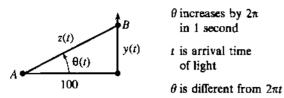passes c (the speed of light) when $\sec^2\theta$ passes _____. Such a speed is impossible—we forget that light takes time to reach B.

θ increases by 2π in 1 second

t is arrival time of light

θ is different from $2\pi t$

18 (Explanation by ID) Light travels from A to B in time z/c, distance over speed. Its arrival time is $t = \theta/2\pi + z/c$ so $\theta'/2\pi = 1 - z'/c$. Then $z' = y'\sin\theta$ and $y' = 100\sec^2\theta\,\theta'$ (all these are ID) lead to

$$y' = 200\pi c/(c\,\cos^2\theta + 200\pi\,\sin\theta)$$

As θ approaches $\pi/2$, this speed approaches _____.
Note: y' still exceeds c for some negative angle. That is for Einstein to explain. See the 1985 *College Math Journal*, page 186, and the 1960 *Scientific American*, "Things that go faster than light."

19 If a plane follows the curve $y = f(x)$, and its ground speed is $dx/dt = 500$ mph, how fast is the plane going up? How fast is the plane going?

20 Why can't we differentiate $x = 7$ and reach $1 = 0$?

Problems 21–29 are applications of related rates.

21 (Calculus classic) The bottom of a 10-foot ladder is going away from the wall at $dx/dt = 2$ feet per second. How fast is the top going down the wall? Draw the right triangle to find dy/dt when the height y is (a) 6 feet (b) 5 feet (c) zero.

22 The top of the 10-foot ladder can go faster than light. At what height y does $dy/dt = -c$?

23 How fast does the level of a Coke go down if you drink a cubic inch a second? The cup is a cylinder of radius 2 inches—first write down the volume.

24 A jet flies at 8 miles up and 560 miles per hour. How fast is it approaching you when (a) it is 16 miles from you; (b) its

shadow is 8 miles from you (the sun is overhead); (c) the plane is 8 miles from you (exactly above)?

25 Starting from a 3–4–5 right triangle, the short sides increase by 2 meters/second but the angle between them decreases by 1 radian/second. How fast does the area increase or decrease?

26 A pass receiver is at $x = 4$, $y = 8t$. The ball thrown at $t = 3$ is at $x = c(t - 3)$, $y = 10c(t - 3)$.

    (a) Choose c so the ball meets the receiver.

    *(b) At that instant the distance D between them is changing at what rate?

27 A thief is 10 meters away (8 meters ahead of you, across a street 6 meters wide). The thief runs on that side at 7 meters/second, you run at 9 meters/second. How fast are you approaching if (a) you follow on your side; (b) you run toward the thief; (c) you run away on your side?
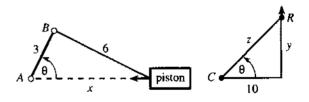
28 A spherical raindrop evaporates at a rate equal to twice its surface area. Find dr/dt.

29 Starting from $P = V = 5$ and maintaining $PV = T$, find dV/dt if $dP/dt = 2$ and $dT/dt = 3$.

30 (a) The crankshaft AB turns twice a second so $d\theta/dt =$ _____.

    (b) Differentiate the cosine law $6^2 = 3^2 + x^2 - 2(3x\cos\theta)$ to find the piston speed dx/dt when $\theta = \pi/2$ and $\theta = \pi$.

31 A camera turns at C to follow a rocket at R.
    (a) Relate dz/dt to dy/dt when $y = 10$.
    (b) Relate $d\theta/dt$ to dy/dt based on $y = 10\tan\theta$.
    (c) Relate $d^2\theta/dt^2$ to d^2y/dt^2 and dy/dt.

4.3 Inverse Functions and Their Derivatives

There is a remarkable special case of the chain rule. It occurs when $f(y)$ and $g(x)$ are "*inverse functions.*" That idea is expressed by a very short and powerful equation: $f(g(x)) = x$. Here is what that means.
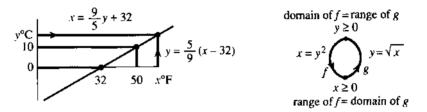
Inverse functions: Start with any input, say $x = 5$. *Compute* $y = g(x)$, say $y = 3$. Then compute $f(y)$, and *the answer must be* 5. What one function does, the inverse function

undoes. If $g(5) = 3$ then $f(3) = 5$. *The inverse function f takes the output y back to the input x.*

EXAMPLE 1 $g(x) = x - 2$ and $f(y) = y + 2$ are inverse functions. Starting with $x = 5$, the function g subtracts 2. That produces $y = 3$. Then the function f adds 2. *That brings back $x = 5$.* To say it directly: *The inverse of $y = x - 2$ is $x = y + 2$.*

EXAMPLE 2 $y = g(x) = \frac{5}{9}(x - 32)$ and $x = f(y) = \frac{9}{5}y + 32$ are inverse functions (for temperature). Here x is degrees Fahrenheit and y is degrees Celsius. From $x = 32$ (freezing in Fahrenheit) you find $y = 0$ (freezing in Celsius). The inverse function takes $y = 0$ back to $x = 32$. Figure 4.4 shows how $x = 50°F$ matches $y = 10°C$.

Notice that $\frac{5}{9}(x - 32)$ subtracts 32 *first*. The inverse $\frac{9}{5}y + 32$ adds 32 *last*. In the same way g multiplies last by $\frac{5}{9}$ while f multiplies first by $\frac{9}{5}$.

Fig. 4.4 °F to °C to °F. Always $g^{-1}(g(x)) = x$ and $g(g^{-1}(y)) = y$. If $f = g^{-1}$ then $g = f^{-1}$.

The inverse function is written $f = g^{-1}$ and pronounced "g inverse." It is not $1/g(x)$.

If the demand y is a function of the price x, then the price is a function of the demand. Those are inverse functions. *Their derivatives obey a fundamental rule: dy/dx times dx/dy equals* 1. In Example 2, dy/dx is $5/9$ and dx/dy is $9/5$.

There is another important point. When f and g are applied in the *opposite order*, they still come back to the start. First f adds 2, then g subtracts 2. The chain $g(f(y)) = (y + 2) - 2$ brings back y. *If f is the inverse of g then g is the inverse of f.* The relation is completely symmetric, and so is the definition:

Inverse function: If $y = g(x)$ then $x = g^{-1}(y)$. If $x = g^{-1}(y)$ then $y = g(x)$.

