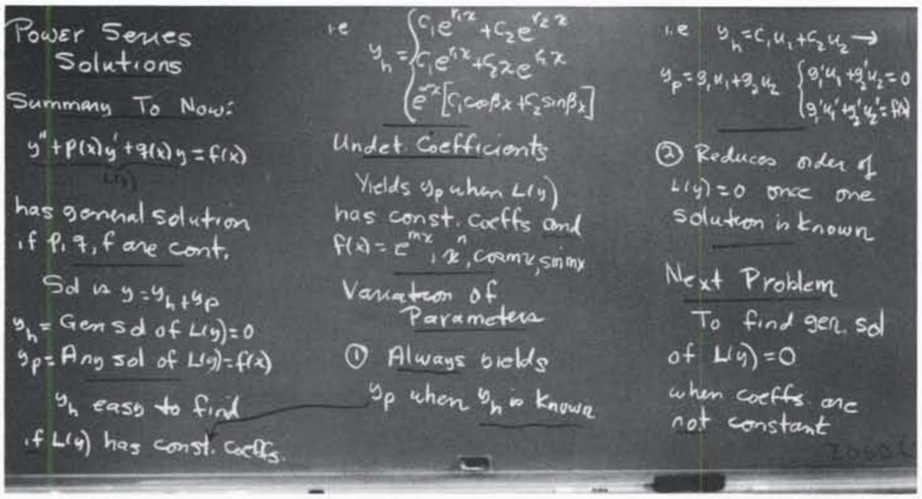
Unit 8: The Use of Power Series

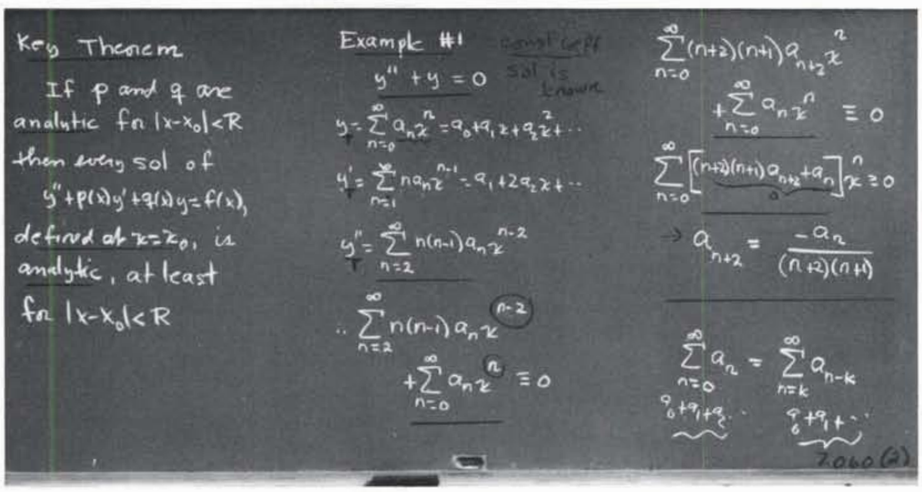
1. Overview

The method of variation of parameters quarantees us the general solution of $L(y) = f(x)$ once we know the general solution of $L(y) = 0$. There is a very general class of equations of the form $L(y) = 0$ for which we can not only be sure the general solution exists but for which we can also construct the general solution in the form of a power series. Among other things, therefore, this unit supplies us with another important application of power series.

2. Lecture 2.060



a.



