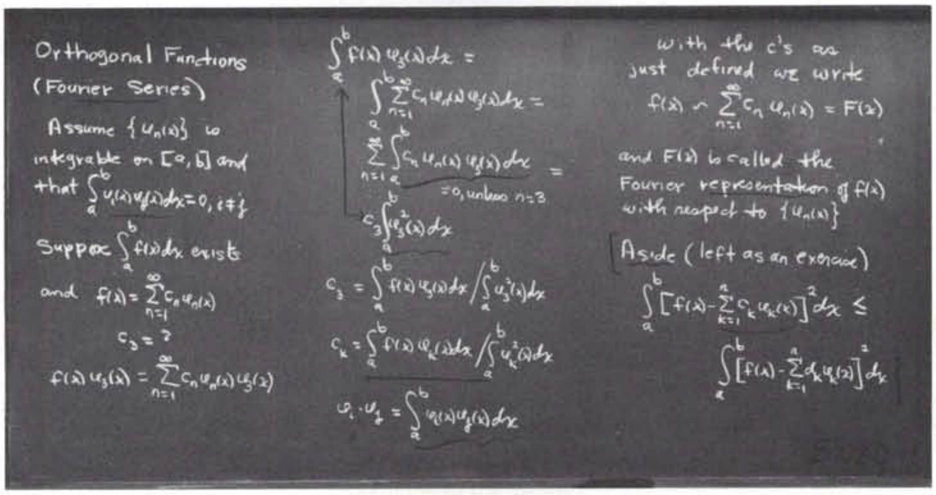
Unit 8: Fourier Series

1. Overview

   We conclude our present course with a brief excursion into Fourier series. Aside from their great practical value, Fourier series give us a chance to see an excellent application of the dot product in terms of a function space. It is in this context that Fourier series play as important a role in pure mathematics as they do in applied mathematics.

2. Lecture 3.080



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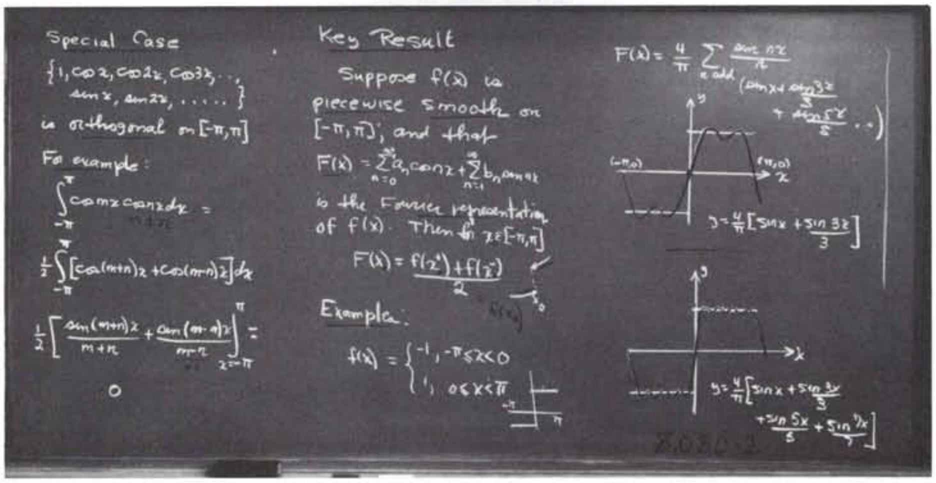
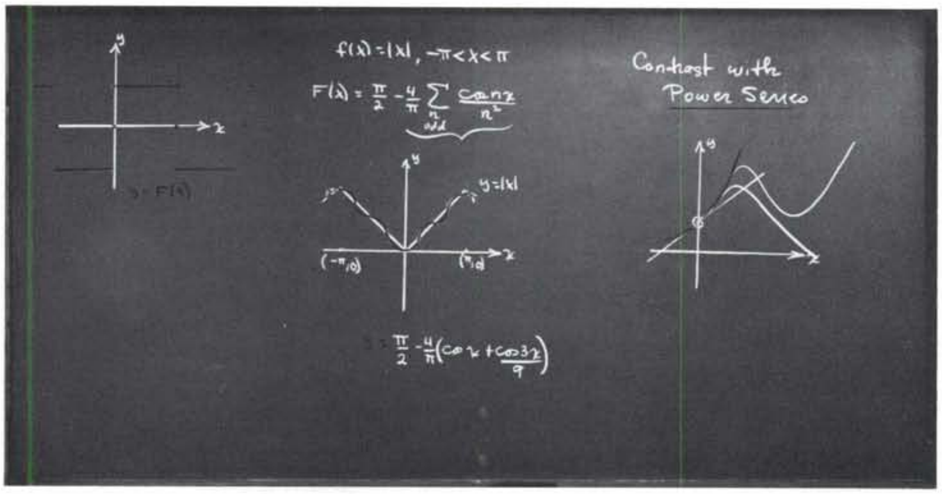


b.

Lecture 3.080 continued

c. 

