Massachusetts Institute of Technology

Department of Physics

Course:	8.701 — Introduction to Nuclear and Particle Physics
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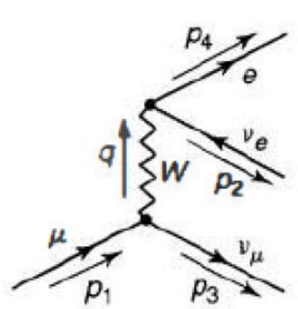
Problem Set 4

handed out October 21st, 2020

Problem 1: Muon decay [20 points]

Consider the decay $\mu \to e\nu_\mu\bar{\nu}_e$.



$$\tau = \frac{192\pi^3\hbar^7}{G_F^2 m_\mu^5 c^4}$$

Describe the necessary steps to calculate the lifetime of the μ as given in the formula above and highlight assumptions you might make in the calculation. [Bonus: you have all tools at hand to carry out the full calculation. Challenge yourself!]

- Assumption 1: The momentum transfer is much smaller than the W-boson mass, so the W-boson propagator:

$$\frac{-i(g_{\mu\nu} - q_\mu q_\nu/m_W^2)}{q^2 - m_W^2} \approx \frac{ig_{\mu\nu}}{m_W^2}$$

Summon the Feynman rules, we can write the matrix element as:

$$\begin{aligned}\mathcal{M} &= \left(\frac{ig_W}{2\sqrt{2}}\right)^2 \left[\bar{u}(p_3)\gamma^\mu(1-\gamma^5)u(p_1)\right]\frac{ig_{\mu\nu}}{m_W^2}\left[\bar{u}(p_4)\gamma^\nu(1-\gamma^5)v(p_2)\right] \\ &= \frac{-ig_W^2}{8m_W^2}\left[\bar{u}(p_3)\gamma^\mu(1-\gamma^5)u(p_1)\right]\left[\bar{u}(p_4)\gamma_\mu(1-\gamma^5)v(p_2)\right].\end{aligned}$$

So,

$$\begin{aligned}|\mathcal{M}|^2 &= \mathcal{M}\mathcal{M}^\dagger \\ &= \left(\frac{g_W^2}{8m_W^2}\right)^2 \left[\bar{u}(p_3)\gamma^\mu(1-\gamma^5)u(p_1)\right]\left[\bar{u}(p_1)\gamma^\nu(1-\gamma^5)u(p_3)\right] \\ &\qquad \left[\bar{u}(p_4)\gamma_\mu(1-\gamma^5)v(p_2)\right]\left[\bar{v}(p_2)\gamma_\nu(1-\gamma^5)u(p_4)\right].\end{aligned}$$

Sum over spins, and make the second assumption that neutrinos are massless:

$$\begin{aligned}\sum_{spins}|\mathcal{M}|^2 &= \mathrm{Tr}\{\not{p}_3\gamma^\mu(1-\gamma^5)(\not{p}_1+m_\mu)\gamma^\nu(1-\gamma^5) \\ &\quad \mathrm{Tr}\{\not{p}_2\gamma^\mu(1-\gamma^5)(\not{p}_4+m_e)\gamma^\nu(1-\gamma^5)\end{aligned}$$

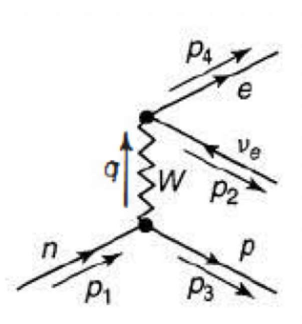
Invoke the results from Problem 9.2 in Griffiths, and average over the initial muon spin, we get:

$$\langle|\mathcal{M}|^2\rangle = 2\left(\frac{g_W}{m_W}\right)^4(p_1\cdot p_2)(p_3\cdot p_4)$$

Next we use Fermi's Golden Rule for decay (Equation 6.21 in Griffiths), and carry out the kinematic integral calculations shown in great details by Griffiths pg. 311-314, we acquire the muon lifetime as given by this problem.