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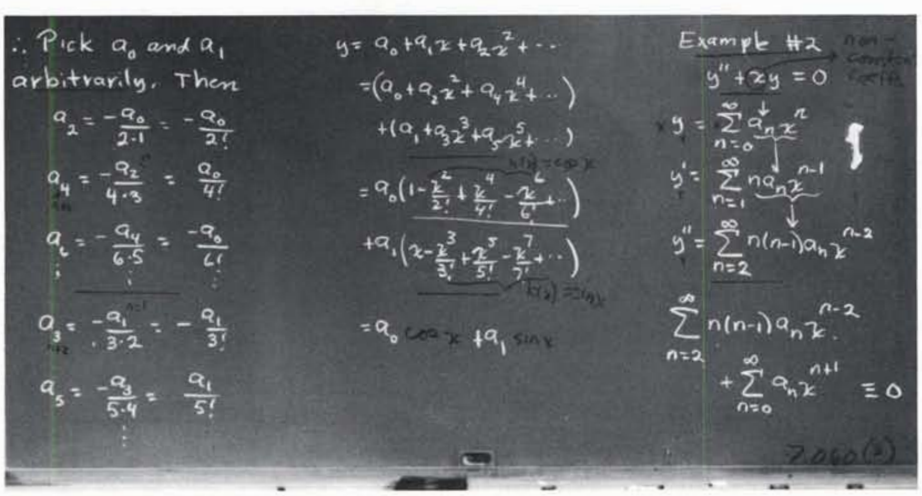
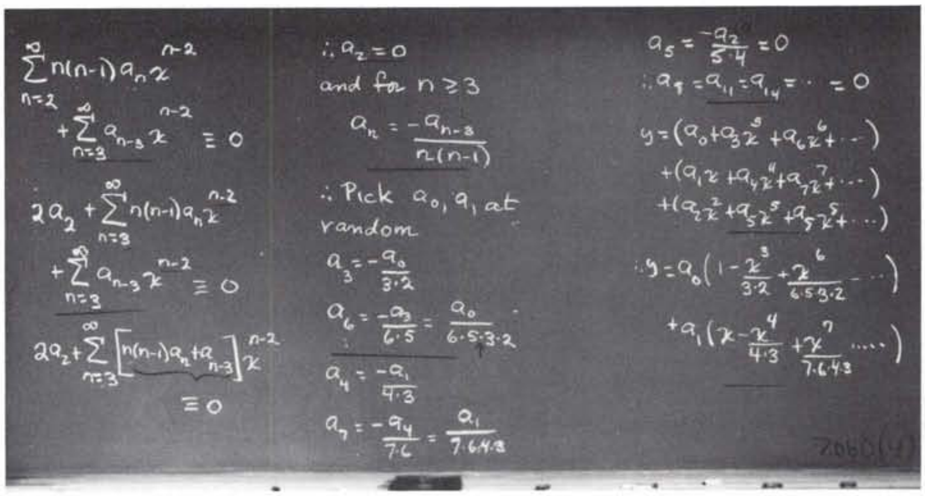


c.

2. Lecture 2.060 continued



d.

3. Do the Exercises. (Every exercise in this Unit is designated as a learning exercise. The main reason is that the material is not covered in the text).

Note #1:
All exercises are given in the form $L(y) = 0$. The reason for this is that we can use variation of parameters to solve $L(y) = f(x)$ once the general solution of $L(y) = 0$ is known.

Note #2:
To help supply you with a brief review of convergence as well as with a few manipulative devices which are helpful in the study of series solutions, we have included at the end of the exercises a special preface as a "preamble" to the solutions of the exercises in this unit. Feel free to read this preface before you begin working on the exercises.

4. Exercises:

2.8.1(L)

a. Rewrite

$$\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n$$

in an equivalent form which involves a single infinite series.

b. Determine each of the coefficients $a_o, a_1, \ldots a_n, \ldots$, if

$$\sum_{n=1}^{\infty} na_n x^{n-1} - \sum_{n=1}^{\infty} na_n x^n + \sum_{n=0}^{\infty} a_n x^n \equiv 0.$$

c. Let

$$y = \sum_{n=0}^{\infty} a_n x^n$$

and use the power series technique to find the general solution of $(1 - x)dy/dx + y = 0$.

2.8.2(L)

Use the power series technique, letting

$$y = \sum_{n=0}^{\infty} a_n x^n$$

to find the general solution of $(1 - x)dy/dx - y = 0$.

2.8.3(L)

Find the general solution of $dy/dx - 2xy = 0$ in the form

$$y = \sum_{n=0}^{\infty} a_n x$$

where a_o is arbitrary.

2.8.4(L)

Find all the solutions of

$$x \frac{dy}{dx} + y = 0$$

which can be written in the form

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

Read the optional note which follows the solution of this exercise for a deeper look at what happened here.

2.8.5(L)

Use series to find the general solution of

$$(1 - x^2)y'' - xy' + y = 0, \text{ where } |x| < 1$$

in the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

with a_o and a_1 arbitrary.

2.8.6(L)

Find the general solution of $(1 + x^2)y'' - xy' + y = 0$ in the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

with a_o and a_1 arbitrary.