The loop in the figure goes from x to y to x. The composition $g^{-1}(g(x))$ is the "identity function." Instead of a new point z it returns to the original x. This will make the chain rule particularly easy—leading to $(dy/dx)(dx/dy) = 1$.

EXAMPLE 3 $y = g(x) = \sqrt{x}$ and $x = f(y) = y^2$ are inverse functions.

Starting from $x = 9$ we find $y = 3$. The inverse gives $3^2 = 9$. The square of $\sqrt{x}$ is $f(g(x)) = x$. In the opposite direction, the square root of y^2 is $g(f(y)) = y$.

Caution That example does not allow x to be negative. The domain of g—the set of numbers with square roots—is restricted to $x \geqslant 0$. This matches the range of g^{-1}. The outputs y^2 are nonnegative. With *domain of g = range of g^{-1}*, the equation $x = (\sqrt{x})^2$ is possible and true. The nonnegative x goes into g and comes out of g^{-1}.

In this example y is also nonnegative. You might think we could square anything, but y must come back as the square root of y^2. So $y \geqslant 0$.

To summarize: *The domain of a function matches the range of its inverse*. The inputs to g^{-1} are the outputs from g. The inputs to g are the outputs from g^{-1}.

If $g(x) = y$ then solving that equation for x gives $x = g^{-1}(y)$:

$$\text{if } y = 3x - 6 \quad \text{then } x = \tfrac{1}{3}(y + 6) \quad (\text{this is } g^{-1}(y))$$

$$\text{if } y = x^3 + 1 \quad \text{then } x = \sqrt[3]{y - 1} \quad (\text{this is } g^{-1}(y))$$

In practice that is how g^{-1} is computed: *Solve* $g(x) = y$. This is the reason inverses are important. Every time we solve an equation we are computing a value of g^{-1}.
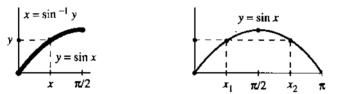
Not all equations have one solution. *Not all functions have inverses.* For each y, the equation $g(x) = y$ is only allowed to produce one x. That solution is $x = g^{-1}(y)$. If there is a second solution, then g^{-1} will not be a function—because a function cannot produce two x's from the same y.

EXAMPLE 4 There is more than one solution to $\sin x = \tfrac{1}{2}$. Many angles have the same sine. On the interval $0 \leqslant x \leqslant \pi$, the inverse of $y = \sin x$ is not a function. Figure 4.5 shows how two x's give the same y.

Prevent x from passing $\pi/2$ and the sine has an inverse. Write $x = \sin^{-1} y$.

The function g has no inverse if two points x_1 and x_2 give $g(x_1) = g(x_2)$. Its inverse would have to bring the same y back to x_1 and x_2. No function can do that; $g^{-1}(y)$ cannot equal both x_1 and x_2. There must be only one x for each y.

To be invertible over an interval, g must be steadily increasing or steadily decreasing.

Fig. 4.5 Inverse exists (one x for each y). No inverse function (two x's for one y).

THE DERIVATIVE OF g^{-1}

It is time for calculus. Forgive me for this very humble example.

EXAMPLE 5 (ordinary multiplication) The inverse of $y = g(x) = 3x$ is $x = f(y) = \tfrac{1}{3}y$.

This shows with special clarity the rule for derivatives: *The slopes* $dy/dx = 3$ *and* $dx/dy = \tfrac{1}{3}$ *multiply to give* 1. This rule holds for all inverse functions, even if their slopes are not constant. It is a crucial application of the chain rule to the derivative of $f(g(x)) = x$.

4C (*Derivative of inverse function*) From $f(g(x)) = x$ the chain rule gives $f'(g(x))g'(x) = 1$. Writing $y = g(x)$ and $x = f(y)$, this rule looks better:

$$\frac{dx}{dy}\frac{dy}{dx} = 1 \qquad \textit{or} \qquad \frac{dx}{dy} = \frac{1}{dy/dx}. \tag{1}$$

The slope of $x = g^{-1}(y)$ times the slope of $y = g(x)$ equals one.
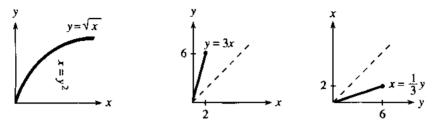

This is the chain rule with a special feature. Since $f(g(x)) = x$, *the derivative of both sides is* 1. If we know g' we now know f'. That rule will be tested on a familiar example. In the next section it leads to totally new derivatives.

EXAMPLE 6 The inverse of $y = x^3$ is $x = y^{1/3}$. We can find dx/dy two ways:

directly: $\dfrac{dx}{dy} = \dfrac{1}{3} y^{-2/3}$ indirectly: $\dfrac{dx}{dy} = \dfrac{1}{dy/dx} = \dfrac{1}{3x^2} = \dfrac{1}{3y^{2/3}}.$

The equation $(dx/dy)(dy/dx) = 1$ is not ordinary algebra, but it is true. Those derivatives are limits of fractions. The fractions are $(\Delta x/\Delta y)(\Delta y/\Delta x) = 1$ and we let $\Delta x \to 0$.

Fig. 4.6 Graphs of inverse functions: $x = \frac{1}{3}y$ is the mirror image of $y = 3x$.

Before going to new functions, I want to draw graphs. Figure 4.6 shows $y = \sqrt{x}$ and $y = 3x$. What is special is that *the same graphs also show the inverse functions.* The inverse of $y = \sqrt{x}$ is $x = y^2$. The pair $x = 4$, $y = 2$ is the same for both. That is the whole point of inverse functions—if $2 = g(4)$ then $4 = g^{-1}(2)$. Notice that the graphs go steadily up.

The only problem is, the graph of $x = g^{-1}(y)$ is on its side. To change the slope from 3 to $\frac{1}{3}$, you would have to turn the figure. After that turn there is another problem—the axes don't point to the right and up. You also have to look in a mirror! (The typesetter refused to print the letters backward. He thinks it's crazy but it's not.) To keep the book in position, and the typesetter in position, we need a better idea.

