

Massachusetts Institute of Technology

Department of Physics

Course: 8.701 — Introduction to Nuclear and Particle Physics

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Instructor: Markus Klute

TA : Tianyu Justin Yang

Problem Set 5

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Problem 1: The T2K Experiment[50 points]

The T2K experiment uses an off-axis ν_μ beam from $\pi^+ \rightarrow \mu^+ \nu_\mu$ decays. Consider the case where the pion has velocity β along the z -direction in the laboratory frame and a neutrino with energy E^* is produced at an angle θ^* with respect to the z' -axis in the π^+ rest frame.

(a) Show that the neutrino energy in the pion rest frame is $p^* = (m_\pi^2 - m_\mu^2)/2m_\pi$.

- In the pion rest frame, the momenta of the particles are:

$$\begin{aligned} p_\pi^* &= (m_\pi, \vec{0}) \\ p_\mu^* &= (\sqrt{p^{*2} + m_\mu^2}, \vec{p}^*) \\ p_\nu^* &= (p^{*2}, -\vec{p}^*) \end{aligned}$$

So, we have:

$$\begin{aligned}
p_\pi^{*2} &= (p_\mu^* + p_\nu^*)^2 \\
m_\pi^2 &= (p^* + \sqrt{p^{*2} + m_\mu^2})^2 \\
m_\pi^2 &= 2p^{*2} + 2p^* \sqrt{p^{*2} + m_\mu^2} + m_\mu^2 \\
m_\pi^2 - m_\mu^2 &= 2p^*(p^* + \sqrt{p^{*2} + m_\mu^2}) \\
p^* &= \frac{m_\pi^2 - m_\mu^2}{2(p^* + \sqrt{p^{*2} + m_\mu^2})}
\end{aligned}$$

But, $p_\pi^* = p_\mu^* + p_\nu^* \Rightarrow m_\pi = p^* + \sqrt{p^{*2} + m_\mu^2}$. Therefore,

$$p^* = \frac{m_\pi^2 - m_\mu^2}{2m_\pi}. \quad (1)$$

(b) Show that the energy E and angle of the production θ of the neutrino in the laboratory frame are $E = \gamma E^*(1 + \beta \cos \theta^*)$ and $E \cos \theta = \gamma E^*(\cos \theta^* + \beta)$ where $\gamma = E_\pi/m_\pi$

- Same as part (a), here we assume that the neutrinos are massless. So, in the pion rest frame, $p_\nu^* = p^*(1, \sin \theta^*, 0, \cos \theta^*)$. Lorentz-transforming this vector to the lab frame, which is a boost in the z-direction with velocity $-\beta$, we have:

$$\begin{aligned}
p_\nu &= \Lambda p_\nu^* \\
&= p^* \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 \\ \sin \theta^* \\ 0 \\ \cos \theta^* \end{pmatrix} \\
&= \begin{pmatrix} \gamma p^* + \gamma\beta \sin \theta^* p^* \\ \cos \theta^* p^* \\ 0 \\ \gamma\beta p^* + \gamma \cos \theta^* p^* \end{pmatrix}
\end{aligned}$$

Since p_ν is the momentum of the neutrino in the lab frame, we can also write it as $p_\nu = E(1, \sin \theta, 0, \cos \theta)$. Equating the two expression for the neutrino momentum, we immediate acquire:

$$E = \gamma E^*(1 + \beta \cos \theta^*) \quad (2)$$

$$E \cos \theta = \gamma E^*(\cos \theta^* + \beta). \quad (3)$$

Note that here $p^* \equiv E^*$.

(c) Using the expressions for E^* and θ^* in terms of E and θ , show that $\gamma^2(1 - \beta \cos \theta)(1 + \beta \cos \theta^*) = 1$.

- Dividing Equation 3 by Equation 2, we have:

$$\cos \theta = \frac{\cos \theta^* + \beta}{1 + \beta \cos \theta^*} \quad (4)$$

$$1 - \beta \cos \theta = \frac{1 - \beta^2}{1 + \beta \cos \theta^*}$$

$$1 - \beta \cos \theta = \frac{\gamma^{-2}}{1 + \beta \cos \theta^*}$$

$$(1 - \beta \cos \theta)(1 + \beta \cos \theta^*)\gamma^2 = 1 \quad (5)$$

(d) Show that the maximum value of θ in the laboratory frame is $\theta_{\max} = 1/\gamma$.

- θ is maximum when $\theta^* = \frac{\pi}{2}$; i.e., the neutrino trajectory is perpendicular to the boost direction. Using Equation 4 and $\cos \frac{\pi}{2} = 0$, we see that:

$$\cos \theta_{\max} = \beta = \sqrt{1 - \gamma^{-2}}.$$

Since neutrinos are nearly massless, we can safely assume $\theta_{\max} \approx 0$ and $\gamma \gg 1$. We then Taylor expand both sides of the equation above:

$$\begin{aligned} 1 - \frac{\theta_{\max}^2}{2} &= 1 - \frac{\gamma^{-2}}{2} \\ \theta_{\max} &= \gamma^{-1} \end{aligned}$$

(e) In the limit $\theta \ll 1$ show that $E \approx 0.43E_\pi \frac{1}{1 + \beta\gamma^2\theta^2}$ and therefore on-axis ($\theta = 0$) the neutrino energy spectrum follows that of the pions.