2.8.7(L)

a. Find the general solution of $y'' - xy = 0$ in the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

where a_o and a_1 are arbitrary.

b. Find the particular solution, $y = f(x)$, of the equation $y'' - xy = 0$, given that $f(0) = 0$ and $f'(0) = 1$.

c. With $f(x)$ as in part (b) compute $f(1)$ to the nearest hundredth.

2.8.8(L)

Find all solutions of $x^3y'' + xy' - y = 0$ which are analytic at $x = 0$ (i.e., find all solutions which have the form

$$y = \sum_{n=0}^{\infty} a_n x^n).$$

Preface: Part 1 - A Brief Review of Convergence

1. Suppose $\{f_n(x): n = 1, 2, 3, \ldots\}$ is a sequence of functions defined on some common interval, I.

Definition #1

The sequence $\{f_n(x)\}$ is said to be pointwise convergent on I if

$\lim_{n\to\infty} f_n(x_o)$

exists for each $x_o \varepsilon I$.

2. If the sequence $\{f_n(x)\}$ is pointwise convergent on I, we define the limit function, f(x), of this sequence by

$$f(x_o) = \lim f_n(x_o), \text{ for each } x_o \varepsilon I. \quad (1)$$

3. Notice that in computing $\lim_{n\to\infty} f_n(x_o)$, our "tolerance limit", ε, usually depends on x_o. That is, to say that $f(x_o) = \lim_{n\to\infty} f_n(x_o)$ means that given $\varepsilon>0$ there exists N such that $n \geq N \to |f(x_o) - f_n(x_o)| < \varepsilon$; and for the same ε, different values of x_o may determine different values of N.

4. If I had a finite number of points (which it doesn't since an interval is a connected segment of the x-axis, and hence has infinitely many points), the fact that N dependent on x_o would be irrelevant since we could examine each N and then choose the greatest. But with an infinite set there need not be a greatest member. For example, if $0 < x_o < 1$ and for a given x_o, $N = 1/x_o$, then as $x_o \to 0$, $N \to \infty$.

This in itself is not bad. What is bad is that certain "self-evident" results need not be true. By way of illustration, recall the example in Part 1 of our course in which we defined f_n on [0,1] by

$$f_n(x) = x^n, \ 0 \leq x \leq 1.$$

Then, since $\lim_{n\to\infty} f_n(x) = 0$, if $0 \leq x < 1$ and 1, if x = 1, the limit function f(x) exists and is indeed given by

$$f(x) = \begin{cases} 0, & 0 \leq x < 1 \\ 1, & x = 1 \end{cases}.$$

Thus, each member of the sequence $f_n(x)$ is continuous on [0,1] but the limit function f is discontinuous at x = 1.

This violates our "intuitive" feeling that the limit of a sequence of continuous functions should be a continuous function.

5. Since the afore mentioned trouble arose because N depended on x_o, we try to eliminate the trouble by defining a stronger* type of convergence in which N will not depend on x_o. This leads to Definition #2.

Definition #2

$\{f_n(x)\}$ is said to be uniformly convergent to $f(x)$ on I if for every $\varepsilon > 0$ there exists an N such that $n > N \to |f(x) - f_n(x)| < \varepsilon$ for every $x \in I$.

6. Thus, in uniform convergence the choice of N depends only on the choice of ε not on the choice of x_o. This "slight" modification is enough to quarantee the following results.

If $\{f_n(x)\}$ converges uniformly to $f(x)$ on $a \leq x \leq b (= [a,b] = I)$ and each f_n is continuous on $[a,b]$, then f is also continuous on $[a,b]$.

If $\{f_n(x)\}$ converges uniformly to $f(x)$ on $[a,b]$, then for each $x \in [a,b]$,

$$\lim_{n\to\infty} \int_a^x f_n(t)dt = \int_a^x \lim_{n\to\infty} f_n(t)dt = \int_a^x f(t)dt.$$

If $\{f_n(x)\}$ is a sequence of continuously differentiable functions which converge point-wise to $f(x)$ on $[a,b]$ and if $\{f_n'(x)\}$ converges uniformly on $[a,b]$, then $f'(x)$ exists on $[a,b]$ and may be computed by $f'(x) = \lim_{n\to\infty} f_n'(x)$.