3. Read Thomas, Section 18.8.

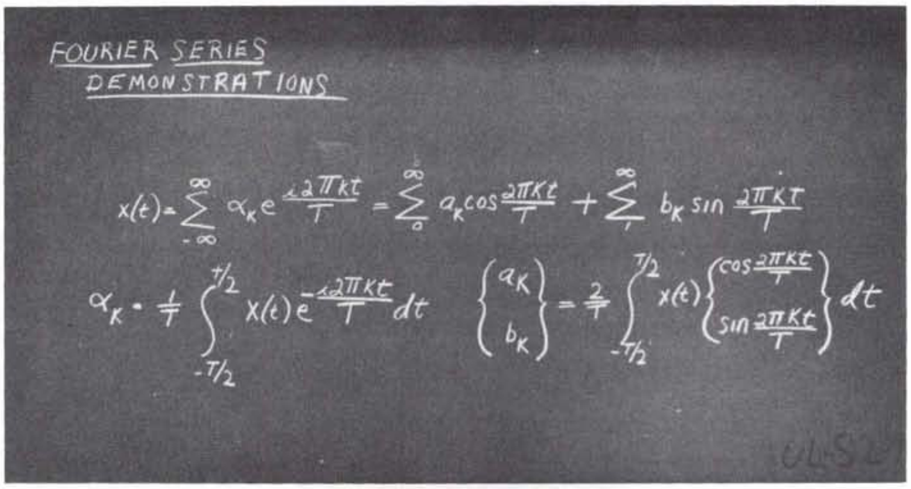
   The key difference between our approach and that of the text is that we prefer to use the interval $[-\pi,\pi]$ while the text uses $[0,2\pi]$. While this in itself is a minor point, it should be noted that using $[-\pi,\pi]$ allows us to utilize properties of even and odd functions. This has been mentioned in the lecture and is further discussed in the exercises.

4. Do the exercises.

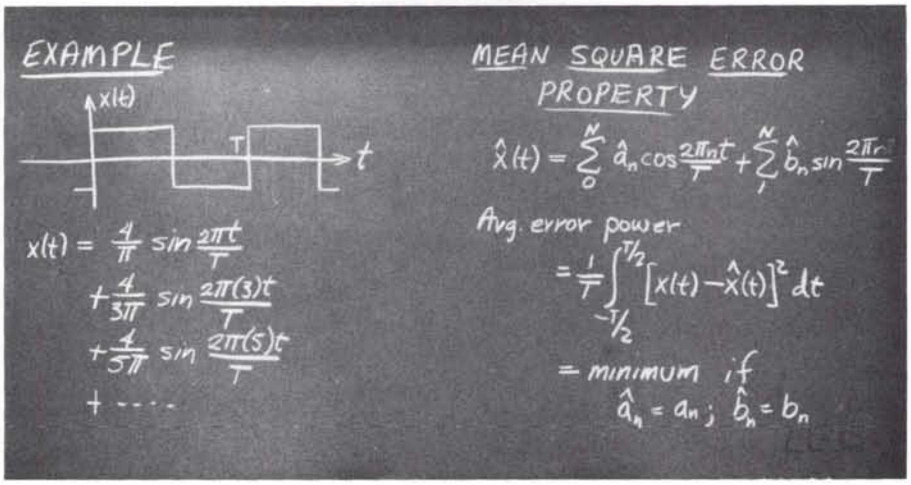
5. Lecture 3.090

   This lecture is by Professor William Siebert of the Electrical Engineering Department at M.I.T. It was developed independently of this course, but it is such an excellent example of how Fourier series can "be brought to life" electronically in a way that can never be approached from a purely analytical point of view that we felt compelled to end our course on this note. It is particularly important, in our opinion, that you do the exercises before you watch this lecture. For one thing, doing the exercises will give you more familiarity with what is going on in the lecture, and for another thing, doing the exercises from a purely analytic, mathematical point of view will help you to appreciate better the approach used by Professor Siebert.

Lecture 3.090



a.



b.

Exercises:

3.8.1(L)

Show that the set of functions

$\{1, \cos x, \cos 2x, \ldots, \cos nx, \ldots, \sin x, \sin 2x, \ldots, \sin nx, \ldots\}$

is orthogonal on the interval $[-\pi, \pi]$.

3.8.2(L)

Suppose f(x) is integrable on $[-\pi,\pi]$ and let

$$F(x) = \sum_{n=0}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx^*$$

be the Fourier representation of f(x) relative to $\{1, \cos x, \cos 2x, \ldots, \sin x, \sin 2x, \ldots\}$. Use the method described in the lecture to determine each a_n and b_n.

3.8.3(L)

Define f on $[-\pi,\pi]$ by

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 < x < \pi \end{cases}$$

a. Derive the Fourier representation of f(x).

b. Use part (a) to evaluate the sum

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}.$$

3.8.4

a. Find the Fourier representation of f where $f(x) = |x|$, $-\pi \leqslant x \leqslant \pi$.

b. Use the result of part (a) to compute $\displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}$.

