LINEAR INDEPENDENCE

A

Preface

The concept of linear independence (and linear dependence) transcends the study of differential equations. For example, in studying systems of n linear algebraic equations in n unknowns, we saw that there was "trouble" if one of the equations was a linear combination of some of the others in the system. By way of brief review, we saw that the system

$$
\begin{aligned}
x + 2y + 3z &= a \\
4x + 5y + 6z &= b \\
7x + 8y + 9z &= c
\end{aligned}
$$

either had infinitely many solutions or else no solutions depending upon whether or not $c = 2b - a$. That is, in the given system of equations, the third equation was a linear combination of the first and the second; in particular (and we used row-reduced matrix techniques for carrying out these computations), the third equation is twice the second minus the first.

Now, in this portion of our course (see, for example, Exercise 2.4.5), we again see the danger of one function being a linear combination of some others when we seek the general solution of a linear differential equation. We shall dwell on this point in a moment. For now, however, let us admit that the concept of one element being a linear combination of others plays an important role in any topic that can be described as a linear system, and the study of linear systems is a major topic of investigation in the study of (n-dimensional) vector spaces. In Block 3, the final block of our course, we shall study vector spaces a bit more thoroughly than we have up to now, and at that time, we shall revisit linear independence from a more general point of view.

Since this topic is rather subtle and often requires much experience on the part of the student, we prefer to introduce it with some of the general ramifications that apply to all vector spaces. In this way, not only will we get a better understanding of the structure of linear differential equations, but we shall be better fortified to discuss the structure and behavior of any vector space when we get to this study in Block 3.

Introduction

Based on loose intuition, if one were trying to formulate a definition of what is meant by the general solution of an nth order differential equation, it is natural to assume that one would have begun with a much less sophisticated definition than the one chosen by us (replete with initial conditions, etc.). Most likely, we intuitively view an nth order differential equation as one which we must integrate n times in order to obtain the solution [and, indeed, this intuitive approach is perfectly accurate in the ultra-simple case in which the equation has the form $\frac{d^n y}{dx^n} = f(x)$]. Each time we integrate, we get another arbitrary constant of integration. Thus, we might have been tempted to define the general solution of an nth order differential equation as any solution which contains n arbitrary constants.

What we showed in Exercise 2.4.5 was that such a definition failed to capture the difference between arbitrary and independent arbitrary constants. For those who have not looked at Exercise 2.4.5 or who have but do not wish to interrupt this reading to return to it, let us give another example. Clearly,

$$y = x + c_1 + c_2 \tag{1}$$

contains two arbitrary constants. That is, how we choose c_1 in no way restricts how we may choose c_2. On the other hand, it should be clear that (1) is, in a somewhat disguised form, a 1-parameter family of curves, not a 2-parameter family. That is, since the sum of two arbitrary constants is an arbitrary constant, we may replace the sum $c_1 + c_2$ in (1) by the single arbitrary constant c, to obtain

$$y = x + c. \tag{2}$$

Equation (2) represents a 1-parameter family of curves which is equivalent to the family represented by equation (1), even though two arbitrary constants appear in (1).

To make this discussion more pertinent to our present study of linear differential equations in general, we have seen in the present lesson that if

$$L(y) = 0 \tag{3}$$

is any nth order linear differential equation, and if $y_1 = u_1(x)$, ..., and $y = u_n(x)$ were each solutions of (3), then so also was any linear combination of u_1, ..., and u_n. (In the lecture, we stressed the case $n = 2$, but the result holds for any order.)

In other words, under the given conditions,

$$y = c_1 u_1(x) + \dots + c_n u_n(x) \tag{4}$$

is also a solution of (3), where c_1, ..., and c_n are n arbitrary constants.

The question that we wish to address ourselves to in this section is under what conditions can we tell that the constants in (4) cannot be "condensed," or from the opposite perspective, under what conditions can a family of curves defined with less than n-parameters be the same family as that named by (4) with its n arbitrary constants.

In order not to make this discussion too abstract, let us start with a more concrete illustration. Suppose that equation (3) represented a third-order linear differential equation, and that by "hook or crook" we had somehow discovered that $y = \sin^2 x$, $y = \cos^2 x$, and $y = \cos 2x$ were each solutions of (3). Certainly then, we could conclude that for any choice of constants c_1, c_2, and c_3,

$$y = c_1 \sin^2 x + c_2 \cos^2 x + c_3 \cos 2x \tag{5}$$

was also a solution of (3).*

However, if we now introduce the trigonometric identity

$$\cos 2x \equiv \cos^2 x - \sin^2 x, \tag{6}$$

into (5), we obtain

---

*Do not try to find the specific <u>linear</u> equation for which $\sin^2 x$, $\cos^2 x$ and $\cos 2x$ are solutions. It is often a difficult job, but more importantly, it is irrelevant to our present discussion. It is shown in Exercise 2.5.6 and reinforced in Section C that $\sin^2 x$ and $\cos^2 x$ <u>cannot</u> be solutions of (3) if (3) has <u>constant</u> coefficients. Notice also that it is important to stress <u>linearity</u>. For example, it is simple to construct non-linear differential equations which have given solutions. By way of illustration, the trivial zeroth-order non-linear equation $(y - \sin^2 x)(y - \cos^2 x)(y - \cos 2x) = 0$ admits $\sin^2 x$, $\cos^2 x$ $\cos 2x$ as solutions, but in this case, (5) is not a solution. That is, keep in mind the fact that for linear combinations of solutions to again be solutions requires that the equation be linear.