The graph of $x = \frac{1}{3}y$ comes from **turning the picture across the $45°$ line**. The y axis becomes horizontal and x goes upward. The point $(2, 6)$ on the line $y = 3x$ goes into the point $(6, 2)$ on the line $x = \frac{1}{3}y$. The eyes see a reflection across the $45°$ line (Figure 4.6c). The mathematics sees the same pairs x and y. The special properties of g and g^{-1} allow us to know two functions—and draw two graphs—at the same time.† **The graph of $x = g^{-1}(y)$ is the mirror image of the graph of $y = g(x)$.**

EXPONENTIALS AND LOGARITHMS

I would like to add two more examples of inverse functions, because they are so important. Both examples involve the *exponential* and the *logarithm*. One is made up of linear pieces that imitate 2^x; it appeared in Chapter 1. The other is the true function 2^x, which is not yet defined—and it is not going to be defined here. The functions b^x and $\log_b y$ are so overwhelmingly important that they deserve and will get a whole chapter of the book (at least). But you have to see the graphs.

The slopes in the linear model are powers of 2. So are the heights y at the start of each piece. *The slopes $1, 2, 4, \ldots$ equal the heights $1, 2, 4, \ldots$ at those special points.*

The inverse is a discrete model for the logarithm (to base 2). The logarithm of 1 is 0, because $2^0 = 1$. The logarithm of 2 is 1, because $2^1 = 2$. The logarithm of 2^j is the exponent j. Thus the model gives the correct $x = \log_2 y$ at the breakpoints $y = 1, 2, 4, 8, \ldots$. The slopes are $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots$ because $dx/dy = 1/(dy/dx)$.

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†I have seen graphs with $y = g(x)$ and also $y = g^{-1}(x)$. For me that is wrong: it has to be $x = g^{-1}(y)$. If $y = \sin x$ then $x = \sin^{-1} y$.
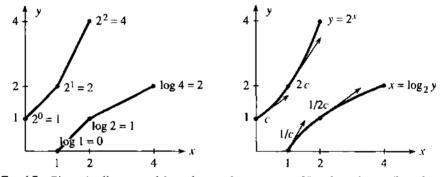
The model is good, but the real thing is better. The figure on the right shows the true exponential $y = 2^x$. At $x = 0, 1, 2, \ldots$ the heights y are the same as before. But now the height at $x = \frac{1}{2}$ is the number $2^{1/2}$, which is $\sqrt{2}$. The height at $x = .10$ is the tenth root $2^{1/10} = 1.07\ldots$ *The slope at $x = 0$ is no longer* 1—*it is closer to* $\Delta y/\Delta x = .07/.10$. The exact slope is a number c (near $.7$) that we are not yet prepared to reveal.

The special property of $y = 2^x$ is that the slope at all points is cy. **The slope is proportional to the function.** The exponential solves $dy/dx = cy$.

Now look at the inverse function—*the logarithm*. Its graph is the mirror image:

If $y = 2^x$ then $x = \log_2 y$. If $2^{1/10} \approx 1.07$ then $\log_2 1.07 \approx 1/10$.

What the exponential does, the logarithm undoes—and vice versa. **The logarithm of 2^x is the exponent x.** Since the exponential starts with slope c, the logarithm must start with slope $1/c$. Check that numerically. The logarithm of 1.07 is near $1/10$. The slope is near $.10/.07$. The beautiful property is that $dx/dy = 1/cy$.

Fig. 4.7   Piecewise linear models and smooth curves: $y = 2^x$ and $x = \log_2 y$. Base $b = 2$.

I have to mention that calculus avoids logarithms to base 2. The reason lies in that mysterious number c. It is the "natural logarithm" of 2, which is $.693147\ldots$—and who wants that? Also $1/.693147\ldots$ enters the slope of $\log_2 y$. Then $(dx/dy)(dy/dx) = 1$. The right choice is to use "natural logarithms" throughout. In place of 2, they are based on the special number e:

$$y = e^x \text{ is the inverse of } x = \ln y. \tag{2}$$

The derivatives of those functions are sensational—they are saved for Chapter 6. Together with x^n and $\sin x$ and $\cos x$, they are the backbone of calculus.

Note   It is almost possible to go directly to Chapter 6. The inverse functions $x = \sin^{-1} y$ and $x = \tan^{-1} y$ can be done quickly. The reason for including integrals first (Chapter 5) is that they solve differential equations with no guesswork:

$$\frac{dy}{dx} = y \text{ or } \frac{dx}{dy} = \frac{1}{y} \text{ leads to } \int dx = \int \frac{dy}{y} \text{ or } x = \ln y + C.$$

Integrals have applications of all kinds, spread through the rest of the book. But do not lose sight of 2^x and e^x. They solve $dy/dx = cy$—the key to applied calculus.

THE INVERSE OF A CHAIN $h(g(x))$

The functions $g(x) = x - 2$ and $h(y) = 3y$ were easy to invert. For g^{-1} we added 2, and for h^{-1} we divided by 3. Now the question is: If we create the composite function $z = h(g(x))$, or $z = 3(x - 2)$, what is its inverse?

Virtually all known functions are created in this way, from chains of simpler functions. *The problem is to invert a chain using the inverse of each piece.* The answer is one of the fundamental rules of mathematics:

4D The inverse of $z = h(g(x))$ is a chain of inverses *in the opposite order*:

$$x = g^{-1}(h^{-1}(z)). \tag{3}$$

h^{-1} is applied first because h was applied last: $g^{-1}(h^{-1}(h(g(x)))) = x$.

That last equation looks like a mess, but it holds the key. In the middle you see h^{-1} and h. That part of the chain does nothing! The inverse functions cancel, to leave $g^{-1}(g(x))$. But that is x. The whole chain collapses, when g^{-1} and h^{-1} are in the correct order—which is opposite to the order of $h(g(x))$.

EXAMPLE 7 $z = h(g(x)) = 3(x - 2)$ and $x = g^{-1}(h^{-1}(z)) = \frac{1}{3}z + 2$.

First h^{-1} divides by 3. Then g^{-1} adds 2. The inverse of $h \circ g$ is $g^{-1} \circ h^{-1}$. *It can be found directly by solving* $z = 3(x - 2)$. A chain of inverses is like writing in prose—we do it without knowing it.

