MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department
Earth, Atmospheric, and Planetary Sciences Department

Astronomy 8.282J-12.402J April 19, 2006

Quiz 2

Name Solutions (please print)
Last **First**

1. Work any **7** of the **9** problems - indicate clearly which 7 you want graded.
2. Spend 10 or 15 minutes reviewing the problems, and select 7.
3. Immediately after this you have two continuous hours to work the problems.
4. All problems are worth 14 points; everyone gets 2 points just for taking the exam.
5. Closed book exam; you may use two pages of notes, and a calculator.
6. Wherever possible, try to solve the problems using general analytic expressions. Plug in numbers only as a last step.
7. If you have any questions, e-mail the Instructor.
8. Turn exam in during lecture, Friday, April 21.

Time Started: Time Stopped:

Signature ______________________________

Problem	Grade	Grader
1		
2		
3		
4		
5		
6		
7		
8		
9		
Total		

APPROXIMATE VALUES OF USEFUL CONSTANTS

Constant	cgs units		mks units	
c (speed of light)	3×10^{10}	cm/sec	3×10^{8}	m/sec
G (gravitation constant)	7×10^{-8}	dyne-cm^2/g^2	7×10^{-11}	N-m^2/kg^2
k (Boltzmann's constant)	1.4×10^{-16}	erg/K	1.4×10^{-23}	J/K
h (Planck's constant)	6.6×10^{-27}	erg-sec	6.6×10^{-34}	J-sec
$m_{\rm proton}$	1.6×10^{-24}	g	1.6×10^{-27}	kg
eV (electron Volt)	1.6×10^{-12}	erg	1.6×10^{-19}	J
$M_\odot$ (solar mass)	2×10^{33}	g	2×10^{30}	kg
$L_\odot$ (solar luminosity)	4×10^{33}	erg/sec	4×10^{26}	J/sec
$R_\odot$ (solar radius)	7×10^{10}	cm	7×10^{8}	m
σ (Stefan-Boltzmann cons)	6×10^{-5}	erg/cm^2-sec-K^4	6×10^{-8}	J/m^2-sec-K^4
Å(Angstrom)	10^{-8}	cm	10^{-10}	m
km (kilometer)	10^{5}	cm	10^{3}	m
pc (parsec)	3×10^{18}	cm	3×10^{16}	m
kpc (kiloparsec)	3×10^{21}	cm	3×10^{19}	m
Mpc (megaparsec)	3×10^{24}	cm	3×10^{22}	m
year	3×10^{7}	sec	3×10^{7}	sec
day	86400	sec	86400	sec
AU	1.5×10^{13}	cm	1.5×10^{11}	m
$1'$ (arc minute)	1/3400	rad	1/3400	rad
$1''$ (arc second)	1/200,000	rad	1/200,000	rad

Problem 1 (Short Answer Questions on Magnitudes)

a. A globular cluster has 10^6 stars each of *apparent* magnitude +8. What is the combined *apparent* magnitude of the entire cluster?

$$+8 = -2.5\log(F/F_0)$$
$$F = 6.3 \times 10^{-4} F_0$$
$$F_{\text{cluster}} = 10^6 \times 6.3 \times 10^{-4} F_0 = 630\ F_0$$
$$m_{\text{cluster}} = -2.5\log(630) = -7$$

b. Find the distance modulus to the Andromeda galaxy (M31). Take the distance to Andromeda to be 750 kpc.