$$y = c_1 \sin^2 x + c_2 \cos^2 x + c_3(\cos^2 x - \sin^2 x)$$

$$= (c_1 - c_3)\sin^2 x + (c_2 + c_3)\cos^2 x. \qquad (7)$$

Since c_1, c_2, and c_3 are arbitrary constants, so also are $c_1 - c_3$ and $c_2 + c_3$. Letting $k_1 = c_1 - c_3$ and $k_2 = c_2 + c_3$, equation (7) becomes

$$y = k_1 \sin^2 x + k_2 \cos^2 x. \qquad (8)$$

Thus, we see that (8), an equation with <u>two</u> arbitrary constants, names the same family of curves as does (5), an equation with <u>three</u> arbitrary constants. In other words, the constants in (5), while arbitrary, are not independent in the sense that they may be condensed. That is, there is an equation with fewer than n constants [e.g. equation (8)] which defines the same family of curves as does equation (5).

Our central question is what caused this to come about? Hopefully, the answer is clear at least in terms of our concrete example. Namely, one of our solutions (in this case, $u_3 = \cos 2x$) was a linear combination of the previous* solutions ($u_1 = \sin^2 x$, $u_2 = \cos^2 x$). In particular,

$$u_3 = u_2 - u_1$$

in our present illustration.

In other words, returning to the more general case, if any specific member, u_k, of $\{u_1, \ldots, u_n\}$ may be expressed as a linear combination of the preceding members of the set, then the term $c_k u_k(x)$ is "redundant" in equation (4).

*The term "previous" suggests that the solutions are <u>ordered</u>. For example, had we ordered the terms by $u_1 = \cos^2 x$, $u_2 = \cos 2x$, and $u_3 = \sin^2 x$, our identity tells us that $u_2 = u_1 - u_3$. Now, however, u_2 is not a linear combination of "previous" solutions since u_3 is listed after u_2. Notice, however, that since $u_2 = u_1 - u_3$, we may write that $u_3 = u_1 - u_2$ (i.e. $\sin^2 x = \cos^2 x - \cos 2x$) and now u_3 is a linear combination of terms which come before it. In other words, if any member of $\{u_1, \ldots, u_n\}$ is a linear combination of some others, we can always rewrite the equality so that one member is a linear combination of the preceding ones, no matter how the members are listed.

Namely, if

$$u_k(x) = a_1u_1(x) + \dots + a_{k-1}u_{k-1}(x) \tag{9}$$

[where one or more of the a's may be 0], then by (9),

$$y = c_1u_1(x) + \dots + c_{k-1}u_{k-1}(x) + c_ku_k(x) + \dots + c_nu_n(x)$$

may be written as

$$y = c_1u_1(x) + \dots + c_{k-1}u_{k-1}(x) + c_k[a_1u_1(x) + \dots + a_{k-1}u_{k-1}(x)] + $$

$$+ \dots + c_nu_n(x)$$

or

$$y = (c_1 + a_1c_k)u_1(x) + \dots + (c_{k-1} + a_{k-1}c_k)u_{k-1}(x) +$$

$$+ \dots + c_nu_n(x), \tag{10}$$

where in (10) there are only n-1 arbitrary constants since the term involving $u_k(x)$ has vanished.

Again, more concretely, suppose we let $n = 5$ and $u_4 = u_1 + 2u_3$ $(= 1u_1 + 0u_2 + 2u_3)$. Then

$$y = c_1u_1 + c_2u_2 + c_3u_3 + c_4u_4 + c_5u_5 \rightarrow$$

$$y = c_1u_1 + c_2u_2 + c_3u_3 + c_4(u_1 + 2u_3) + c_5u_5 \rightarrow$$

$$y = (c_1 + c_4)u_1 + c_2u_2 + (c_3 + 2c_4)u_3 + c_5u_5 \tag{11}$$

Letting $k_1 = c_1 + c_4$, $k_2 = c_2$, $k_3 = c_3 + 2c_4$, and $k_5 = c_5$, (11) becomes

$$y = k_1u_1 + k_2u_2 + k_3u_3 + k_5u_5$$

and this equation has <u>four</u>, not five, arbitrary constants.

The preceding discussion is usually formalized as follows.

Given the set of functions of x $\{u_1, \dots, u_n\}$, the set is called <u>linearly dependent</u> if <u>at least one member</u> of the set is a <u>linear</u>

combination of the others. (Whenever this happens, it is also true that the one member can be chosen to be a linear combination of the preceding members, no matter in what order the members are listed.)

If the set is not linearly dependent, then it is called linearly independent.*

From an esthetic point of view (i.e. because the definition doesn't depend on how the set is ordered), many authors prefer to begin with a definition of linear independence and then define $\{u_1, \ldots, u_n\}$ to be linearly dependent if it is not linearly independent. When this is done, the usual definition is:

The set $\{u_1, \ldots, u_n\}$ is said to be linearly independent if and only if $c_1u_1 + \ldots + c_nu_n = 0 \rightarrow c_1 = \ldots = c_n = 0$.** That is, $\{u_1, \ldots, u_n\}$ is linearly independent $\leftrightarrow$ the only way a linear combination of $u_1, \ldots, u_n$ can be 0 is if all the coefficients are 0.