7. While we have thus far been talking about sequences of functions, notice that our results also apply to series since every series is a sequence of partial sums.
That is, when we write

$$\sum_{n=1}^{\infty} f_n(x)$$

*Recall that "stronger" means a definition which includes the previous one. In other words, everything that obeys the stronger definition obeys the weaker one, but not everything which obeys the weaker one obeys the stronger one.

what we mean is;

$$\lim_{k\to\infty} F_k(x) \text{ where } F_k(x) = f_1(x) +\ldots+ f_k(x) = \sum_{n=1}^{k} f_n(x).$$

One very important series is the power series

$$\sum_{n=0}^{\infty} a_n x^n{}^{*}.$$

The usefulness of power series stems from the following theorem:

Theorem 1:

If the power series

$$\sum_{n=0}^{\infty} a_n x^n$$

converges for some value of $x = x_o$, then it converges absolutely for every x such that $|x|<|x_o|$, and this convergence is also uniform on every closed interval defined by $|x| \leq |x_1|<|x_o|$.

8. If

$$\sum_{n=0}^{\infty} a_n x^n$$

converges uniformly and absolutely for $|x|< R$, we can take the same liberties with the power series as we could have taken with polynomials. That is, by the properties of absolute convergence we can add series term-by-term, we can re-arrange the terms, etc.

For example, if

$$\sum_{n=0}^{\infty} a_n x^n$$

converges absolutely we may, if we wish, rewrite the series as the sum of two series, one of which contains the terms in which the exponents are even and the other, in which the exponents are odd.

*More generally, one usually talks about $\sum_{n=0}^{\infty} a_n(x - b)^n$ but as usual we take the special case b = 0 simply for convenience.

That is,

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{n \text{ even}} a_n x^n + \sum_{n \text{ odd}} a_n x^n$$

$$= (a_o + a_2x^2 + a_4x^4 + \ldots) + (a_1x + a_3x^3 + a_5x^5 + \ldots),$$

etc.

Moreover by uniform convergence, we may differentiate and integrate power series term-by-term.

9. In particular, the results we shall use most in this Unit are:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\left(\sum_{n=0}^{\infty} a_n x^n\right)'' = \left(\sum_{n=1}^{\infty} n a_n x^{n-1}\right)' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} (a_n + b_n) x^n$$

$$cx^k \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} c a_n x^{n+k}$$

$$\sum_{n=0}^{\infty} a_n x^n \equiv \sum_{n=0}^{\infty} b_n x^n \leftrightarrow a_n = b_n \text{ for each } n.$$

Preface: Part 2

Now that we have reviewed the theory behind power series, we should like to present two computational techniques that are employed very often when we look for series solutions of differential equations.

The first, and probably easier, of these techniques involves adding series when the summation starts at different values of n. For example, suppose we want to express

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=2}^{\infty} b_n x^n$$

as a single series. What we observe is that if the first term had been

$$\sum_{n=2}^{\infty} a_n x^n$$

we could have simply added the two series term-by-term to obtain

$$\sum_{n=2}^{\infty} (a_n + b_n) x^n .$$

We, therefore, convert

$$\sum_{n=0}^{\infty} a_n x^n$$

into

$$\sum_{n=2}^{\infty} a_n x^n$$

by "splitting off" its first two terms. That is,

$$\begin{aligned}\sum_{n=0}^{\infty} a_n x^n &= a_o + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots \\ &= (a_o + a_1 x) + (a_2 x^2 + \dots + a_n x^n + \dots \\ &= a_o + a_1 x + \sum_{n=2}^{\infty} a_n x^n .\end{aligned}$$

Thus,

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=2}^{\infty} b_n x^n$$

$$= a_o + a_1x + \sum_{n=2}^{\infty} a_nx^n + \sum_{n=2}^{\infty} b_nx^n$$

$$= a_o + a_1x + \sum_{n=2}^{\infty} (a_n + b_n)x^n.$$

Notice that the procedure used here is applicable to all finite sums quite in general. Namely,

$$\sum_{n=1}^{\infty} c_n = c_1 + c_2 + \ldots + c_k + c_{k+1} + \ldots + c_m$$

$$= (c_1 + \ldots + c_k) + (c_{k+1} + \ldots + c_m)$$

$$= \sum_{n=1}^{k} c_n + \sum_{n=k+1}^{m} c_n.$$

We use absolute convergence when we allow ourselves the luxury of assuring that we may regroup terms at will, even when the sum involves infinitely many terms.