EXAMPLE 8 Invert $z = \sqrt{x - 2}$ by writing $z^2 = x - 2$ and then $x = z^2 + 2$.

The inverse adds 2 and takes the square—*but not in that order*. That would give $(z + 2)^2$, which is wrong. The correct order is $z^2 + 2$.

The domains and ranges are explained by Figure 4.8. We start with $x \geqslant 2$. Subtracting 2 gives $y \geqslant 0$. Taking the square root gives $z \geqslant 0$. Taking the square brings back $y \geqslant 0$. Adding 2 brings back $x \geqslant 2$—which is in the original domain of g.

Fig. 4.8 The chain $g^{-1}(h^{-1}(h(g(x)))) = x$ is one-to-one at every step.

EXAMPLE 9 Inverse matrices $(AB)^{-1} = B^{-1}A^{-1}$ (this linear algebra is optional).

Suppose a vector x is multiplied by a square matrix B: $y = g(x) = Bx$. The inverse function multiplies by the *inverse matrix*: $x = g^{-1}(y) = B^{-1}y$. It is like multiplication by $B = 3$ and $B^{-1} = 1/3$, except that x and y are vectors.

Now suppose a second function multiplies by another matrix A: $z = h(g(x)) = ABx$. The problem is to recover x from z. The first step is to invert A, because that came last: $Bx = A^{-1}z$. Then the second step multiplies by B^{-1} and brings back $x = B^{-1}A^{-1}z$. *The product $B^{-1}A^{-1}$ inverts the product AB.* The rule for matrix inverses is like the rule for function inverses—in fact it is a special case.

I had better not wander too far from calculus. The next section introduces the inverses of the sine and cosine and tangent, and finds their derivatives. Remember that the ultimate source is the chain rule.

4.3 EXERCISES

Read-through questions

The functions $g(x) = x - 4$ and $f(y) = y + 4$ are __a__ functions, because $f(g(x)) = $ __b__. Also $g(f(y)) = $ __c__. The notation is $f = g^{-1}$ and $g = $ __d__. The composition __e__ is the identity function. By definition $x = g^{-1}(y)$ if and only if $y = $ __1__. When y is in the range of g, it is in the __g__ of g^{-1}. Similarly x is in the __h__ of g when it is in the __i__ of g^{-1}. If g has an inverse then $g(x_1)$ __j__ $g(x_2)$ at any two points. The function g must be steadily __k__ or steadily __l__.

The chain rule applied to $f(g(x)) = x$ gives $(df/dy)($ __m__ $) = $ __n__. The slope of g^{-1} times the slope of g equals __o__. More directly $dx/dy = 1/$ __p__. For $y = 2x + 1$ and $x = \frac{1}{2}(y - 1)$, the slopes are $dy/dx = $ __q__ and $dx/dy = $ __r__. For $y = x^2$ and $x = $ __s__, the slopes are $dy/dx = $ __t__ and $dx/dy = $ __u__. Substituting x^2 for y gives $dx/dy = $ __v__. Then $(dx/dy)(dy/dx) = $ __w__.

The graph of $y = g(x)$ is also the graph of $x = $ __x__, but with x across and y up. For an ordinary graph of g^{-1}, take the reflection in the line __y__. If $(3, 8)$ is on the graph of g, then its mirror image (__z__) is on the graph of g^{-1}. Those particular points satisfy $8 = 2^3$ and $3 = $ __A__.

The inverse of the chain $z = h(g(x))$ is the chain $x = $ __B__. If $g(x) = 3x$ and $h(y) = y^3$ then $z = $ __C__. Its inverse is $x = $ __D__, which is the composition of __E__ and __F__.

Solve equations 1–10 for x, to find the inverse function $x = g^{-1}(y)$. When more than one x gives the same y, write "no inverse."

1 $y = 3x - 6$	2 $y = Ax + B$
3 $y = x^2 - 1$	4 $y = x/(x - 1)$ [solve $xy - y = x$]
5 $y = 1 + x^{-1}$	6 $y = |x|$
7 $y = x^3 - 1$	8 $y = 2x + |x|$
9 $y = \sin x$	10 $y = x^{1/5}$ [draw graph]

11 Solving $y = \dfrac{1}{x - a}$ gives $xy - ay = 1$ or $x = \dfrac{1 + ay}{y}$. Now solve that equation for y.

12 Solving $y = \dfrac{x + 1}{x - 1}$ gives $xy - y = x + 1$ or $x = \dfrac{y + 1}{y - 1}$. Draw the graph to see why f and f^{-1} are the same. Compute dy/dx and dx/dy.

13 Suppose f is increasing and $f(2) = 3$ and $f(3) = 5$. What can you say about $f^{-1}(4)$?

14 Suppose $f(2) = 3$ and $f(3) = 5$ and $f(5) = 5$. What can you say about f^{-1}?

15 Suppose $f(2) = 3$ and $f(3) = 5$ and $f(5) = 0$. How do you know that there is no function f^{-1}?

16 **Vertical line test:** If no vertical line touches its graph twice then $f(x)$ is a *function* (one y for each x). **Horizontal line test:** If no horizontal line touches its graph twice then $f(x)$ is *invertible* because _____.

17 If $f(x)$ and $g(x)$ are increasing, which two of these might not be increasing?

$$f(x) + g(x) \qquad f(x)g(x) \qquad f(g(x)) \qquad f^{-1}(x) \qquad 1/f(x)$$

18 If $y = 1/x$ then $x = 1/y$. If $y = 1 - x$ then $x = 1 - y$. The graphs are their own mirror images in the 45° line. Construct two more functions with this property $f = f^{-1}$ or $f(f(x)) = x$.

19 For which numbers m are these functions invertible?

    (a) $y = mx + b$ (b) $y = mx + x^3$ (c) $y = mx + \sin x$

20 From its graph show that $y = |x| + cx$ is invertible if $c > 1$ and also if $c < -1$. The inverse of a piecewise linear function is piecewise _____.

In 21–26 find dy/dx in terms of x and dx/dy in terms of y.

