Effective Field Theory (8.851) Spring 2013
Homework 5

NOTE: In thinking of good SCET problems for the last problem set I encounter the issue that either the solutions can be found in the literature, or the solution of the problem is on the difficult side and can not be found readily in the literature. Therefore I will give you the following choice. Either do TWO out of three of the problems 1, 2, 3. Or do ONE OF the problems 4, 5, or 6. If you pick one of the last 3 problems then you will be graded on having shown progress towards a solution, rather than having the complete answer. (Problem 6 is something that is publishable if pushed to the end. If after the term is over there is interest by several students in carrying through on this problem I will be happy to facilitate forming a collaboration of the relevant subset of people.)

Problem 1) Decoupling of Ultrasoft gluons in $\mathbf{SCET_I}$

Consider a Wilson line built from usoft gluons

$$Y_n(x) = P \exp\left(ig \int_{-\infty}^{0} ds\, n\cdot A_{us}(sn^\mu + x^\mu) \right). \tag{1}$$

It satisfies $Y_n^\dagger Y_n = 1$ and has the equation of motion $n\cdot D_{us}\, Y_n = 0$. Start with the leading order Lagrangian, $\mathcal{L}^{(0)}$ for a collinear quark in $\mathrm{SCET_I}$. This action gives eikonal couplings to ultrasoft gluons. Make the field redefinitions $\xi_n = Y_n \xi_n^{(0)}$ and $A_n = Y_n A_n^{(0)} Y_n^\dagger$ to obtain a Lagrangian $\mathcal{L}^{(0)}(\xi_n^{(0)}, A_n^{(0)})$. Show explicitly that this new Lagrangian has no coupling to $n\cdot A_{us}$ gluons.

Problem 2) Pion Light-Cone Distribution Function

Factorization of degrees of freedom allows us to describe processes in terms of simpler (universal) objects. One such object is the pion light-cone distribution function. In QCD we can define

$$\begin{aligned}\left\langle \pi^a(p) \right| \bar\psi(y)\gamma^\mu\gamma^5 \frac{\tau^b}{\sqrt{2}}\, Y(y,x)\, \psi(x) \left|0\right\rangle \;=\; & -if_\pi \delta^{ab} p^\mu \int_0^1 dz\, e^{i[zp\cdot y + (1-z)p\cdot x]}\, \phi_\pi(z) \\ & + \ldots , \end{aligned} \tag{2}$$

Here $f_\pi \simeq 131\,\mathrm{MeV}$ and the field ψ denotes the isospin doublet (u,d). Consider the path from y^μ to x^μ to be light-like, $(y-x)^2 = 0$, with $y^\mu = y\bar n^\mu$, $x^\mu = x\bar n^\mu$. $Y(y,x)$ is a Wilson line along this path. For the leading order collinear operator in SCET

$$\big\langle \pi^a_{n,p}\big|\bar\xi_{n,p_1} W\, \bar{n}\!\!\!/ \gamma_5 \frac{\tau^b}{\sqrt{2}}\, \delta(\omega - \bar{\mathcal{P}}_+)\, W^\dagger \xi_{n,p_2}\big|0\big\rangle \\ = -i f_\pi\, \bar n\!\cdot\! p\, \delta^{ab} \int_0^1 dz\, \delta[\omega - (2z-1)\bar n\!\cdot\! p]\, \phi_\pi(z)\,, \qquad (3)$$

where $\bar{\mathcal{P}}_+ = \bar{\mathcal{P}}^\dagger + \bar{\mathcal{P}}$. Eq. (3) can be obtained from Eq. (2) by projecting onto the LO term with collinear quarks and taking a Fourier transform with $x = -y$.

a) Using the power counting for the collinear fields in $\lambda = \Lambda_{\rm QCD}/Q$, together with mass dimensions, count the powers of $\Lambda_{\rm QCD}$ and Q on the LHS and RHS of Eq. (3) and verify that $\phi_\pi(z)$ is $\mathcal{O}(\lambda^0)$ and is dimensionless.

