Massachusetts Institute of Technology

Department of Physics

Course: 8.701 – Introduction to Nuclear and Particle Physics
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Discussion Problems

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Problem 1: Unitarity

Show that the CKM matrix is unitary for any real number θ_{12}, θ_{23},θ_{13}, and δ, i.e. show that $(VV^\dagger)_{11} = 1$ and $(VV^\dagger)_{12} = 0$ and so on. .

$$VV^\dagger = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$
$$\times \begin{pmatrix} c_{12}c_{13} & -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{-i\delta} & s_{12}s_{23} - c_{12}c_{23}s_{13}e^{-i\delta} \\ s_{12}c_{13} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{-i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{-i\delta} \\ s_{13}e^{i\delta} & s_{23}c_{13} & c_{23}c_{13} \end{pmatrix}$$

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\begin{aligned}
(VV^\dagger)_{11} &= c_{12}c_{13}c_{12}c_{13} + s_{12}c_{13}s_{12}c_{13} + s_{13}e^{-i\delta}s_{13}e^{i\delta} \\
&= \cos^2\theta_{13}(\cos^2\theta_{12} + \sin^2\theta_{12}) + \sin^2\theta_{13} = \cos^2\theta_{13} + \sin^2\theta_{13} = 1. \\
(VV^\dagger)_{12} &= c_{12}c_{13}[\cancel{-s_{12}c_{23}} - c_{12}s_{23}s_{13}e^{-i\delta}] + s_{12}c_{13}[\cancel{c_{12}c_{23}} - s_{12}s_{23}s_{13}e^{-i\delta}] \\
&\quad + s_{13}e^{-i\delta}s_{23}c_{13} = c_{13}s_{13}s_{23}e^{-i\delta}[-c_{12}^2 - s_{12}^2 + 1] = 0. \\
(VV^\dagger)_{13} &= c_{12}c_{13}[\cancel{s_{12}s_{23}} - c_{12}c_{23}s_{13}e^{-i\delta}] + s_{12}c_{13}[\cancel{-c_{12}s_{23}} - s_{12}c_{23}s_{13}e^{-i\delta}] \\
&\quad + s_{13}e^{-i\delta}c_{23}c_{13} = c_{13}s_{13}c_{23}e^{-i\delta}[-c_{12}^2 - s_{12}^2 + 1] = 0. \\
(VV^\dagger)_{21} &= [\cancel{-s_{12}c_{23}} - c_{12}s_{23}s_{13}e^{i\delta}]c_{12}c_{13} + [\cancel{c_{12}c_{23}} - s_{12}s_{23}s_{13}e^{i\delta}]s_{12}c_{13} \\
&\quad + s_{23}c_{13}s_{13}e^{i\delta} = s_{23}c_{13}s_{13}e^{i\delta}[-c_{12}^2 - s_{12}^2 + 1] = 0. \\
(VV^\dagger)_{22} &= [-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta}][-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{-i\delta}] \\
&\quad + [c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta}][c_{12}c_{23} - s_{12}s_{23}s_{13}e^{-i\delta}] + s_{23}c_{13}s_{23}c_{13} \\
&= s_{12}^2c_{23}^2 + c_{12}^2s_{23}^2s_{13}^2 + c_{12}^2c_{23}^2 + s_{12}^2s_{23}^2s_{13}^2 + s_{23}^2c_{13}^2 \\
&= c_{23}^2 + s_{23}^2s_{13}^2 + s_{23}^2c_{13}^2 = 1. \\
(VV^\dagger)_{23} &= [-s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta}][s_{12}s_{23} - c_{12}c_{23}s_{13}e^{-i\delta}] \\
&\quad + [c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta}][-c_{12}s_{23} - s_{12}c_{23}s_{13}e^{-i\delta}] + s_{23}c_{13}c_{23}c_{13} \\
&= -s_{12}^2s_{23}c_{23} + c_{12}^2s_{12}^2s_{23}c_{23} - c_{12}^2s_{23}c_{23} + s_{12}^2s_{13}^2s_{23}c_{23} + c_{13}^2s_{23}c_{23} \\
&= s_{23}c_{23}[-1 + s_{13}^2 + c_{13}^2] = 0. \\
(VV^\dagger)_{31} &= [\cancel{s_{12}s_{23}} - c_{12}c_{23}s_{13}e^{i\delta}]c_{12}c_{13} + [\cancel{-c_{12}s_{23}} - s_{12}c_{23}s_{13}e^{i\delta}]s_{12}c_{13} \\
&\quad + c_{23}c_{13}s_{13}e^{i\delta} = c_{23}c_{13}s_{13}e^{i\delta}[-c_{12}^2 - s_{12}^2 + 1] = 0.
\end{aligned}
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Problem 2: CKM Parameter

Show that as long as the CKM matrix is unitary, the GIM mechanism for eliminating $K^0 \to \mu^+\mu^-$ works for three generations or any number of generations. Note: $u \to d + W^+$ carries a CKM factor V_{ud} and $d \to u + W^-$ carries a factor V_{ud}^*

amplitude is proportional to

$$[V_{ud}^* V_{us} + V_{cd}^* V_{cs} + V_{td}^* V_{ts} + \cdots] = \sum_{j=1}^{n} V_{jd}^* V_{js} = \sum_{j=1}^{n} \tilde{V}_{dj}^* V_{js} = (V^\dagger V)_{ds}.$$

But if V is unitary, then $(V^\dagger V)_{ds} = \delta_{ds} = 0$. QED