21 $y = x^5$	22 $y = 1/(x - 1)$
23 $y = x^3 - 1$	24 $y = 1/x^3$
25 $y = \dfrac{x}{x - 1}$	26 $y = \dfrac{ax + b}{cx + d}$

27 If $dy/dx = 1/y$ then $dx/dy = $ _____ and $x = $ _____.

28 If $dx/dy = 1/y$ then $dy/dx = $ _____ (these functions are $y = e^x$ and $x = \ln y$, soon to be honored properly).

29 The slopes of $f(x) = \frac{1}{3}x^3$ and $g(x) = -1/x$ are x^2 and $1/x^2$. Why isn't $f = g^{-1}$? What is g^{-1}? Show that $g'(g^{-1})' = 1$.

30 At the points x_1, x_2, x_3 a piecewise constant function jumps to y_1, y_2, y_3. Draw its graph starting from $y(0) = 0$. The mirror image is piecewise constant with jumps at the points _____ to the heights _____. Why isn't this the inverse function?

In 31–38 draw the graph of $y = g(x)$. Separately draw its mirror image $x = g^{-1}(y)$.

31 $y = 5x - 10$	32 $y = \cos x, \ 0 \leqslant x \leqslant \pi$
33 $y = 1/(x + 1)$	34 $y = |x| - 2x$
35 $y = 10^x$	36 $y = \sqrt{1 - x^2}, \ 0 \leqslant x \leqslant 1$
37 $y = 2^{-x}$	38 $y = 1/\sqrt{1 - x^2}, \ 0 \leqslant x < 1$

In 39–42 find dx/dy at the given point.

39 $y = \sin x$ at $x = \pi/6$	40 $y = \tan x$ at $x = \pi/4$
41 $y = \sin x^2$ at $x = 3$	42 $y = x - \sin x$ at $x = 0$

43 If y is a decreasing function of x, then x is a _____ function of y. Prove by graphs and by the chain rule.

44 If $f(x) > x$ for all x, show that $f^{-1}(y) < y$.

45 *True or false*, with example:

(a) If $f(x)$ is invertible so is $h(x) = (f(x))^2$.

(b) If $f(x)$ is invertible so is $h(x) = f(f(x))$.

(c) $f^{-1}(y)$ has a derivative at every y.

In the chains 46–51 write down $g(x)$ and $f(y)$ and their inverses. Then find $x = g^{-1}(f^{-1}(z))$.

46 $z = 5(x - 4)$ **47** $z = (x^m)^r$

48 $z = (6 + x)^3$ **49** $z = 6 + x^3$

50 $z = \frac{1}{2}(\frac{1}{2}x + 4) + 4$ **51** $z = \log(10^x)$

52 Solving $f(x) = 0$ is a large part of applied mathematics. Express the solution x^* in terms of f^{-1}: $x^* = $ _____.

53 (a) Show by example that d^2x/dy^2 is not $1/(d^2y/dx^2)$.

(b) If y is in meters and x is in seconds, then d^2y/dx^2 is in _____ and d^2x/dy^2 is in _____.

54 Newton's method solves $f(x^*) = 0$ by applying a linear approximation to f^{-1}:

$$f^{-1}(0) \approx f^{-1}(y) + (df^{-1}/dy)(0 - y).$$

For $y = f(x)$ this is Newton's equation $x^* \approx x + $ _____.

55 If the demand is $1/(p + 1)^2$ when the price is p, then the demand is y when the price is _____. If the range of prices is $p \geqslant 0$, what is the range of demands?

56 If $dF/dx = f(x)$ show that the derivative of $G(y) = yf^{-1}(y) - F(f^{-1}(y))$ is $f^{-1}(y)$.

57 For each number y find the maximum value of $yx - 2x^4$. This maximum is a function $G(y)$. Verify that the derivatives of $G(y)$ and $2x^4$ are inverse functions.
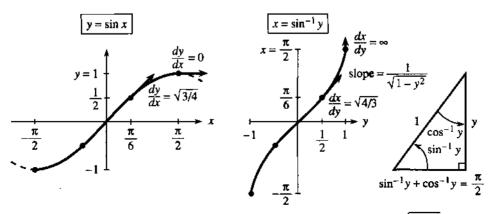
58 (for professors only) If $G(y)$ is the maximum value of $yx - F(x)$, prove that $F(x)$ is the maximum value of $xy - G(y)$. Assume that $f(x) = dF/dx$ is increasing, like $8x^3$ in Problem 57.

59 Suppose the richest x percent of people in the world have $10\sqrt{x}$ percent of the wealth. Then y percent of the wealth is held by _____ percent of the people.

■■■■■■■ **4.4 Inverses of Trigonometric Functions** ■■■■■■■

Mathematics is built on basic functions like the sine, and on basic ideas like the inverse. Therefore *it is totally natural to invert the sine function*. The graph of $x = \sin^{-1}y$ is a mirror image of $y = \sin x$. This is a case where we pay close attention to the domains, since the sine goes up and down infinitely often. We only want *one piece* of that curve, in Figure 4.9.

For the hold line the domain is restricted. *The angle x lies between $-\pi/2$ and $+\pi/2$.* On that interval the sine is increasing, so *each y comes from exactly one angle x*. If the whole sine curve is allowed, infinitely many angles would have sin $x = 0$. The sine

Fig. 4.9 Graphs of $\sin x$ and $\sin^{-1}y$. Their slopes are $\cos x$ and $1/\sqrt{1 - y^2}$.

function could not have an inverse. By restricting to an interval where sin x is increasing, we make the function invertible.

The inverse function brings y back to x. It is $x = \sin^{-1}y$ (the *inverse sine*):

$$x = \sin^{-1}y \text{ when } y = \sin x \text{ and } |x| \leqslant \pi/2. \tag{1}$$

The inverse starts with a number y between -1 and 1. It produces an angle $x = \sin^{-1}y$—*the angle whose sine is y.* The angle x is between $-\pi/2$ and $\pi/2$, with the required sine. Historically x was called the "arc sine" of y, and *arcsin* is used in computing. The mathematical notation is $\sin^{-1}$. *This has nothing to do with* $1/\sin x$.