Problem 2: Neutron decay [20 points]

Consider the decay of a neutron $n \to p e \nu_e$.



Compare this decay with the decay of the μ. Highlight the differences between the two processes and compare the expected electron energy spectra.

- Treating the neutron and the proton as elementary particles, the neutron decay process is very similar to the muon decay except that the proton is massive. Details of the neutron lifetime calculation can be found on Griffiths pg. 316-318.

  Now we compare the electron energy spectrum in both cases:

$$\left.\frac{\mathrm{d}\Gamma}{\mathrm{d}E}\right|_{\mu} \propto E^2\left(1-\frac{4E}{3m_\mu}\right) \tag{1}$$

$$\left.\frac{\mathrm{d}\Gamma}{\mathrm{d}E}\right|_{n} \propto E\sqrt{E^2-m_e^2}\left[(m_n-m_p)-E\right]^2 \tag{2}$$

  Both distributions have a cutoff, with that of the muon decay at $m_\mu/2 \approx 52.5\,\mathrm{MeV}$ while that of the neutron decay at $m_n - m_p \approx 1.29\,\mathrm{MeV}$. The electron energy distribution of muon decay also reaches peak near the cutoff energy while that of neutron decay are more on the lower energy side.

Problem 3: CKM Matrix [20 points]

The purpose of this exercise is review properties of unitary matrices. How many independent real parameters are there in a general 3×3 unitary matrix? How about $n \times n$?. [Hint: It helps to know that any unitary matrix (U) can be written in the form $U = e^{iH}$, wher H is a hermitian matrix. So an equivalent question is, how many independent real parameters are there in the general hermitian matrix.] How many independent real parameters are there in a general 3×3 (real) orthogonal matrix? How about $n \times n$?

- For an $n \times n$ Hermitian matrix H, we have $H = H^\dagger = H^{T*}$. So, the n diagonal elements satisfy: $H_{ii} = H_{ii}^*$, so there is 1 independent real parameter for each diagonal element. For the off-diagonal elements, we have $H_{ij} = H_{ji}^*$, which puts 2 constraints on the 4 real parameters describing each pair of them. Therefore, there are n^2 independent real parameters in an $n \times n$ unitary matrix. That is 9 parameters for a 3×3 unitary matrix.

For a real $n \times n$ orthogonal matrix O, we have $O = O^*$ and $OO^T = 1$. So $O(O^*)^T = OO^\dagger = 1$, which means O is unitary and can thus be written as $O = e^{-iA}$, where A is a Hermitian matrix. Also, since O is real, we have $-iA = iA*$, which says that A is purely imaginary. So there are n^2 independent real parameters left. Furthermore, $A^* = -A, A = A^\dagger \Rightarrow A = (A^T)^* = -A^T$, which says A is antisymmetric. So, A has $\frac{n^2-n}{2}$ independent real parameters, and so is O. For a 3×3 O, there are 3 parameters.

Problem 4: Neutrino generations [20 points]

The LEP collider operated initially at $\sqrt{s} = m_Z$ to produce the Z boson at the Z pole. The measurement of the cross section allows the estimate of the number of *active* neutrino generations. Explain how this information can be derived without the detection of Z boson decays to neutrinos.

- The cross section for Z-boson produced at center-of-mass energy $\sqrt{s}$ decay to a fermion pair $f\bar{f}$ is:

$$\sigma_{f\bar{f}}(s) = \frac{12\pi}{M_Z^2} \frac{\Gamma_l \Gamma_{f\bar{f}}}{\Gamma_{tot}^2} \frac{s\Gamma_{tot}^2}{(s - M_Z^2)^2 + \left(\frac{s\Gamma_{tot}}{M_Z}\right)^2}.$$