b) Under charge conjugation, $C^{-1}\xi_{n,p}(x)C = -[\bar\xi_{n,-p}(x)\mathcal{C}]^T$ where $\mathcal{C}$ is the usual charge conjugation matrix. For a π^0 state use charge conjugation together with Eq. (3) to prove that $\phi_{\pi^0}(z) = \phi_{\pi^0}(1-z)$. Physically what does this mean? What do you have to assume to prove that this is true for the π^+ and π^-?

c) Carry out the steps to go from Eq. (2) to Eq. (3) ignoring the ellipses.

Problem 3) One-Loop Jet Function

The jet function is defined by the following vacuum matrix element of a purely collinear operator

$$J(k^+\omega) = -\frac{1}{\pi\,\omega}\, {\rm Im} \int d^4x\, e^{ik\cdot x}\, i\, \big\langle 0\big|\, {\rm T}\, \bar\chi_{n,\omega,0_\perp}(0)\, \frac{\bar{n}\!\!\!/}{4N_c}\, \chi_n(x)\big|0\big\rangle\,, \qquad (4)$$

where it suffices to take $k^\mu = k^+\bar n^\mu/2$. Compute all the one-loop diagrams to derive an expression for $J(k^+\omega)$ at one-loop order.

Problem 4) General Covariant Gauge

Carry out the 1-loop matching calculation for the Wilson coefficient $C(\bar{\mathcal{P}})$ appearing in the leading order $b \to s\gamma$ calculation, but do it in a general covariant gauge rather than Feynman gauge (where the results were quoted in lecture). Continue your exploration of the gauge dependence and independence of various intermediate results by also carrying out the one-loop matching computation for the lowest order operator for $e^+e^- \to$ dijets, namely $\bar\chi_n\gamma^\mu_\perp\chi_{\bar n}$ (again it is in the literature for Feynman gauge). Many of the diagrams in these two computations are the same.

Problem 5) RGE for nonforward collinear partons

Use Feynman gauge and carry out the renormalization of the operator

$$O(\omega, \omega') = \bar{\chi}_{n,\omega} \frac{\bar{n}\!\!\!/}{2} \chi_{n,\omega'} \,. \tag{5}$$

For simplicity take the quarks to be non-diagonal in flavor to avoid operator mixing with a $\mathcal{B}^{\mu}_{n\perp}\mathcal{B}_{n\perp\,\mu}$ type operator. For the case $\sum_{\omega'} O(\omega,\omega')$ we carried out this computation in Lecture, and showed that the renormalization is given by the standard Alterelli-Parisi RGE with the splitting function as the anomalous dimension. The analogous computation for the above operator will yield a renormalization group equation for the so-called generalized parton distribution functions (non-forward). Besides containing the standard splitting function, as another special case of your answer you can take a limit to find the Brodsky-Lepage anomalous dimension equation for $\phi_\pi(z)$ in problem 2 above.

Problem 6) Power Suppressed Operators for Dijets

Construct a complete basis of power suppressed operators for the process $e^+e^- \to$ dijets, and carry out the one-loop matching computation to determine their Wilson coefficients. (Use RPI to fix some of the coefficients without computation, if possible. Some information on the complete basis is available in the literature.) The lowest order current operator is $\bar{\chi}_n \gamma_\perp^\mu \chi_{\bar{n}}$, and the leading power suppressed operators will include an extra ${\mathcal{P}_\perp}^\alpha$ or an extra ${\mathcal{B}_\perp}^\alpha$ which can be either n-collinear or $\bar{n}$-collinear.

Also enumerate all time ordered products of $\mathrm{SCET_I}$ Lagrangians and current operators that are the same order in the power counting as your power suppressed current operator. (To make this problem publishable at the level where it would have a non-trivial impact on the literature one should carry out the analysis to $\mathcal{O}(\lambda^2)$ since the $\mathcal{O}(\lambda)$ terms will vanish for the simplest observables.)

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