Trivially, if $c_1, = \ldots = c_n = 0$, then $c_1u_1 + \ldots + c_nu_n = 0$, but the converse must also be true for linear independence.

For example, returning to $\{\sin^2x, \cos^2x, \cos 2x\}$, clearly

$$0 \sin^2x + 0 \cos^2x + 0 \cos 2x \equiv 0.$$

However, the fact that $\cos 2x \equiv \cos^2x - \sin^2x$ means that

$$\sin^2x - \cos^2x + \cos 2x \equiv 0$$

or

$$\underset{\substack{\uparrow \\ c_1}}{(1)}\sin^2x + \underset{\substack{\uparrow \\ c_2}}{(-1)}\cos^2x + \underset{\substack{\uparrow \\ c_3}}{(1)}\cos 2x \equiv 0. \qquad (12)$$

From (12), we see that we have a linear combination of $\sin^2x$, $\cos^2x$,

*As a matter of semantics, one sometimes says that $u_1, u_2, \ldots,$ and u_n are linearly independent (dependent) rather than $\{u_1, \ldots, u_n\}$ is linearly independent (dependent). We shall not worry about the choice of wording and will use whatever expression seems more comfortable at the given time.

**Recall that $u = 0$ means $u(x) \equiv 0$, i.e. $u = 0$ means $u(x) = 0$ for every x in the domain of u. Thus, what we really mean is that $c_1u_1(x) + \ldots + c_nu_n(x) \equiv 0 \rightarrow c_1 = \ldots = c_n = 0$. For example, $x^2 - 4x + 3 = 0 \leftrightarrow x = 1$ or $x = 3$. Hence, if we let $u(x) = x^2 - 4x + 3$, we would not say that $u = 0$ since it is false that $x^2 - 4x + 3 \equiv 0$.

and cos 2x which is identically zero but not all the coefficients [in fact, in this illustration, (12) shows that none of the coefficients] is 0. Hence, $\{\sin^2 x, \cos^2 x, \cos 2x\}$ would still be linearly dependent under the new definition.

In summary, the two methods of defining linear dependence are equivalent, but by beginning with the one for linear independence, we do not have to become involved with the discussion of the order in which the members of the set are listed.

Which ever definition we choose, the point is that our constants are "condensable" in (4) if and only if $\{u_1, \ldots, u_n\}$ is linearly dependent.

Of course, with our more sophisticated definition of "general solution," we are still not sure that the mere fact that $\{u_1, \ldots, u_n\}$ is linearly independent is enough to guarantee that we can meet the initial conditions. Thus, at this stage of the game, all we can say is that linear independence is necessary (but perhaps not sufficient) for (4) to be the general solution of (3). We shall talk about this in the next section, but for now, we would like to point out one very major property of a linearly independent set.

Recall that in our study of partial fractions, we often invoked the technique that if

$$a_o + a_1 x + a_2 x^2 + \ldots + a_n x^n \equiv b_o + b_1 x + b_2 x^2 + \ldots + b_n x^n$$

then

$$a_o = b_o,\ a_1 = b_1,\ a_2 = b_2,\ \ldots,\ \text{and } a_n = b_n.$$

While this may seem very natural, its validity depends on the fact that $\{1, x, x^2, \ldots, x^n\}$ is a linearly independent set. [As mentioned in Part 1, the easiest way to prove that $\{1, x, x^2, \ldots, x^n\}$ is linearly independent is by successive differentiation. By way of review, let's analyze the particular case n = 3. If

(i) $a_o + a_1 x + a_2 x^2 + a_3 x^3 \equiv 0$

then

(ii) $a_1 + 2a_2 x + 3a_3 x^2 \equiv 0$

(iii) $2a_2 + 6a_3x \equiv 0$

and

(iv) $6a_3 \equiv 0$.

From (iv), $a_3 = 0$. Setting $a_3 = 0$ in (iii) yields that $a_2 = 0$, and knowing that both a_2 and $a_3 = 0$ implies, from (ii), that $a_1 = 0$. Once $a_1 = a_2 = a_3 = 0$, (i) implies that a_o is also 0. Hence, we have shown that the only linear combination of 1, x, x^2, and x^3 which is 0 is $0 + ox + ox^2 + ox^3$.]

The crucial point is that we may equate like coefficients if and only if $\{u_1, \ldots, u_n\}$ is linearly independent. For example, suppose

$$A \sin^2 x + B \cos^2 x + C \cos 2x \equiv 0 \equiv 0 \sin^2 x + 0 \cos^2 x + 0 \cos 2x.$$

If we equate coefficients of like terms, we would conclude that

$$A = 0,\ B = 0,\ \text{and } C = 0,$$

but this need not be true!

In particular, since $\cos 2x \equiv \cos^2 x - \sin^2 x$, it follows that

$$\sin^2 x - \cos^2 x + \cos 2x \equiv 0$$

from which we see that

$$A \sin^2 x + B \cos^2 x + C \cos 2x \equiv 0$$

if $A = 1$, $B = -1$, and $C = 1$.

More generally, in this example, since $\cos 2x \equiv \cos^2 x - \sin^2 x$, we may always rewrite

$$A \sin^2 x + B \cos^2 x + C \cos 2x$$

as

$$A \sin^2 x + B \cos^2 x + C(\cos^2 x - \sin^2 x) = (A - C)\sin^2 x + (B + C)\cos^2 x.$$

Consequently, if

$$\begin{cases} A - C = 0 \\ B + C = 0 \end{cases}$$

which is equivalent to $A = C = -B$, then

$$A \sin^2 x + B \cos^2 x + C \cos 2x \equiv 0.$$

That is,

$$A(\sin^2 x - \cos^2 x + \cos 2x) \equiv 0$$

for any value of A.