At the Z pole $s = M_Z^2$, this reduces to:

$$\sigma_{f\bar{f}}^{\text{pole}} = \frac{12\pi}{M_Z^2} \frac{\Gamma_l \Gamma_{f\bar{f}}}{\Gamma_{tot}^2},$$

where $\Gamma_{tot} = \Gamma_{\text{hadrons}} + 3\Gamma_{\text{lepton}} + N_\nu \Gamma_\nu$. Considering only the hadronic cross section, we have:

$$\Gamma_{tot} = \sqrt{\frac{12\pi \Gamma_{\text{lepton}} \Gamma_{\text{hadrons}}}{M_Z^2 \sigma_{\text{hadrons}}^{\text{pole}}}}.$$

Using the Standard Model prediction $\Gamma_l / \Gamma_\nu = 1/2$, we have:

$$\begin{aligned} N_\nu &= \frac{\Gamma_{tot} - \Gamma_{\text{hadrons}}}{2\Gamma_{\text{lepton}}} - \frac{3}{2} \\ &= \frac{1}{2\Gamma_{\text{lepton}}} \left[\sqrt{\frac{12\pi \Gamma_{\text{lepton}} \Gamma_{\text{hadrons}}}{M_Z^2 \sigma_{\text{hadrons}}^{\text{pole}}}} - \Gamma_{\text{hadrons}} \right] - \frac{3}{2} \end{aligned}$$

Problem 5: Deep inelastic scattering [20 points]

The HERA collider at DESY allowed the study of collisions of 27.5 GeV electrons on 820 GeV proton beams. Calculate the kinematic variables Q^2 ,x, and y in terms of the scattered $3^\circ < \theta'_e < 177^\circ$ calculate the kinematic region (x_{min}, x_{max}) and (Q_{min}, Q_{max}) covered by HERA.

- Let E_p and E_e be the energy of the incoming proton and electron Note that at the energy scale of HERA, we can neglect the particle masses and write the momenta as:

$$\begin{aligned} p &= E_P(1,0,0,1) \quad \text{for the incoming proton} \\ k &= E_e(1,0,0,-1) \quad \text{for the incoming electron} \\ k' &= E'_e(1,\sin\theta'_e,0,-\cos\theta'_e) \quad \text{for the scattered electron} \\ q &= k-k' \quad \text{the momentum transfer,} \end{aligned}$$

which gives us:

$$\begin{aligned} Q^2 &= -q^2 = 2k' = 2E_eE'_e(1-\cos\theta'_e) = 4E_eE'_e\sin^2\frac{\theta'_e}{2} \\ y &= \frac{p\cdot q}{p\cdot k} = 1-\frac{p\cdot k'}{p\cdot k} = 1-\frac{E_pE'_e(1+\cos\theta'_e)}{2E_pE_e} = 1-\frac{E'_e}{E_e}\cos^2\frac{\theta'_e}{2} \\ x &= \frac{Q^2}{2p\cdot q} = \frac{Q^2}{2y(p\cdot k)} = \frac{4E_eE'_e\sin^2\frac{\theta'_e}{2}}{2(2E_pE_e)(1-\frac{E'_e}{E_e}\cos^2\frac{\theta'_e}{2})} = \frac{E'_e\sin^2\frac{\theta'_e}{2}}{E_p(1-\frac{E'_e}{E_e}\cos^2\frac{\theta'_e}{2})}. \end{aligned}$$

So, for the scattering angle $3^\circ < \theta'_e < 177^\circ$ we have:

$$\begin{aligned} Q^2_{\text{min}} &= Q^2(\theta'_e=3^\circ) = 4E_eE'_e\sin^2 1.5^\circ = 0.075E'_e\,\text{GeV} \\ x_{\text{min}} &= x(\theta'_e=3^\circ) = \frac{0.00069E'_e}{820(1-0.036E'_e)} \\ Q^2_{\text{max}} &= Q^2(\theta'_e=177^\circ) = 4E_eE'_e\sin^2 88.5^\circ = 109.0E'_e\,\text{GeV} \\ x_{\text{max}} &= x(\theta'_e=177^\circ) = \frac{0.9993E'_e}{820(1-0.000025E'_e)} \end{aligned}$$

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