*Do not be alarmed that one sum includes n = 0 while the other begins with n = 1. The point quite simply, is that sin 0x = 0, but cos 0x = 1. In other words,

$$\sum_{n=0}^{\infty} a_n \cos nx = a_o + \sum_{n=1}^{\infty} \cos nx.$$

3.8.5(L)

a. Find the Fourier series expansion of the function

$$f(x) = \begin{cases} 0, & -\pi < x < 0 \\ x^2, & 0 \leqslant x < \pi \end{cases}$$

b. Sketch $y = F(x)$ where F is the Fourier representation of f.

c. Use (a) to evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

3.8.6 (Optional)

Note

This exercise leads to the proof of the result mentioned in the lecture about the Fourier representation of a function being the best mean square approximation to the function. This result is more general than the special case we are pursuing in this unit; namely, the case in which $f(x)$ is piecewise smooth on $[-\pi,\pi]$ and our orthogonal series is the trigonometric series. It is this result that indicates why the Fourier series approximates the function over the entire interval rather than merely in the neighborhood of a given point.

Let $\{\phi_n\}$ be orthonormal on $[a,b]$ and let

$$s_n(x) = \sum_{k=0}^{n} c_k\phi_k(x)$$

be the nth partial sum of the Fourier series of f relative to $\{\phi_n\}$. Suppose that $t_n(x) = \sum_{k=0}^{n} \gamma_k\phi_k(x)$ is any other linear combination of $\phi_o(x)$, $\phi_1(x)$, ..., and $\phi_n(x)$. Prove that

$$\int_a^b [f(x) - s_n(x)]^2dx \leqslant \int_a^b [f(x) - t_n(x)]^2dx$$

and that the equality holds if and only if $c_k = \gamma_k$ for $k = 0,1,\ldots,n$.

3.8.7 (Optional)

The main aim of this exercise is to show that we may remove the restriction of having to work on the interval $[-\pi,\pi]$.

Let f be defined by $f(x) = x$, $-1 < x < 1$. Express f as a trigonometric series.

Quiz

1. Suppose $V = [u_1,u_2,u_3,u_4]$ and that

$$v_1 = u_1 + u_2 + u_3 + u_4$$

$$v_2 = u_1 + 2u_2 + 3u_3 + 3u_4$$

$$v_3 = 3u_1 + 4u_2 + 6u_3 + 6u_4$$

$$v_4 = 2u_1 + 3u_2 + 4u_3 + 5u_4$$

(a) Express the u's as linear combinations of the v's, and thus show that the set $\{v_1,v_2,v_3,v_4\}$ is also a basis for V.

(b) Suppose $w \in V$ is given by the 4-tuple (4,3,2,1) relative to the basis $[u_1,u_2,u_3,u_4]$. Express w as a 4-tuple relative to the basis $[v_1,v_2,v_3,v_4]$.

2. Let V be as in Problem 1. Let W be that subspace of V which is spanned by $w_1 = (1,2,1,0)$, $w_2 = (-1,1,-4,3)$, $w_3 = (1,8,-5,6)$, and $w_4 = (1,-4,7,-6)$ [where the 4-tuples are relative to the basis $\{u_1,u_2,u_3,u_4\}$].

(a) Find the dimension of W.

(b) Show explicitly how w_3 and w_4 may be expressed in terms of w_1 and w_2.

(c) How must x and y be chosen if $3u_1 + 5u_2 + xu_3 + yu_4$ is to be a member of W?

3. Let V and W be as in Problem 2, and let U be the subspace of V which is spanned by (1,1,1,1), (2,1,5,-3), and (3,2,7,-4).

(a) Find the dimension of U + W.

(b) Find the dimension of $U \cap W$.

(c) Verify in this example that

$$\dim U + \dim W - \dim(U \cap W) = \dim(U + W).$$

4. Let T be the linear transformation of 3-space defined by

$$T(\vec{i}) = 2\vec{i} + \vec{j} + \vec{k}$$

$$T(\vec{j}) = -\vec{i} + 2\vec{j} + 7\vec{k}$$

$$T(\vec{k}) = \vec{i} - \vec{j} - 4\vec{k}$$

(a) Describe the null space of T.

(b) Describe the image of T.

5. Let A be the 5 by 5 matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 & 3 \\ 2 & 3 & 3 & 3 & 4 \\ 3 & 4 & 5 & 4 & 5 \\ 4 & 5 & 4 & 4 & 4 \end{bmatrix}$$

Use row reduction techniques to evaluate the determinant of A.

6. Let $V = [u_1, u_2, u_3]$ and let $T:V \to V$ be the linear transformation defined be

$$T(u_1) = 3u_1 + 2u_2 + 2u_3$$

$$T(u_2) = u_1 + 2u_2 + 2u_3$$

$$T(u_3) = -u_1 - u_2$$

Find all non-zero vectors $v \varepsilon V$ such that $T(v) = cv$ for some scalar c.

7. Let $u_1 = (3,0,4)$, $u_2 = (-1,0,7)$ and $u_3 = (2,9,11)$ be a basis of E^3, where we assume that E^3 is equipped with the "usual" inner product. Use the Gram-Schmidt process to find an orthonormal basis for E^3 which has u_1 as one basis vector and another basis vector in the place spanned by u_1 and u_2.