$$\text{DM} = 5\log\left(\frac{d}{10\,\text{pc}}\right) = 5\log(75,000) = 24.4$$

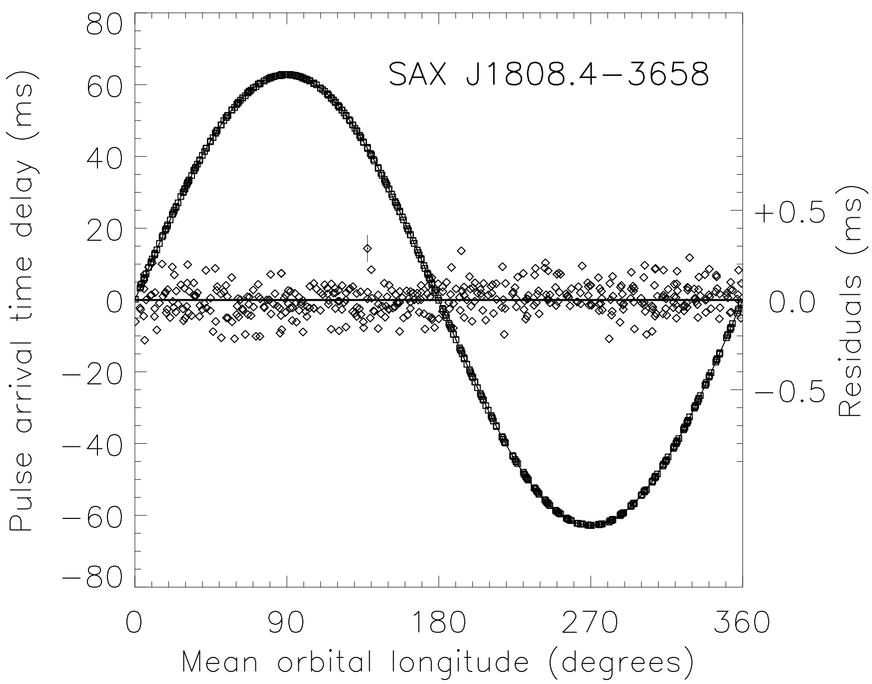
c. An eclipsing binary consists of two stars of different radii and effective temperatures. Star 1 has radius R_1 and T_1, and Star 2 has $R_2 = 0.5R_1$ and $T_2 = 2T_1$. Find the change in bolometric magnitude of the binary, Δm_{bol}, when the smaller star is behind the larger star. (Consider only bolometric magnitudes so you don't have to worry about color differences.)

$$\mathcal{F}_{1\&2} = 4\pi\sigma\left(T_1^4 R_1^2 + T_2^4 R_2^2\right)$$
$$\mathcal{F}_{\text{eclipse}} = 4\pi\sigma T_1^4 R_1^2$$
$$\Delta m = -2.5\log\left(\frac{\mathcal{F}_{1\&2}}{\mathcal{F}_{\text{eclipse}}}\right)$$
$$\Delta m = -2.5\log\left(1 + \frac{T_2^4 R_2^2}{T_1^4 R_1^2}\right)$$
$$\Delta m = -2.5\log\left(1 + \frac{16}{4}\right) = -1.75$$

So, the binary is 1.75 magnitudes brighter out of eclipse than when star 2 is behind star 1.

Problem 2 (Binary System)

The orbit of the first accretion-powered millisecond X-ray pulsar was measured here at M.I.T. using data from the Rossi X-Ray Timing Explorer Satellite. The Doppler delay curve for the motion of the neutron star is given in the figure below. The amplitude of the sine curve indicates the projected light travel time across the orbit of the neutron star around the center of mass of the binary.



(a) Use the figure to estimate the value of $a_{\rm ns}\sin(i)$ for this system, where i is the orbital inclination angle. Express your answer in cm or m.

We can read off $a_{\rm ns}\sin i$ from the graph $= 0.063$ ℓt-sec $= 1.9\times 10^9$cm.