The figure shows the $30°$ angle $x = \pi/6$. Its sine is $y = \frac{1}{2}$. *The inverse sine of $\frac{1}{2}$ is $\pi/6$.* Again: The symbol $\sin^{-1}(1)$ stands for the angle whose sine is 1 (this angle is $x = \pi/2$). We are seeing $g^{-1}(g(x)) = x$:

$$\sin^{-1}(\sin x) = x \text{ for } -\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2} \qquad \sin(\sin^{-1}y) = y \text{ for } -1 \leqslant y \leqslant 1.$$

EXAMPLE 1 (important) If sin $x = y$ find a formula for cos x.

Solution We are given the sine, we want the cosine. The key to this problem must be $\cos^2x = 1 - \sin^2x$. When the sine is y, the cosine is the square root of $1 - y^2$:

$$\cos x = \cos(\sin^{-1}y) = \sqrt{1 - y^2}. \tag{2}$$

This formula is crucial for computing derivatives. We use it immediately.

THE DERIVATIVE OF THE INVERSE SINE

The calculus problem is to find the slope of the inverse function $f(y) = \sin^{-1}y$. The chain rule gives *(slope of inverse function)* $= 1/$*(slope of original function)*. Certainly the slope of sin x is cos x. To switch from x to y, use equation (2):

$$y = \sin x \text{ gives } \frac{dy}{dx} = \cos x \text{ so that } \frac{dx}{dy} = \frac{1}{\cos x} = \frac{1}{\sqrt{1 - y^2}}. \tag{3}$$

This derivative $1/\sqrt{1 - y^2}$ gives a new v–f pair that is extremely valuable in calculus:

$$velocity \quad v(t) = 1/\sqrt{1 - t^2} \qquad distance \quad f(t) = \sin^{-1}t.$$

Inverse functions will soon produce two more pairs, from the derivatives of $\tan^{-1}y$ and $\sec^{-1}y$. *The table at the end lists all the essential facts.*

EXAMPLE 2 The slope of $\sin^{-1}y$ at $y = 1$ is *infinite*: $1/\sqrt{1 - y^2} = 1/0$. Explain.

At $y = 1$ the graph of $y = \sin x$ is horizontal. The slope is zero. So its mirror image is vertical. The slope $1/0$ is an extreme case of the chain rule.

Question What is $d/dx (\sin^{-1}x)$?   Answer $1/\sqrt{1 - x^2}$. I just changed letters.

THE INVERSE COSINE AND ITS DERIVATIVE

Whatever is done for the sine can be done for the cosine. But the domain and range have to be watched. The graph cannot be allowed to go up and down. Each y from -1 to 1 should be the cosine of *only one angle x.* That puts x between 0 and π. Then the cosine is steadily decreasing and $y = \cos x$ has an inverse:

$$\cos^{-1}(\cos x) = x \text{ and } \cos(\cos^{-1}y) = y. \tag{4}$$

The cosine of the angle $x = 0$ is the number $y = 1$. The inverse cosine of $y = 1$ is the angle $x = 0$. Those both express the same fact, that $\cos 0 = 1$.

For the slope of $\cos^{-1}y$, we could copy the calculation that succeeded for $\sin^{-1}y$. The chain rule could be applied as in (3). But there is a faster way, because of a special relation between $\cos^{-1}y$ and $\sin^{-1}y$. **Those angles always add to a right angle**:

$$\cos^{-1}y + \sin^{-1}y = \pi/2. \tag{5}$$

Figure 4.9c shows the angles and Figure 4.10c shows the graphs. The sum is $\pi/2$ (the dotted line), and its derivative is zero. So the derivatives of $\cos^{-1}y$ and $\sin^{-1}y$ must add to zero. *Those derivatives have opposite sign.* There is a **minus** for the inverse cosine, and its graph goes downward:

$$\textbf{\textit{The derivative of }} \ x = \cos^{-1}y \ \textbf{\textit{is}} \ \ dx/dy = -1/\sqrt{1-y^2}. \tag{6}$$
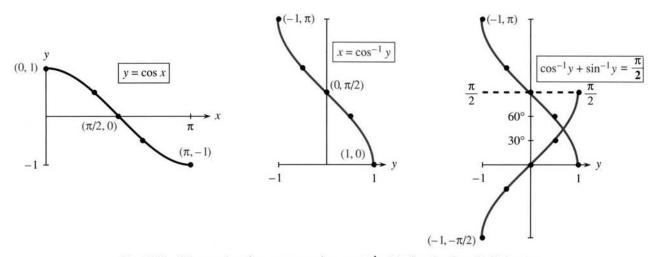

Fig. 4.10 The graphs of $y = \cos x$ and $x = \cos^{-1}y$. Notice the domain $0 \leqslant x \leqslant \pi$.

Question How can two functions $x = \sin^{-1}y$ and $x = -\cos^{-1}y$ have the *same derivative*?

Answer $\sin^{-1}y$ must be the same as $-\cos^{-1}y + C$. Equation (5) gives $C = \pi/2$.
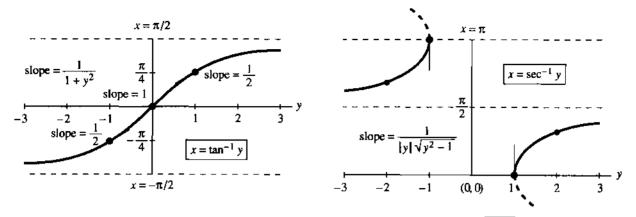
THE INVERSE TANGENT AND ITS DERIVATIVE

The tangent is $\sin x/\cos x$. The inverse tangent is *not* $\sin^{-1}y/\cos^{-1}y$. The inverse function produces *the angle whose tangent is* y. Figure 4.11 shows that angle, which is between $-\pi/2$ and $\pi/2$. The tangent can be any number, but the inverse tangent is in the *open interval* $-\pi/2 < x < \pi/2$. (The interval is "open" because its endpoints are not included.) The tangents of $\pi/2$ and $-\pi/2$ are not defined.

The slope of $y = \tan x$ is $dy/dx = \sec^2 x$. What is the slope of $x = \tan^{-1}y$?

