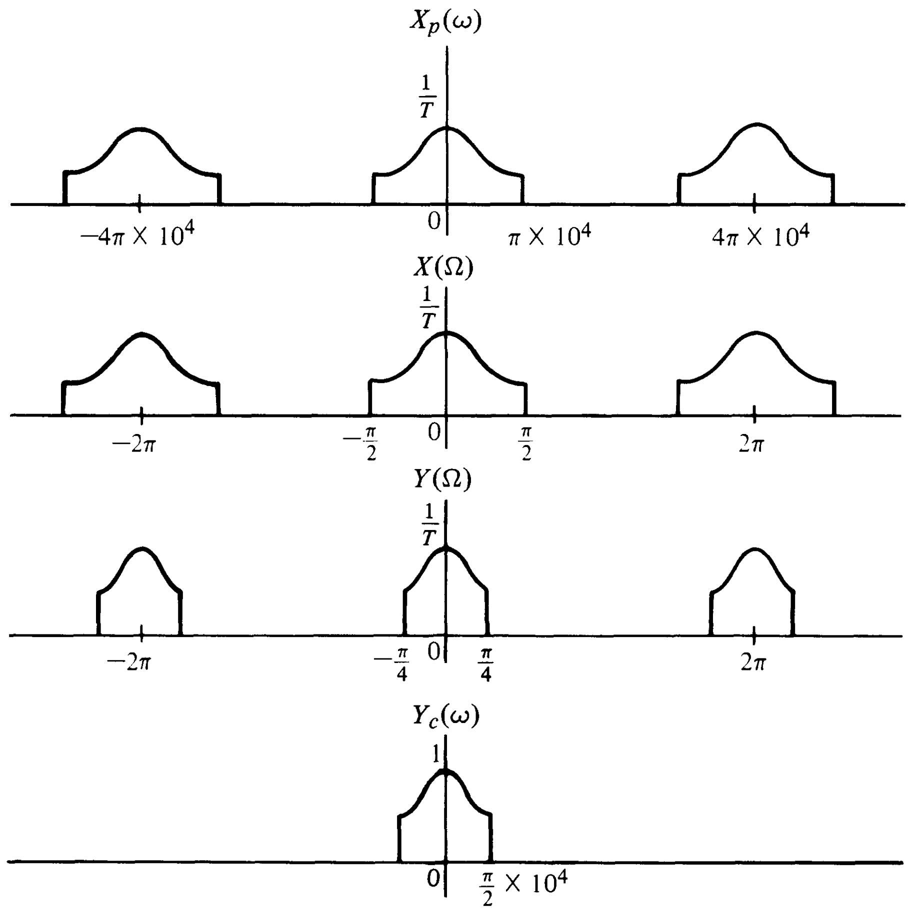

Figure S17.5

Solutions to Optional Problems

S17.6

(a) We want $h_0[n]$ such that $x_p[n] * h_0[n] = x_0[n]$. We sketch $h_0[n]$ in Figure S17.6-1.

Figure S17.6-1

(b) $x_0[n] = x_p[n] * h_0[n]$

If there is no aliasing, we can recover $x[n]$ from $x_0[n]$ by proper filtering. Since

$$H_0(\Omega) = \frac{1 - e^{-j\Omega N}}{1 - e^{-j\Omega}} = e^{-j\Omega(N-1)/2} \frac{\sin \Omega N/2}{\sin \Omega/2},$$

we require

$$H(\Omega) = \begin{cases} N/H_0(\Omega), & |\Omega| \le \dfrac{\pi}{N}, \\ 0, & \dfrac{\pi}{N} < |\Omega| \le \pi \end{cases}$$

$$= \begin{cases} Ne^{j\Omega(N-1)/2} \dfrac{\sin \Omega/2}{\sin \Omega N/2}, & |\Omega| \le \dfrac{\pi}{N} \\ 0, & \dfrac{\pi}{N} < |\Omega| \le \pi \end{cases}$$

(c) $h_1[n] = (1/N)\,(h_0[n] * h_0[-n])$, so $h_1[n]$ is a triangular discrete-time pulse, as shown in Figure S17.6-2.

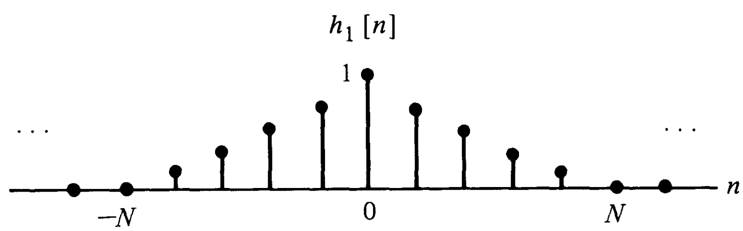

Figure S17.6-2

(d) We require that

$$H(\Omega) = \begin{cases} \dfrac{N}{H_1(\Omega)}, & \text{for} |\Omega| \le \dfrac{\pi}{N}, \\ 0\,, & \text{for} \dfrac{\pi}{N} < |\Omega| < \pi \end{cases}$$