The key point is that if

$$a_1 u_1 + \ldots + a_n u_n \equiv b_1 u_1 + \ldots + b_n u_n$$

then

$$(a_1 - b_1)u_1 + \ldots + (a_n - b_n)u_n \equiv 0, \quad (13)$$

regardless of whether $\{u_1, \ldots, u_n\}$ is linearly independent.

<u>However</u>, since (13) represents a linear combination of $u_1, \ldots,$ and u_n which is identically zero, then <u>all</u> the coefficients <u>must</u> be zero if and only if $\{u_1, \ldots, u_n\}$ is linearly <u>independent</u>. In other words, then, we conclude from (13) that

$$(a_1 - b_1) = (a_2 - b_2) = \ldots (a_n - b_n) = 0 \quad (14)$$

if and only if $\{u_1, \ldots, u_n\}$ is linearly independent.

Clearly (14) is equivalent to

$$a_1 = b_1,\ a_2 = b_2,\ \ldots, \text{ and } a_n = b_n.$$

Thus, in any method involving undetermined coefficients, the success of the method hinges on linear independence. In summary, then, if

$$\sum_{k=1}^{n} a_k u_k \equiv \sum_{k=1}^{n} b_k u_k$$

and if $\{u_1, \ldots, u_n\}$ is linearly independent, then

$a_1 = b_1$, $a_2 = b_2$, ..., and $a_n = b_n$.

We shall have more to say about this as a note at the end of the next section.

C

Application to Linear Differential Equations

Up to now, our discussion has been rather general concerning the concept of linear dependence. In Block 3, we shall try to prove theorems about properties of linearly independent vectors, and we shall also see what other properties are possessed by linearly independent sets of vectors. For our present purposes, however, such a discussion takes us too far a field of our immediate concern of differential equations.

Let us, therefore, return to our study of linear homogeneous differential equations.

Given the nth-order equation

$$L(y) = 0, \tag{1}$$

let us assume that $y = u_1(x)$, ..., and $y = u_n(x)$ are each particular solutions of (1). What we would like to know is whether the set $\{u_1, \ldots, u_n\}$ is linearly independent.

In general, this is a very difficult question to answer, but in this particular case, we have some powerful calculus at our disposal. For example, to be a solution of (1), a function must possess at least its first n derivatives (since the term $\frac{d^n y}{dx^n}$ occurs). For what we have in mind, even possession of the first (n-1) derivatives would be sufficient.

Briefly outlined, our method of attack is the following.

(i) To show that $\{u_1, \ldots, u_n\}$ is linearly independent, we must prove that if $c_1u_1 + \ldots + c_nu_n = 0$ then $c_1 = \ldots = c_n = 0$, or stated in terms of identities, we must prove that if

$$c_1u_1(x) + \ldots + c_nu_n(x) \equiv 0 \tag{2}$$

then

$c_1 = \ldots = c_n = 0.$

(ii) We, therefore, assume that

$$c_1u_1(x) + c_2u_2(x) + \ldots + c_3u_n(x) \equiv 0.$$

Since (2) is an identity, we may equate the derivatives of both sides of the equation to obtain

$$c_1u_1'(x) + c_2u_2'(x) + \ldots + c_3u_n'(x) \equiv 0. \qquad (3)$$

We may continue in this manner until we obtain

$$c_1u_1^{(n-1)}(x) + \ldots + c_nu_n^{(n-1)}(x) \equiv 0,$$

at which time we obtain the system of equations

$$\left.\begin{array}{l} c_1u_1(x) + \ldots + c_nu_n(x) \equiv 0 \\ \vdots \\ c_1u_1^{(n-1)}(x) + \ldots + c_nu_n^{(n-1)}(x) \equiv 0 \end{array}\right\} . \qquad (4)$$

Clearly, one solution of (4) is $c_1 = \ldots = c_n = 0$. How can we be sure that there are no others?

Well, one thing we could do is replace x in (4) by some specific value, say, $x = x_o$. If we do this, (4) becomes a system of n linear algebraic equations in the n unknowns c_1, ..., and c_n. That is, (4) becomes

$$\left.\begin{array}{l} c_1u_1(x_o) + \ldots + c_nu_n(x_o) = 0 \\ \vdots \\ c_1u_1^{(n-1)}(x_o) + \ldots + c_nu_n^{(n-1)}(x_o) = 0 \end{array}\right\} . \qquad (4')$$

Since $c_1 = \ldots = c_n = 0$ is one solution of (4') and since a linear system has a unique solution if and only if the determinant of coefficients in (4') is not zero, we need only check that the determinant of coefficients in (4') is not zero.

Now, the determinant of coefficients in (4') is

$$\begin{vmatrix} u_1(x_o) & \cdots\cdots & u_n(x_o) \\ u_1'(x_o) & \cdots\cdots & u_n'(x_o) \\ \vdots & & \\ u_1^{(n-1)}(x_o) & \cdots\cdots & u_n^{(n-1)}(x_o) \end{vmatrix}$$

and clearly its value depends on x_o.