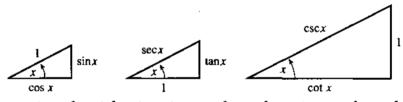$$\text{By the chain rule} \quad \frac{dx}{dy} = \frac{1}{\sec^2 x} = \frac{1}{1 + \tan^2 x} = \frac{1}{1 + y^2}. \tag{7}$$

4E *The derivative of* $f(y) = \tan^{-1}y$ *is* $\dfrac{df}{dy} = \dfrac{1}{1 + y^2}.$ $\qquad$ (8)

Fig. 4.11 $x = \tan^{-1}y$ has slope $1/(1 + y^2)$. $x = \sec^{-1}y$ has slope $1/|y|\sqrt{y^2 - 1}$.

EXAMPLE 3 The tangent of $x = \pi/4$ is $y = 1$. We check slopes. On the inverse tangent curve, $dx/dy = 1/(1 + y^2) = \frac{1}{2}$. On the tangent curve, $dy/dx = \sec^2 x$. At $\pi/4$ the secant squared equals 2. The slopes $dx/dy = \frac{1}{2}$ and $dy/dx = 2$ multiply to give 1.

Important Soon will come the following question. *What function has the derivative* $1/(1 + x^2)$? One reason for reading this section is to learn the answer. The function is in equation (8)—*if we change letters. It is* $f(x) = \tan^{-1}x$ *that has slope* $1/(1 + x^2)$.

Fig. 4.12 $\cos^2 x + \sin^2 x = 1$ and $1 + \tan^2 x = \sec^2 x$ and $1 + \cot^2 x = \csc^2 x$.

INVERSE COTANGENT, INVERSE SECANT, INVERSE COSECANT

There is no way we can avoid completing this miserable list! But it can be painless. The idea is to use $1/(dy/dx)$ for $y = \cot x$ and $y = \sec x$ and $y = \csc x$:

$$\frac{dx}{dy} = \frac{-1}{\csc^2 x} \quad \text{and} \quad \frac{dx}{dy} = \frac{1}{\sec x \tan x} \quad \text{and} \quad \frac{dx}{dy} = \frac{-1}{\csc x \cot x}. \tag{9}$$

In the middle equation, replace $\sec x$ by y and $\tan x$ by $\pm\sqrt{y^2 - 1}$. Choose the sign for positive slope (compare Figure 4.11). That gives the middle equation in (10):
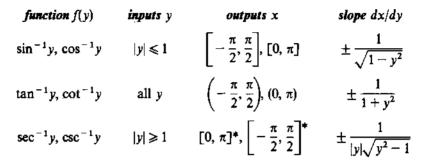
The derivatives of $\cot^{-1}y$ *and* $\sec^{-1}y$ *and* $\csc^{-1}y$ *are*

$$\frac{d}{dy}(\cot^{-1}y) = \frac{-1}{1 + y^2} \quad \frac{d}{dy}(\sec^{-1}y) = \frac{1}{|y|\sqrt{y^2 - 1}} \quad \frac{d}{dy}(\csc^{-1}y) = \frac{-1}{|y|\sqrt{y^2 - 1}}. \tag{10}$$

Note about the inverse secant When y is negative there is a choice for $x = \sec^{-1}y$. We selected the angle in the second quadrant (between $\pi/2$ and π). Its cosine is negative, so its secant is negative. This choice makes $\sec^{-1}y = \cos^{-1}(1/y)$, which matches $\sec x = 1/\cos x$. It also makes $\sec^{-1}y$ an increasing function, where $\cos^{-1}y$ is a decreasing function. So we needed the absolute value $|y|$ in the derivative.

Some mathematical tables make a different choice. The angle x could be in the third quadrant (between $-\pi$ and $-\pi/2$). Then the slope omits the absolute value.

Summary For the six inverse functions it is only necessary to learn three derivatives. The other three just have minus signs, as we saw for $\sin^{-1}y$ and $\cos^{-1}y$. Each inverse function and its "cofunction" add to $\pi/2$, so their derivatives add to zero. Here are the six functions for quick reference, with the three new derivatives.

function $f(y)$	inputs y	outputs x	slope dx/dy
$\sin^{-1}y,\ \cos^{-1}y$	$|y|\leqslant 1$	$\left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right],\ [0,\pi]$	$\pm\dfrac{1}{\sqrt{1-y^2}}$
$\tan^{-1}y,\ \cot^{-1}y$	all y	$\left(-\dfrac{\pi}{2},\dfrac{\pi}{2}\right),\ (0,\pi)$	$\pm\dfrac{1}{1+y^2}$
$\sec^{-1}y,\ \csc^{-1}y$	$|y|\geqslant 1$	$[0,\pi]^*,\ \left[-\dfrac{\pi}{2},\dfrac{\pi}{2}\right]^*$	$\pm\dfrac{1}{|y|\sqrt{y^2-1}}$

If $y=\cos x$ or $y=\sin x$ then $|y|\leqslant 1$. For $y=\sec x$ and $y=\csc x$ the opposite is true; we must have $|y|\geqslant 1$. The graph of $\sec^{-1}y$ misses all the points $-1<y<1$.

Also, that graph misses $x=\pi/2$—where the cosine is zero. The secant of $\pi/2$ would be $1/0$ (impossible). Similarly $\csc^{-1}y$ misses $x=0$, because $y=\csc 0$ cannot be $1/\sin 0$. The asterisks in the table are to remove those points $x=\pi/2$ and $x=0$.

The column of derivatives is what we need and use in calculus.

4.4 EXERCISES

Read-through questions

The relation $x=\sin^{-1}y$ means that __a__ is the sine of __b__. Thus x is the angle whose sine is __c__. The number y lies between __d__ and __e__. The angle x lies between __f__ and __g__. (If we want the inverse to exist, there cannot be two angles with the same sine.) The cosine of the angle $\sin^{-1}y$ is $\sqrt{\underline{\quad h\quad}}$. The derivative of $x=\sin^{-1}y$ is $dx/dy=\underline{\quad i\quad}$.

The relation $x=\cos^{-1}y$ means that y equals __j__. Again the number y lies between __k__ and __l__. This time the angle x lies between __m__ and __n__ (so that each y comes from only one angle x). The sum $\sin^{-1}y+\cos^{-1}y=\underline{\quad o\quad}$. (The angles are called __p__, and they add to a __q__ angle.) Therefore the derivative of $x=\cos^{-1}y$ is $dx/dy=\underline{\quad r\quad}$, the same as for $\sin^{-1}y$ except for a __s__ sign.

