Massachusetts Institute of Technology

Department of Physics

Course:	8.701 – Introduction to Nuclear and Particle Physics
Term:	Fall 2020
Instructor:	Markus Klute
TA :	Tianyu Justin Yang

Discussion Problems

from recitation on **September 8rd, 2020**

Problem 1: Cosmic rays

Greisen, Zatespin, and Kuzmin (GZK) predicted that there is a maximum energy for which any cosmic ray (which are predominantly protons) will be observed. Their argument was that if energetic enough, cosmic rays will lose energy via the process $p\gamma_{CMB} \rightarrow p\pi^0$, where γ_{CMB} is a cosmic microwave background photon, whose energies are $E_{CMB} \approx 3 \times 10^{-4}$eV. Therefore, the threshold energy for which this process can occur sets the maximum expected energy a cosmic ray should be observed to have (called the GZK cutoff). Calculate the GZK cutoff energy in eV, and compare your result to the observed cosmic ray energy spectrum (see Fig. 29.8 of the PDG review on Cosmic Rays or below). [Hint: You can take the energy to be much larger than the proton mass and assume that the frame in which we want to know this energy is the one where the proton and CMB photon collide head on.]

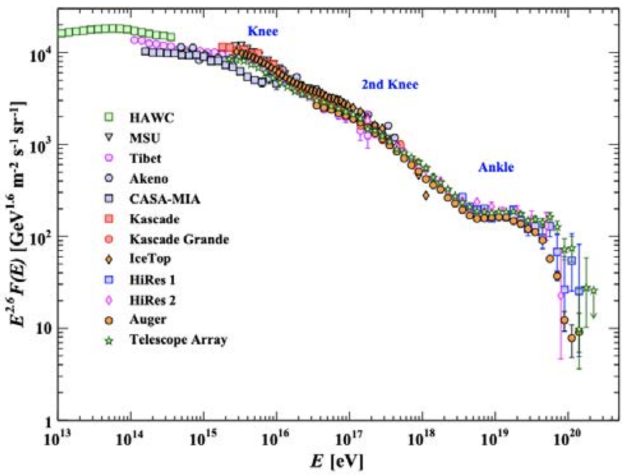

Figure 29.8: The all-particle spectrum as a function of E (energy-per-nucleus) from air shower measurements [95–105]

7) $(p_{p_i} + p_\gamma)^2 = (p_{p_f} + p_\pi)^2$ [Lorentz Inv.]

min. $E \Rightarrow p_{p_f} = \begin{pmatrix} m_p \\ \vec{0} \end{pmatrix}$ $p_\pi = \begin{pmatrix} m_\pi \\ \vec{0} \end{pmatrix}$

$\therefore (p_{p_i} + p_\gamma)^2 = (m_p + m_\pi)^2 = m_p^2 + m_\pi^2 + 2 m_p m_\pi$

Also, $\hookrightarrow m_p^2 + \cancel{m_\gamma^2}^{\,0} + 2 p_{p_i} \cdot p_\gamma$

$E_{p_i} \gg m_p \Rightarrow p_{p_i} \approx \begin{pmatrix} E_{p_i} \\ 0 \\ 0 \\ E_{p_i} \end{pmatrix}$ $p_\gamma \approx \begin{pmatrix} E_\gamma \\ 0 \\ 0 \\ -E_\gamma \end{pmatrix}$ [Assume frame is $\xrightarrow{p} \xleftarrow{\gamma}$]

$\Rightarrow p_{p_i} \cdot p_\gamma \approx 2 E_{p_i} E_\gamma$

$\Rightarrow 4 E_{p_i} E_\gamma = m_\pi^2 + 2 m_p m_\pi$

$\therefore E_{p_i} \approx \dfrac{m_\pi^2 + 2 m_p m_\pi}{4 E_\gamma} \approx 2 \times 10^{20}$ eV

Problem 2: Mandelstam variables

Consider the reaction between two particles with 4-momenta p_1^μ and p_2^μ. The outgoing particles have 4-momenta p_3^μ and p_4^μ. Discuss the variables s, t, and u for the center-of-mass frame and a fixed-target frame where the second particle is at rest. Assume that the masses involved are much smaller than the energies ($m_i \ll E_i$).

$s = (p_1^\mu + p_2^\mu)^2 = (p_3^\mu + p_4^\mu)^2$	$4\,{E_1^*}^2$	$2\,m_2\,E_1$
$t = (p_1^\mu - p_3^\mu)^2 = (p_4^\mu - p_2^\mu)^2$	$\frac{s}{2}\,(\cos\theta^* - 1)$	$2\,m_2\,(E_3 - E_1)$
$u = (p_1^\mu - p_4^\mu)^2 = (p_3^\mu - p_2^\mu)^2$	$-\frac{s}{2}\,(\cos\theta^* + 1)$	$-\,2\,m_2\,E_3$

$$s + t + u = \sum_{i=1}^{4} m_i^2$$

- Kinematic in center-of-mass frame:

$$\begin{aligned} \vec{p}_1^{\,*} + \vec{p}_2^{\,*} = \vec{p}_3^{\,*} + \vec{p}_4^{\,*} = 0 \\ E_1^* + E_2^* = E_3^* + E_4^* \end{aligned} \tag{1}$$