- Starting from Equation 2, and citing results from Equation 1 and 5, we have:

$$E = \frac{E_\pi}{m_\pi} \frac{m_\pi^2 - m_\mu^2}{2m_\pi} \frac{1}{(1 - \beta \cos \theta)\gamma^2}. \quad (6)$$

In the limit $\theta \gg 1$, we know that $\cos \theta \approx 1 - \frac{\theta^2}{2}$ and that $\beta \approx 1$. So,

$$\begin{aligned} (1 - \beta \cos \theta)\gamma^2 &\approx \gamma^2(1 - \beta) + \frac{\gamma^2\theta^2\beta}{2} \\ &= \frac{1}{1 + \beta} + \frac{\gamma^2\theta^2\beta}{2} \\ &\approx \frac{1}{2} + \frac{\gamma^2\theta^2\beta}{2} \end{aligned}$$

So, Equation 6 becomes:

$$E = E_\pi \frac{m_\pi^2 - m_\mu^2}{m_\pi^2} \frac{1}{1 + \gamma^2 \theta^2 \beta}$$

$$E \approx 0.43 E_\pi \frac{1}{1 + \gamma^2 \theta^2 \beta} \quad (7)$$

(f) Assuming that the pions have a flat spectrum in the range 1-5 GeV, sketch the form of the resulting neutrino energy spectrum at the T2K far detector (Super-Kamiokande), which is off-axis at $\theta = 2.5^\circ$. Given that the Super-Kamiokande detector is 295 km from the beam, explain why this angle was chosen.

- By Equation 7, the neutrino energies at $\theta = 0^\circ$ and $\theta = 2.5^\circ$ for E_π between 1-5 GeV are: So, the neutrino energy spectrum at $\theta = 2.5^\circ$ is narrowly peaked, as

E_π	E_ν at $\theta = 0^\circ$	E_ν at $\theta = 2.5^\circ$
1.0 GeV	0.43 GeV	0.39 GeV
1.5 GeV	0.65 GeV	0.53 GeV
2.0 GeV	0.86 GeV	0.62 GeV
2.5 GeV	1.08 GeV	0.67 GeV
3.0 GeV	1.29 GeV	0.68 GeV
3.5 GeV	1.50 GeV	0.68 GeV
4.0 GeV	1.72 GeV	0.67 GeV
4.5 GeV	1.93 GeV	0.65 GeV
5.0 GeV	2.15 GeV	0.62 GeV

opposed to the on-axis neutrinos which have a flat energy spectrum. This narrow peak maximizes the effect of neutrino oscillation at the T2K detector.

Problem 2: Nuclear Stability [30 points]

The Weizäcker formula or semi-empirical mass formula is a parametrization of nuclear mass as a function of A and Z . Following this formula, the mass of an atom with Z protons and N neutrons is given by the following:

$$M(A, Z) = NM_n + ZM_p + Zm_e - a_V A + a_s A^{2/3} + a_c \frac{Z^2}{A^{1/3}} + a_a \frac{(N-Z)^2}{4A} + \frac{\delta}{A^{1/2}}$$

with $N = A - Z$.

The exact values of the parameters a_V, a_s, a_c, a_a , and δ depend on the range of masses for which they are optimized. One possible set of parameters is given by the following:

$a_V = 15.67 \text{ MeV}/c^2$, $a_s = 17.23 \text{ MeV}/c^2$, $a_c = 0.714 \text{ MeV}/c^2$, $a_a = 93.15 \text{ MeV}/c^2$ and $\delta = -11.2, 0, +11.2 \text{ MeV}/c^2$ for even Z and Z , odd A , or odd Z and N , respectively.

For fixed A find the proton number Z for the most stable nucleus, and plot Z as a function of A . Each term captures an aspect of the atom. Explain briefly how the individual terms can be interpreted.

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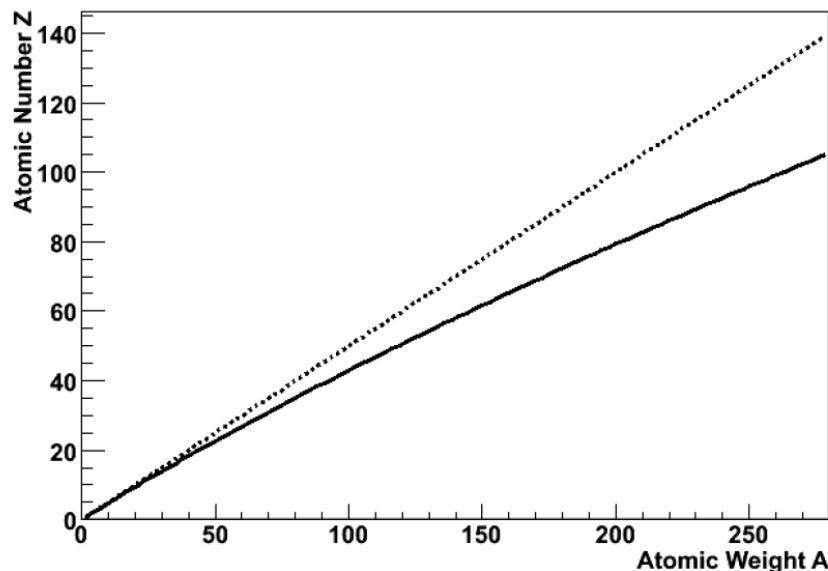
The most stable nucleus will have the highest binding energy, which is:

$$\begin{aligned} E_B(A, Z) &= NM_n + ZM_p + Zm_e - M(A, Z) \\ &= a_V A - a_s A^{2/3} - a_c \frac{Z^2}{A^{1/3}} - a_a \frac{(N - Z)^2}{4A} - \frac{\delta}{A^{1/2}} \\ &= \alpha + \beta Z + \gamma Z^2. \end{aligned}$$

We take the derivative of E_B to find the maximum:

$$\begin{aligned} \frac{dE_B(A, Z)}{dZ} &= \beta + 2\gamma Z = 0 \\ \Rightarrow Z_{\text{stable}} &= -\frac{\beta}{2\gamma} = \frac{a_a A}{2(a_c A^{2/3} + a_a)}. \end{aligned}$$

Note that if we assume the only type of decays here are conversion between nuclides with fixed A , then we should take derivative w.r.t. $M(A, Z)$ instead of the binding energy.



The dashed line is $A = 2Z$.

Interpretation of the terms: (also see pp. 20 of *Particles and Nuclei: An Introduction to the Physical Concepts* by Povh, Rith, Scholz and Zetsche, 5th Ed.)

a_V : the volume term. This term, which dominates the binding energy, is proportional to the number of nucleons. Each nucleon in the interior of a (large) nucleus contributes an energy of about 16 MeV.

a_s : the surface term. For nucleons at the surface of the nucleus, which are surrounded by fewer nucleons, the above binding energy is reduced. This contribution is proportional to the surface area of the nucleus.

a_c : the Coulomb term. The electrical repulsive force acting between the protons in the nucleus further reduces the binding energy.

a_a : the asymmetry term. As long as mass numbers are small, nuclei tend to have the same number of protons and neutrons. Heavier nuclei accumulate more and more neutrons, to partly compensate for the increasing Coulomb repulsion by increasing the nuclear force. This creates an asymmetry in the number of neutrons and protons. The dependence of the nuclear force on the surplus of neutrons is described by the asymmetry term.

δ : the pairing term. A systematic study of nuclear masses shows that nuclei are more stable when they have an even number of protons and/or neutrons. This observation is interpreted as a coupling of protons and neutrons in pairs. The pairing energy depends on the mass number, as the overlap of the wave functions of these nucleons is smaller, in larger nuclei.

Problem 3: Decay time dating [20 points]

Naturally occurring uranium is a mixture of the ^{238}U (99.28%) and ^{235}U (0.72%) isotopes.

How old must the material of the solar system be if one assumes that at its creation both isotopes were present in equal quantities? The lifetimes are $\tau(^{235}\text{U}) = 1 \times 10^9$ years and $\tau(^{238}\text{U}) = 6.6 \times 10^9$ years.

- The number of exponentially-decaying nuclei as a function of time is: $N(t) = N_0 e^{-t/\tau}$. Let $\tau_1 = 6.6 \times 10^9$ yrs and $\tau_2 = 1 \times 10^9$ yrs be the lifetime of ^{238}U and ^{235}U respectively, and let α be the ratio of abundance between ^{238}U and ^{235}U . Then,

$$\begin{aligned} \alpha &= \frac{N_0 e^{-t/\tau_1}}{N_0 e^{-t/\tau_2}} \\ \alpha &= e^{(\frac{1}{\tau_2} - \frac{1}{\tau_1})t} \\ t &= \frac{\ln \alpha}{\frac{1}{\tau_2} - \frac{1}{\tau_1}} \\ t &= \frac{\ln \frac{99.28}{0.72}}{\frac{1}{1 \times 10^9 \text{ yrs}} - \frac{1}{6.6 \times 10^9 \text{ yrs}}} \\ t &= 5.8 \times 10^9 \text{ yrs} \end{aligned}$$

How much of the ^{238}U has decayed since the formation of the earth's crust 2.5×10^9 years ago?

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$$\begin{aligned} \frac{N}{N_0} &= e^{-t/\tau_1} \\ \frac{N}{N_0} &= e^{-\frac{2.5 \times 10^9 \text{ yrs}}{6.6 \times 10^9 \text{ yrs}}} \\ \frac{N}{N_0} &\approx 0.685. \end{aligned}$$

So, about 31.5% of the ^{238}U has decayed.

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