From part (c),

$$H(\Omega) = N\left[\frac{N}{|H_0(\Omega)|^2}\right] = \begin{cases} N^2 \left(\dfrac{\sin \Omega/2}{\sin N\Omega/2}\right)^2, & |\Omega| \le \dfrac{\pi}{N}, \\ 0, & \dfrac{\pi}{N} < |\Omega| < \pi \end{cases}$$

S17.7

(a) Taking Fourier transforms of both sides of the LCCDE yields

$$j\omega Y_c(\omega) + Y_c(\omega) = 1,$$

$$Y_c(\omega) = \frac{1}{1 + j\omega},$$

so $y_c(t) = e^{-t}u(t)$. By examining the sampler followed by conversion to an impulse train, we note that

$$y[n] = y_c(nT) = e^{-nT}u[n]$$

(b) Since $y[n] = e^{-nT}u[n]$, we can take the Fourier transform to yield

$$Y(\Omega) = \sum_{n=0}^{\infty} e^{-nT}e^{-j\Omega n}$$
$$= \frac{1}{1 - e^{-(T+j\Omega)}}$$

In order for $w[n] = \delta[n]$, we require $W(\Omega) = 1$ for all Ω. Thus,

$$Y(\Omega)H(\Omega) = W(\Omega) = 1,$$
$$H(\Omega) = 1 - e^{-(T+j\Omega)}$$

Now since $H(\Omega) = \Sigma_{n=-\infty}^{\infty} h[n]e^{-j\Omega n}$, we see by inspection that

$$h[0] = 1,$$
$$h[1] = -e^{-T},$$
$$h[n] = 0, \qquad n \neq 0, 1$$

S17.8

(a) Since $X_p(\omega) = X(\omega)P(\omega)$, we conclude that $X_p(\omega)$ is as shown in Figure S17.8-1.

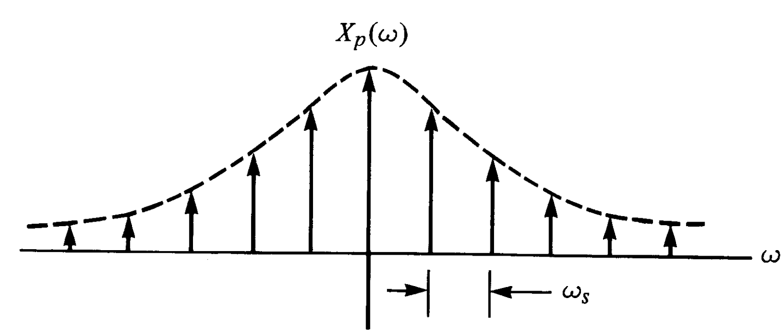

Figure S17.8-1

$$X_p(\omega) = X(\omega) \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$
$$= \sum_{n=-\infty}^{\infty} X(n\omega_s)\,\delta(\omega - n\omega_s)$$

(b) From the convolution theorem,

$$y(t) = x(t) * p(t)$$

and

$$p(t) = \frac{1}{\omega_s} \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{2\pi n}{\omega_s}\right)$$

The fact that

$$\mathcal{F}\{p(t)\} = \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)$$

can be easily verified. Therefore,

$$y(t) = x(t) * \frac{1}{\omega_s} \sum_{n=-\infty}^{\infty} \delta\left(t - \frac{2\pi n}{\omega_s}\right)$$

(c) We see from the sketch in Figure S17.8-2 that for no time-domain aliasing,

$$T \leq \frac{2\pi}{\omega_s} - T, \qquad \text{so} \qquad \omega_s \leq \frac{\pi}{T}$$

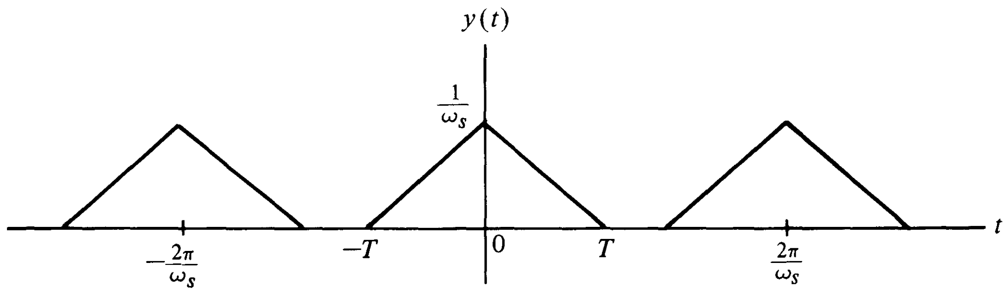

Figure S17.8-2

(d) $x(t)$ may be recovered from $y(t)$, assuming that no time-domain aliasing has occurred, by low-time filtering $y(t)$ from $t = -T$ to T and applying a gain of ω_s.