However, if there is <u>even one</u> number, x_o, for which this determinant is not zero, then for this value of x_o, the system of equations in (4') has a unique solution, and since $c_1 = \ldots = c_n = 0$ is one solution, it must, therefore, be the only one. But since $c_1, \ldots,$ and c_n are <u>constants</u>, the fact that $c_1 = \ldots = c_n = 0$ for one value of x_o means that $c_1 = \ldots = c_n = 0$ for all x.

In summary, then, if $y = u_1(x), \ldots,$ and $y = u_n(x)$ are each solutions of the nth-order linear equation

$$L(y) = 0$$

then $\{u_1, \ldots, u_n\}$ is linearly independent as soon as there exists <u>at least</u> one value of x for which

$$\begin{vmatrix} u_1(x) & \cdots\cdots & u_n(x) \\ \vdots & & \vdots \\ u_1^{(n-1)}(x) & \cdots\cdots & u_n^{(n-1)}(x) \end{vmatrix} \neq 0. \tag{5}$$

The left side of (5) is given a very special name. It is called the Wronskian (determinant) of $\{u_1, \ldots, u_n\}$ and usually abbreviated as $W(u_1, \ldots, u_n)$. In other words, the <u>number</u> $W(u_1, \ldots, u_n)$ is defined by

$$W(u_1, \ldots, u_n) = \begin{vmatrix} u_1(x) & \cdots\cdots & u_n(x) \\ \vdots & & \vdots \\ u_1^{(n-1)}(x) & \cdots\cdots & u_n^{(n-1)}(x) \end{vmatrix} \tag{6}$$

and since u_1, ..., and u_n are each functions of x, so also is $W(u_1, \ldots, u_n)$. [For this reason, one also sees the notation W(x) as well as $W(u_1, \ldots, u_n)$.]

Perhaps the best way to digest the previous remarks is by viewing a few examples.

Example #1

Suppose we solve $y''' - y' = 0$. The auxiliary equation is $r^3 - r = 0$, so that $y = 1$, $y = e^x$, and $y = e^{-x}$ are each solutions of the equation. Letting $u_1(x) = 1$, $u_2(x) = e^x$, and $u_3(x) = e^{-x}$, we obtain $u_1' = u_1'' = 0$, $u_2'(x) = u_2''(x) = e^x$, $u_3'(x) = -e^{-x}$, and $u_3''(x) = e^{-x}$.

Hence,

$$W(u_1,u_2,u_3) = \begin{vmatrix} \overset{(+)}{1} & e^x & e^{-x} \\ \overset{(-)}{0} & e^x & -e^{-x} \\ \overset{(-)}{0} & e^x & e^{-x} \end{vmatrix}$$

$$= \begin{vmatrix} e^x & -e^{-x} \\ e^x & e^{-x} \end{vmatrix}$$

$$= e^x e^{-x} - e^x(-e^{-x})$$

$$= 1 + 1$$

$$= 2. \tag{7}$$

Equation (7) is stronger than what is actually needed. All we needed to show was the existence of at least one value of x for which $W(u_1,u_2,u_3) \neq 0$ and equation (7) shows that for <u>every</u> real x, $W(u_1,u_2,u_3) \neq 0$.

In any event, we conclude from (7) that $\{1, e^x, e^{-x}\}$ is linearly independent.

As a footnote to this exercise, notice that in terms of our discussion at the end of the previous section, we may conclude that if, for example,

$$(A + B + C) + (2A + B - C)e^x + (3A - B + C)e^{-x} = 2e^x - 3e^{-x}$$

$$(= 0 + 2e^x - 3e^{-x})$$

then

$$\left.\begin{aligned} A + B + C &= 0 \\ 2A + B - C &= 2 \\ 3A - B + C &= -3 \end{aligned}\right\}$$

since we may equate the coefficients of "like" terms whenever our set of functions (in this case, $\{1, e^x, e^{-x}\}$) is linearly independent.

Example #2

Find three linearly independent solutions of

$$y''' + y' = 0.$$

Solution

From $r^3 + r = 0 = r(r^2 + 1)$, we conclude that three solutions are $u_1(x) = 1$, $u_2(x) = \sin x$, $u_3(x) = \cos x$. Hence,

$$u_1'(x) = 0, \; u_2'(x) = \cos x, \; u_3'(x) = -\sin x$$

and

$$u_1''(x) = 0, \; u_2''(x) = -\sin x, \; u_3''(x) = -\cos x.$$

Therefore,

$$W(u_1, u_2, u_3) = \begin{vmatrix} 1 & \sin x & \cos x \\ 0 & \cos x & -\sin x \\ 0 & -\sin x & -\cos x \end{vmatrix}$$

$$= \begin{vmatrix} \cos x & -\sin x \\ -\sin x & -\cos x \end{vmatrix}$$

$$= -\cos^2 x - \sin^2 x$$

$$= -1.$$

In particular, then, $W(u_1,u_2,u_3) \not\equiv 0$; hence, $\{1, \sin x, \cos x\}$ is linearly independent. Consequently, $y = c_1 + c_2 \sin x + c_3 \cos x$ is the general solution of $y''' + y' = 0$.

We now want to throw in a word of caution. All we have shown so far is that <u>if</u> $W(u_1, \ldots, u_n) \not\equiv 0$, then, $\{u_1, \ldots, u_n\}$ is linearly independent. We have not shown that if $W(u_1, \ldots, u_n) \equiv 0$ then $\{u_1, \ldots, u_n\}$ is linearly dependent. One reason we have not shown this is that it need not be true! (Quite in general, if $p \rightarrow q$, we cannot conclude that if p is false, then q is also false.) We illustrate this in our next example.