The relation $x=\tan^{-1}y$ means that $y=\underline{\quad t\quad}$. The number y lies between __u__ and __v__. The angle x lies between __w__ and __x__. The derivative is $dx/dy=\underline{\quad y\quad}$. Since $\tan^{-1}y+\cot^{-1}y=\underline{\quad z\quad}$, the derivative of $\cot^{-1}y$ is the same except for a __A__ sign.

The relation $x=\sec^{-1}y$ means that __B__. The number y never lies between __C__ and __D__. The angle x lies between __E__ and __F__, but never at $x=\underline{\quad G\quad}$. The derivative of $x=\sec^{-1}y$ is $dx/dy=\underline{\quad H\quad}$.

In 1–4, find the angles $\sin^{-1}y$ and $\cos^{-1}y$ and $\tan^{-1}y$ in radians.

1 $y=0$ **2** $y=-1$ **3** $y=1$ **4** $y=\sqrt{3}$

5 We know that $\sin\pi=0$. Why isn't $\pi=\sin^{-1}0$?

6 Suppose $\sin x=y$. Under what restriction is $x=\sin^{-1}y$?

7 Sketch the graph of $x=\sin^{-1}y$ and locate the points with slope $dx/dy=2$.

8 Find dx/dy if $x=\sin^{-1}\tfrac{1}{2}y$. Draw the graph.

9 If $y=\cos x$ find a formula for $\sin x$. First draw a right triangle with angle x and near side y—what are the other two sides?

10 If $y=\sin x$ find a formula for $\tan x$. First draw a right triangle with angle x and far side y—what are the other sides?

11 Take the x derivative of $\sin^{-1}(\sin x)=x$ by the chain rule. Check that $d(\sin^{-1}y)/dy=1/\sqrt{1-y^2}$ gives a correct result.

12 Take the y derivative of $\cos(\cos^{-1}y)=y$ by the chain rule. Check that $d(\cos^{-1}y)/dy=-1/\sqrt{1-y^2}$ gives a correct result.

13 At $y=0$ and $y=1$, find the slope dx/dy of $x=\sin^{-1}y$ and $x=\cos^{-1}y$ and $x=\tan^{-1}y$.

14 At $x=0$ and $x=1$, find the slope dx/dy of $x=\sin^{-1}y$ and $x=\cos^{-1}y$ and $x=\tan^{-1}y$.

15 True or false, with reason:

(a) $(\sin^{-1}y)^2 + (\cos^{-1}y)^2 = 1$

(b) $\sin^{-1}y = \cos^{-1}y$ has no solution

(c) $\sin^{-1}y$ is an increasing function

(d) $\sin^{-1}y$ is an odd function

(e) $\sin^{-1}y$ and $-\cos^{-1}y$ have the same slope—so they are the same.

(f) $\sin(\cos x) = \cos(\sin x)$

16 Find $\tan(\cos^{-1}(\sin x))$ by drawing a triangle with sides $\sin x, \cos x, 1$.

Compute the derivatives in 17–28 (using the letters as given).

17 $u = \sin^{-1}x$

18 $u = \tan^{-1}2x$

19 $z = \sin^{-1}(\sin 3x)$

20 $z = \sin^{-1}(\cos x)$

21 $z = (\sin^{-1}x)^2$

22 $z = (\sin^{-1}x)^{-1}$

23 $z = \sqrt{1-y^2}\,\sin^{-1}y$

24 $z = (1+x^2)\tan^{-1}x$

25 $x = \sec^{-1}(y+1)$

26 $u = \sec^{-1}(\sec x^2)$

27 $u = \sin^{-1}y/\cos^{-1}\sqrt{1-y^2}$

28 $u = \sin^{-1}y + \cos^{-1}y + \tan^{-1}y$

29 Draw a right triangle to show why $\tan^{-1}y + \cot^{-1}y = \pi/2$.

30 Draw a right triangle to show why $\tan^{-1}y = \cot^{-1}(1/y)$.

31 If $y = \tan x$ find $\sec x$ in terms of y.

32 Draw the graphs of $y = \cot x$ and $x = \cot^{-1}y$.

33 Find the slope dx/dy of $x = \tan^{-1}y$ at

(a) $y = -3$ (b) $x = 0$ (c) $x = -\pi/4$

34 Find a function $u(t)$ whose slope satisfies $u' + t^2u' = 1$.

35 What is the second derivative d^2x/dy^2 of $x = \sin^{-1}y$?

36 What is d^2u/dy^2 for $u = \tan^{-1}y$?

Find the derivatives in 37–44.

37 $y = \sec \frac{1}{2}x$

38 $x = \sec^{-1}2y$

39 $u = \sec^{-1}(x^n)$

40 $u = \sec^{-1}(\tan x)$

41 $\tan y = (x-1)/(x+1)$

42 $z = (\sin x)(\sin^{-1}x)$

43 $y = \sec^{-1}\sqrt{x^2+1}$

44 $z = \sin(\cos^{-1}x) - \cos(\sin^{-1}x)$

45 Differentiate $\cos^{-1}(1/y)$ to find the slope of $\sec^{-1}y$ in a new way.

46 The domain and range of $x = \csc^{-1}y$ are _____.

47 Find a function $u(y)$ such that $du/dy = 4/\sqrt{1-y^2}$.

48 Solve the differential equation $du/dx = 1/(1+4x^2)$.

49 If $du/dx = 2/\sqrt{1-x^2}$ find $u(1) - u(0)$.

50 (recommended) With $u(x) = (x-1)/(x+1)$, find the derivative of $\tan^{-1}u(x)$. This is also the derivative of _____. So the difference between the two functions is a _____.

51 Find $u(x)$ and $\tan^{-1}u(x)$ and $\tan^{-1}x$ at $x = 0$ and $x = \infty$. Conclusion based on Problem 50: $\tan^{-1}u(x) - \tan^{-1}x$ equals the number _____.

52 Find $u(x)$ and $\tan^{-1}u(x)$ and $\tan^{-1}x$ as $x \to -\infty$. Now $\tan^{-1}u(x) - \tan^{-1}x$ equals _____. Something has happened to $\tan^{-1}u(x)$. At what x do $u(x)$ and $\tan^{-1}u(x)$ change instantly?

Resource: Calculus Online Textbook
Gilbert Strang

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