The second technique involves summing two series which don't "line up" term-by-term. For example, suppose we want to express

$$\sum_{n=0}^{\infty} a_nx^n + \sum_{n=1}^{\infty} b_nx^{n-1} \qquad (1)$$

as a single series. Granted that the summations do not begin at the same value of n, there is even a worse problem in the sense that the general term in the first sum involves x^n while in the second sum the general term involves x^{n-1}.

The key to this problem lies in the fact that given,

$$\sum_{n=1}^{k} c_n,$$

we may replace c_n by c_{n-r} provided we add r to both our lower and upper limits of summation. That is

$$\sum_{n=1}^{k} c_n = \sum_{n=1+r}^{k+r} c_{n-r} .$$

As a more concrete example,

$$\sum_{n=1}^{5} c_r = \sum_{n=4}^{8} c_{n-3} .$$

To check this, we need only observe that

$$\sum_{n=1}^{5} c_n = c_1 + c_2 + c_3 + c_4 + c_5;$$

while

$$\sum_{n=4}^{8} c_{n-3} = c_{4-3} + c_{5-3} + c_{6-3} + c_{7-3} + c_{8-3} = c_1 + c_2 + c_3 + c_4 + c_5.$$

Thus, returning to (1) we notice that to change x^{n-1} into x^n we must add 1 to n. Therefore, we replace each n in $\Sigma\, b_n x^{n-1}$ by n + 1 and adjust for this by subtracting 1 from each of our limits of summation. That is:

$$\sum_{n=1}^{\infty} b_n x^{n-1} = \sum_{n=1-1}^{\infty} b_{n+1} x^n$$

$$= \sum_{n=0}^{\infty} b_{n+1}\, x^n.^{*} \qquad (2)$$

Using (2), (1) becomes

$$\sum_{n=0}^{\infty} a_n x^n + \sum_{n=0}^{\infty} b_{n+1}\, x^n,$$

which, in turn, is

$$\sum_{n=0}^{\infty} (a_n + b_{n+1}) x^n.$$

As a final point, we should observe that "lining up" exponents usually shifts the summation to start at a different value of n. The proper procedure is first to "line up" the exponents

*Again, as a quick check,

$$\sum_{n-1}^{\infty} b_n x^{n-1} = b_1 + b_2 x + b_3 x^2 + b_4 x^3 + \ldots; \quad \text{while}$$

$$\sum_{n=0}^{\infty} b_{n+1} x^n = b_1 + b_2 x + b_3 x^2 + b_4 x^3 + \ldots$$

and then to alter the lower limit of the summation (if necessary). For example, given

$$\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} \qquad (3)$$

we either raise n by 2 in the first series or lower n by 2 in the second series, we must add 2 to the limits of summation. This yields

$$\sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=2}^{\infty} a_{n-2} x^{n-1},$$

so that (3) becomes

$$\sum_{n=1}^{\infty} na_n x^{n-1} + \sum_{n=2}^{\infty} a_{n-2}\, x^{n-1}. \qquad (4)$$

We now "split off" the first term in the first series in (4) to obtain

$$a_1 + \sum_{n=2}^{\infty} na_n x^{n-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n-1},$$

or

$$a_1 + \sum_{n=2}^{\infty} (na_n + a_{n-2}) x^{n-1}. \qquad (5)$$

<u>Note</u>

We could have elected to raise n by 2 in the first series in (3). That is, we could have written

$$\sum_{n=1}^{\infty} na_n x^{n-1} = \sum_{n=-1}^{\infty} (n + 2) a_{n+2} x^{n+1},$$

whereupon (3) would become

$$\sum_{n=-1}^{\infty} (n + 2) a_{n+2} x^{n+1} + \sum_{n=0}^{\infty} a_n x^{n+1}. \qquad (6)$$

We would then rewrite

$$\sum_{n=-1}^{\infty} (n + 2) a_{n+2}\, x^{n+1}$$

as

$$a_1 + \sum_{n=0}^{\infty} (n + 2)a_{n+2}x^{n+1},$$

so that (6) becomes

$$a_1 + \sum_{n=0}^{\infty} [(n + 2)a_{n+2} + a_n]x^{n+1} . \qquad (7)$$

It is left for you to check that (5) and (7) are the same, each being

$$a_1 + (2a_2 + a_o)x + (3a_3 + a_1)x^2 + (4a_4 + a_2)x^3 + \ldots$$

Further drill is left to the exercises.