Example #3

Let $u_1(x) = x^3$ and $u_2(x) = |x|^3$. Compute $W(u_1,u_2)$.

Solution

By definition,

$$W(u_1,u_2) = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$$

The "major" problem here is to compute $u_2'(x)$. Recall that

$$|x| = \begin{cases} x, & \text{if } x \geqslant 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Hence, in this example,

$$u_2(x) = |x|^3 = \begin{cases} x^3, & \text{if } x \geqslant 0 \\ -x^3, & \text{if } x < 0 \end{cases}$$

Hence,

$$u_2'(x) = \begin{cases} 3x^2, & \text{if } x \geqslant 0 \\ -3x^2, & \text{if } x < 0 \end{cases}$$

Since $u_1'(x) = 3x^2$ for all x, we may proceed by cases.

Case 1: $x \geq 0$

Then

$$W(u_1,u_2) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix}$$

$$= \begin{vmatrix} x^3 & x^3 \\ 3x^2 & 3x^2 \end{vmatrix}$$

$$= 3x^5 - 3x^5$$

$$\equiv 0.$$

Case 2: $x < 0$

Then

$$W(u_1,u_2) = \begin{vmatrix} u_1(x) & u_2(x) \\ u_1'(x) & u_2'(x) \end{vmatrix}$$

$$= \begin{vmatrix} x^3 & -x^3 \\ 3x^2 & -3x^2 \end{vmatrix}$$

$$= -3x^5 - 3x^2(-x^3)$$

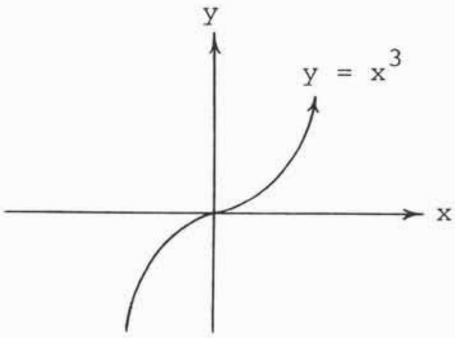
$$\equiv 0.$$

Hence, in this example,

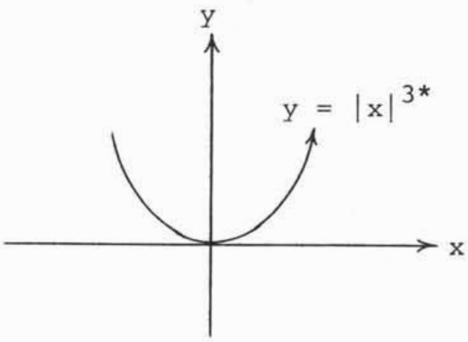
$$W(u_1,u_2) \equiv 0.$$

Yet, $\{u_1,u_2\}$ is linearly independent!

That is, $|x|^3$ is not a constant multiple of x^3. Pictorially, the graph of x^3 is



while the graph of $|x|^3$ is



That is, for $x \geqslant 0$, $|x|^3 = x^3 = 1(x^3)$ and for $x < 0$, $|x^3| = -x^3 = -1(x^3)$ } and since $1 \neq -1$, $|x|^3 \neq kx^3$

Example #4

Compute $W(u_1,u_2,u_3)$ where $u_1(x) = \sin^2 x$, $u_2(x) = \cos^2 x$, and $u_3(x) = \cos 2x$.

Solution

We have

$u_1'(x) = 2 \sin x \cos x = \sin 2x$; $u_1''(x) = 2 \cos 2x$

$u_2'(x) = 2 \cos x\ (- \sin x) = - \sin 2x$; $u_2''(x) = -2 \cos 2x$

$u_3'(x) = -2 \sin 2x$; $u_3''(x) = -4 \cos 2x$.

*Recall that, in general, one obtains $y = |f(x)|$ from $y = f(x)$ by reflecting about the x-axis the portion of $y = f(x)$ which lies below the x-axis.

Hence,

$$W(u_1,u_2,u_3) = \begin{vmatrix} \sin^2 x & \cos^2 x & \cos 2x \\ \sin 2x & -\sin 2x & -\sin 2x \\ 2\cos 2x & -2\cos 2x & -4\cos 2x \end{vmatrix}$$

$$= 2 \sin 2x \cos 2x \begin{vmatrix} \sin^2 x & \cos^2 x & \cos 2x \\ 1 & -1 & -2 \\ 1 & -1 & -2 \end{vmatrix}*$$

$$= 2 \sin 2x \cos 2x \, [0]*$$

$$\equiv 0.$$

The interesting thing about Example #4 is that we have previously seen that the functions u_1, u_2, and u_3 in this example are linearly dependent. Yet, the fact that the Wronskian is zero does not help us come to this conclusion, since in the previous example, we showed that even when the Wronskian is identically zero, the functions may be linearly independent.

It would be nice to be able to use the Wronskian as a necessary and sufficient condition to determine whether or not functions were linearly independent. Yet Examples #3 and #4 show us that this cannot be done. However, let us also observe that up to now, nothing in our examples has made it necessary for the functions to be solutions of a linear homogeneous differential equation.