(b) Assume that the neutron star has a mass of $M_{\rm ns} = 1.4\, M_\odot$ and that the orbital inclination is $i = 30°$. The orbital period of the binary is $P_{\rm orb} = 2$ hours. Compute the mass of the companion star, m_c. Hint: in order to solve for m_c you will have to make the approximation that $m_c/M_{\rm ns} \ll 1$ and, equivalently, $M_{\rm ns}/m_c \gg 1$.

$$\frac{4\pi^2}{P^2} = \frac{GM_{\rm tot}}{a^3} = \frac{G(M_{\rm ns}+M_c)}{(a_{\rm ns}+a_c)^3} \simeq \frac{G(M_{\rm ns}+M_c)}{a_{\rm ns}^3(1+a_c/a_{\rm ns})^3} \simeq \frac{G(M_{\rm ns}+M_c)}{a_{\rm ns}^3(1+M_{\rm ns}/M_c)^3}$$

We can now take $M_{\rm ns} \gg M_c$ to find:

$$\frac{4\pi^2}{P^2} \simeq \frac{GM_c^3}{a_{\rm ns}^3 M_{\rm ns}^2} = \frac{GM_c^3\sin^3 i}{(a_{\rm ns}\sin i)^3 M_{\rm ns}^2}$$

Plugging in numbers, we have:

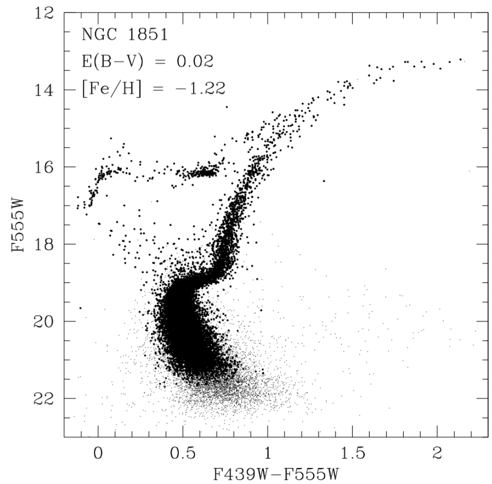
$$\frac{4\pi^2}{7200^2} = \frac{GM_c^3\frac{1}{2^3}}{(1.9\times 10^9)^3(1.4\,M_\odot)^2}$$

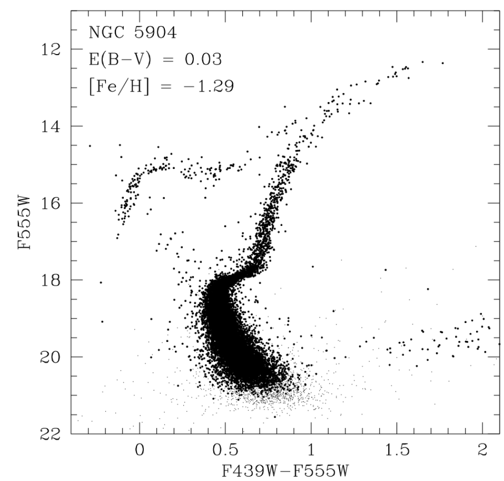
Solve for M_c to find

$$M_c = 0.085\ M_\odot$$

Problem 3 (Hertzsprung-Russell Diagrams)

The figures below show H–R diagrams for two globular clusters, NGC 1851 and NGC 5904. The data were obtained with the Hubble Space Telescope (HST) and reported by Piotto et al. (2002). For purposes of this problem you may consider the HST magnitudes $F439W$ and $F555W$ to be the equivalent of the apparent B and V magnitudes.





a. Use these HR diagrams to find the ratio of the distances to the two clusters.

The best defined feature with which to compare magnitudes between the two HR diagrams is, perhaps, the horizontal branch. I estimate 16th and 15th magnitudes for NGC 1851 and NGC 5904, respectively.

$$\Delta \text{ mag} = 5 \log \frac{d_{1851}}{d_{5904}} \simeq 1$$

Thus, $d_{1851}/d_{5904} \simeq 1.58$.

b. If we take the absolute visual magnitude of a star on the Horizontal Branch to be $V = +0.5$, find the distance to either cluster.

The distance modulus for NGC 1851 is $DM \simeq 16 - 0.5 = 15.5$. Hence the distance is:

$$\text{distance } = 10 \times 10^{15.5/5} \text{ pc } \simeq 12,600 \text{ pc}$$

c. Identify as many phases of stellar evolution as you can on one of the globular cluster diagrams. You may mark directly on the figure. Comment on any physical processes that you know are occurring in the stars at these particular phases.