Kinematic in fixed-target frame:

$$\begin{aligned} &\vec{p}_2 = 0 \\ &\vec{p}_1 = \vec{p}_3 + \vec{p}_4 \\ &E_1 + m_2 = E_3 + E_4 \end{aligned} \tag{2}$$

Assuming that $m_1, m_2 \ll E_1$, i.e. small masses relative to the energies of the scattering process:

$$\begin{aligned} &E \approx |\vec{p}| \\ &E_1^* \approx E_2^* \approx E_3^* \approx E_4^* \end{aligned}$$

For s and t:

$$s = (p_1^\mu + p_2^\mu)^2 = \underbrace{2\, p_1^\mu\, p_{2\,\mu} + m_1^2 + m_2^2 \approx 2 \begin{pmatrix} E_1 \\ \vec{p}_1 \end{pmatrix} \begin{pmatrix} m_2 \\ \vec{0} \end{pmatrix} = 2\, m_2\, E_1}_{\text{Fixed-target frame}}$$

$$= \underbrace{\begin{pmatrix} E_1^* + E_1^* \\ \vec{p}_1^{\,*} - \vec{p}_1^{\,*} \end{pmatrix}^2 = 4\, E_1^{*\,2}}_{\text{Center-of-mass frame}}$$

$$\begin{aligned} t = (p_1^{\mu\,*} - p_3^{\mu\,*})^2 &= -2\, p_1^{\mu\,*}\, p_{3\,\mu}^* + m_1^2 + m_3^2 \\ &\approx -2 \begin{pmatrix} E_1^* \\ \vec{p}_1^{\,*} \end{pmatrix} \begin{pmatrix} E_3^* \\ \vec{p}_3^{\,*} \end{pmatrix} \\ &= -2\Big(E_1^*\, E_3^* - \vec{p}_1^{\,*} \vec{p}_3^{\,*}\Big) = -2\Big(E_1^*\, E_3^* - |\vec{p}_1^{\,*}|\; |\vec{p}_3^{\,*}|\; \cos\theta^*\Big) \\ &= 2\, E_1^{2\,*}\Big(\cos\theta^* - 1\Big) = \frac{s}{2}\Big(\cos\theta^* - 1\Big) \end{aligned}$$

$$\begin{aligned} t = (p_1^\mu - p_3^\mu)^2 &= \begin{pmatrix} E_1 - E_3 \\ \vec{p}_1 - \vec{p}_3 \end{pmatrix}^2 = \begin{pmatrix} E_4 - m_2 \\ \vec{p}_4 \end{pmatrix}^2 \\ &= E_4^2 - \vec{p}_4^{\,2} + m_2^2 - 2\, m_2\, E_4 \approx -2\, m_2\, E_4 \\ &= -2\, m_2\Big(E_1 + m_2 - E_3\Big) \\ &= \;\; 2\, m_2\Big(E_3 - E_1\Big) \end{aligned}$$

Note the range of values for t: $-\infty < t \le 0$. For u and $s + t + u$:

$$
\begin{aligned}
u = (p_1^{\mu\,*} - p_4^{\mu\,*})^2 &= -2\,p_1^{\mu\,*}\,p_{4\,\mu}^{*} + m_1^2 + m_4^2 \\
&\approx -2\,\begin{pmatrix} E_1^* \\ \vec{p}_1^{\,*} \end{pmatrix} \begin{pmatrix} E_4^* \\ \vec{p}_4^{\,*} \end{pmatrix} \\
&= -2\Big(E_1^*\,E_4^* - \vec{p}_1^{\,*}\vec{p}_4^{\,*}\Big) = -2\Big(E_1^*\,E_4^* - |\vec{p}_1^{\,*}|\;|\vec{p}_4^{\,*}|\,\cos\left(\pi + \theta^*\right)\Big) \\
&= -2\,E_1^{2\,*}\Big(\cos\theta^* + 1\Big) = -\frac{s}{2}\Big(\cos\theta^* + 1\Big)
\end{aligned}
$$

$$
\begin{aligned}
u = (p_1^\mu - p_4^\mu)^2 &= \begin{pmatrix} E_1 - E_4 \\ \vec{p}_1 - \vec{p}_4 \end{pmatrix}^2 = \begin{pmatrix} E_3 - m_2 \\ \vec{p}_3 \end{pmatrix}^2 \\
&= E_3^2 - \vec{p}_3^{\,2} + m_2^2 - 2\,m_2\,E_3 \approx -2\,m_2\,E_3
\end{aligned}
$$

$$
\begin{aligned}
s + t + u &= \Big(p_1^\mu + p_2^\mu\Big)^2 + \Big(p_1^\mu - p_3^\mu\Big)^2 + \Big(p_1^\mu - p_4^\mu\Big)^2 \\
&= \sum_{i=1}^{4} m_i^2 + 2\,p_{1\,\mu}\,\underbrace{\Big(p_1^\mu + p_2^\mu - \Big(p_3^\mu + p_4^\mu\Big)\Big)}_{\equiv\,0} = \sum_{i=1}^{4} m_i^2
\end{aligned}
$$

8.701 Introduction to Nuclear and Particle Physics
Fall 2020