What is very interesting is the fact that if $\{u_1, \ldots, u_n\}$ is a set each of whose elements is a <u>solution</u> of a <u>linear homogeneous differential</u> equation, then:

(1) $\{u_1, \ldots, u_n\}$ is linearly dependent if and only if $W(u_1, \ldots, u_n) \equiv 0$; and

(2) $W(u_1, \ldots, u_n)$ is either identically equal to zero or else it is never equal to zero.

---

*In Block 3, we examine the various properties of determinants in greater detail (such as factoring out a common factor of a row or that the determinant is 0 if two rows are the same). But, if you do not know these results, the same answer can be determined by computing the 3 by 3 determinant in the traditional manner.

In other words, as long as $u_1, \ldots, u_n$ are solutions of $L(y) = 0$, then we may test the linear dependence of $\{u_1, \ldots, u_n\}$ simply by computing $W(u_1, \ldots, u_n)$. If $W(u_1, \ldots, u_n) = 0$, then $\{u_1, \ldots, u_n\}$ is linear dependent, otherwise, not. Moreover, in this case, either $W(u_1, \ldots, u_n) \equiv 0$, or else it is never equal to 0 for any value of x.

While any proof of these two remarks will be left as optional notes at the end of this section, it should be pointed out that there is a rather strong connection between the Wronskian determinant and the definition of what is meant by a general solution of a differential equation. In particular, if $y = c_1u_1(x) + \ldots + c_nu_n(x)$ is the general solution of $L(y) = 0$ on some interval I, then for any $x_o \varepsilon I$, we must be able to determine the c's uniquely such that for given numbers $y_o, \ldots,$ and $y_0^{(n-1)}$, we can satisfy the conditions that when $x = x_o$, $y = y_o$, $y' = y_o'$, $\ldots$, and $y^{(n-1)} = y_o^{(n-1)}$. In terms of a system of equations, this means that

$$\left.\begin{aligned} y_o &= c_1u_1(x_o) + \ldots + c_nu_n(x_o) \\ &\;\;\vdots \\ y_o^{(n-1)} &= c_1u_1^{(n-1)}(x_o) + \ldots + c_nu_n^{(n-1)}(x) \end{aligned}\right\}$$

must have a unique solution for $c_1, \ldots,$ and c_n, and this in turn implies that

$$\begin{vmatrix} u_1(x_o) & \ldots\ldots & u_n(x_o) \\ \vdots & & \\ u_1^{(n-1)}(x_o) & \ldots\ldots & u_n^{(n-1)}(x_o) \end{vmatrix} \neq 0 \qquad (8)$$

where the left side of (8) is precisely $W(u_1, \ldots, u_n)$ evaluated at $x = x_o$.

Some Optional Notes

(I) Suppose that $y = u_1(x)$, $y = u_2(x)$, $\ldots$, and $y = u_n(x)$ are solutions (but not necessarily linearly independent) of

$$y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_o(x)y = 0.$$

Let W denote the Wronskian determinant of $\{u_1, \ldots, u_n\}$. Then W satisfies the first order linear equation

$$\frac{dW}{dx} + a_{n-1}(x)W = 0. \tag{1}$$

Since we have not studied n by n determinants in detail as yet, let us limit our "proof" of (1) to the case n = 2, in which we may use "brute force" computation without too much inconvenience.

We have that $y = u_1(x)$ and $y = u_2(x)$ satisfy

$$y'' + a_1(x)y' + a_o(x)y = 0. \tag{2}$$

That is,

$$u_1'' + a_1u_1' + a_ou_1 = u_2'' + a_1u_2' + a_ou_2 = 0. \tag{3}$$

Then, since

$$W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix} = u_1u_2' - u_1'u_2 \tag{4}$$

we have from (4) that

$$\frac{dW}{dx} = \frac{d}{dx}(u_1u_2' - u_1'u_2)$$

$$= (u_1u_2'' + u_1'u_2') - (u_1'u_2' + u_1''u_2)$$

$$= u_1u_2'' - u_1''u_2. \tag{5}$$

Using (4) and (5), we conclude that

$$\frac{dW}{dx} + a_1W = u_1u_2'' - u_1''u_2 + a_1u_1u_2' - a_1u_1'u_2$$

$$= u_1(u_2'' + a_1u_2') - u_2(u_1'' + a_1u_1'). \tag{6}$$

Now, from (3)

$$u_2'' + a_1u_2' = -a_ou_2$$

and

$$u_1'' + a_1 u_1' = -a_o u_1.$$

Putting these results in (6) yields

$$\frac{dW}{dx} + a_1 W = u_1(-a_o u_2) - u_2(-a_o u_1)$$

$$= -a_o u_1 u_2 + a_o u_1 u_2$$

$$\equiv 0.$$

What (1) implies is that $W(x)$ is either identically zero or else it is never zero. Namely,

$$e^{\int a_{n-1}(x)dx}$$

is an integrating factor of (1) whereupon we obtain

$$\frac{d}{dx}\left[e^{\int a_{n-1}(x)dx} W\right] = 0,$$

so that

$$W(x) = ce^{-\int a_{n-1}(x)dx}. \qquad (7)$$

Since the exponential can never be zero, we see from (7) that $W(x) = 0 \leftrightarrow c = 0$, but if $c = 0$, $W(x) \equiv 0$.

Equation (7) establishes the result that if $u_1(x)$, ..., and $u_n(x)$ are solutions of $L(y) = 0$ on $[a,b]$ and if $W(x_o) \neq 0$ for even a single point $x_o \varepsilon (a,b)$, then $W(x) = 0$ for all $x \varepsilon (a,b)$.

(II) In the special case of constant coefficients, we have seen that the only solutions of $L(y) = 0$ are those of the form $x^k e^{\alpha x} \sin \beta x$ or $x^k e^{\alpha x} \cos \beta x$. In fact, if we allow the use of non-real numbers, these solutions have the form $x^k e^{rx}$. Moreover, if $L(e^{rx}) = 0$ has no repeated roots, all solutions are of the form e^{rx}.

Suppose, then, that $y = e^{r_1 x}$, $y = e^{r_2 x}$, ..., and $y = e^{r_n x}$ are all solutions of $L(y) = 0$, where $L(y) = 0$ is an nth-order linear homogeneous differential equation with constant coefficients. Then, $W(e^{r_1x}, \ldots, e^{r_nx})$ is given by

$$\begin{vmatrix} e^{r_1x} & e^{r_2x} & \ldots & e^{r_nx} \\ r_1e^{r_1x} & r_2e^{r_2x} & \ldots & r_ne^{r_nx} \\ r_1^2e^{r_1x} & r_2^2e^{r_2x} & \ldots & r_n^2e^{r_nx} \\ \vdots & & & \\ r_1^{n-1}e^{r_1x} & r_2^{n-1}e^{r_2x} & \ldots & r_n^{n-1}e^{r_nx} \end{vmatrix} \qquad (8)$$

"Factoring out" e^{r_kx} from the kth column of (8) [where k = 1,2,...,n], we see that (8) is equal to

$$e^{r_1x}e^{r_2x} \ldots e^{r_nx} \begin{vmatrix} 1 & 1 & 1 & \ldots & 1 \\ r_1 & r_2 & r_3 & \ldots & r_n \\ r_1^2 & r_2^2 & r_3^2 & \ldots & r_n^2 \\ \vdots & & & & \\ r_1^{n-1} & r_2^{n-1} & r_3^{n-1} & \ldots & r_n^{n-1} \end{vmatrix} \qquad (9)$$

The determinant

$$\begin{vmatrix} 1 & 1 & \ldots & 1 \\ r_1 & r_2 & \ldots & r_n \\ r_1^2 & r_2^2 & \ldots & r_n^2 \\ \vdots & & & \\ r_1^{n-1} & r_2^{n-1} & \ldots & r_n^{n-1} \end{vmatrix}$$

is given the special name of <u>The Vandemonde Determinant</u>. It can be shown that the value of this determinant is given by the product

$$\prod_{i>j} (r_i - r_j).^{*} \tag{10}$$

While we shall not prove this result here, we shall at least verify it in the cases $n = 2$ and $n = 3$.

Namely,

$$\begin{vmatrix} 1 & 1 \\ r_1 & r_2 \end{vmatrix} = r_2 - r_1$$

and

$$\begin{vmatrix} 1 & 1 & 1 \\ r_1 & r_2 & r_3 \\ r_1^2 & r_2^2 & r_3^2 \end{vmatrix} = \begin{vmatrix} r_2 & r_3 \\ r_2^2 & r_3^2 \end{vmatrix} - \begin{vmatrix} r_1 & r_3 \\ r_1^2 & r_3^2 \end{vmatrix} + \begin{vmatrix} r_1 & r_2 \\ r_1^2 & r_2^2 \end{vmatrix}$$

$$= r_2 r_3^2 - r_2^2 r_3 - r_1 r_3^2 + r_1^2 r_3 + r_1 r_2^2 - r_1^2 r_2 \tag{11}$$

and a trivial check shows that (11) is equal to

$$(r_3 - r_2)(r_3 - r_1)(r_2 - r_1)$$

which is $\prod_{i>j} (r_i - r_j)$.

The key point is that once we assume the truth of (10), (9) reveals that

$$W(e^{r_1 x}, \dots, e^{r_n x}) = e^{(r_1 + \dots + r_n)x} \prod_{i>j} (r_i - r_j). \tag{12}$$

Since $e^{(r_1 + \dots + r_n)x}$ can never be zero and since $\prod_{i>j} (r_i - r_j) = 0$ if and only if $r_i = r_j$ (i.e. the only way a product can be zero is if

---

*In the same way that one uses $\prod_{k=1}^{n} a_k$ as an abbreviation for $(a_1 + \dots + a_n)$, one uses $\prod_{k=1}^{n} a_k$ as an abbreviation for the product $a_1 a_2 \dots a_n$.

at least one of the factors is zero), we conclude from (12) that $W(e^{r_1 x}, \ldots, e^{r_n x}) = 0 \leftrightarrow r_i = r_j$ for some $i \neq j$; $i,j = 1, 2, \ldots, n$. In other words, then, if $r_1, \ldots,$ and r_n are all different, $\{e^{r_1 x}, \ldots, e^{r_n x}\}$ is a linearly independent set.

More generally, it can be shown that the only way a set of terms of the form $\{x^k e^{\alpha x} \cos \beta x\}$ can be linearly dependent is if k, α, and β are the same for two different terms.

While we are being a bit skimpy about the details, our main aim is to help you get a better idea of what we mean by linear independence and how the Wronskian determinant plays a key role in determining whether n solutions of an nth order linear equation generate the general solution.