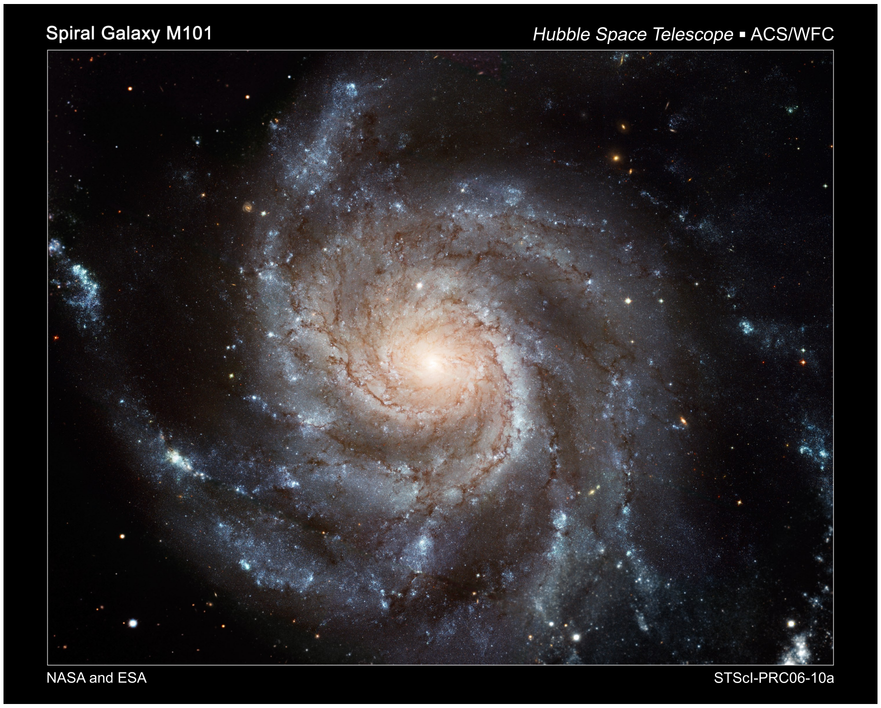
The phases and nuclear burning cycles we were looking for included:

- main sequence; hydrogen core burning
- giant branch; hydrogen shell burning
- horizontal branch; helium core burning

Others that could have been mentioned were asymptotic giant branch, main sequence turnoff, and blue stragglers.

Problem 4 (Galaxy Features)

The image below is of the galaxy M101. Assume that this is a typical spiral galaxy seen nearly face on.



a. Give a short discourse on this image, identifying as many generic features of the galaxy as you can. You may write on the white space provided and draw arrows to the appropriate places on the figure. Indicate approximate dimensions where appropriate. Comment on the stellar populations in various locations and the corresponding metallicities. Roughly where would the Sun be located if this were the Milky Way?

- Dimensions: 30-50 kpc in diameter; perhaps 1 kpc in thickness
- Pop I concentrated in the spiral arms; $Z \simeq 0.02$;
  Pop II concentrated in the bulge; low Z ($\lesssim 0.001$)
- Possible features to point out
  - spiral arms
  - dust lanes
  - galactic bulge
  - star forming regions
  - central black hole
  - globular clusters (hard to discern)
  - dark-matter halo

Problem 5 (Stellar Atmosphere Density Profile)

A star of radius, R, and mass, M, has an atmosphere that obeys a polytropic equation of state:

$$P = K\rho^{5/3} \quad ,$$

where P is the gas pressure, ρ is the gas density (mass per unit volume), and K is a constant throughout the atmosphere. Assume that the atmosphere is sufficiently thin (compared to R) that the gravitational acceleration can be taken to be a constant.

Use the equation of hydrostatic equilibrium to derive the pressure as a function of height z above the surface of the planet. Take the pressure at the surface to be P_0. Sketch your solution $P(z)$.

Start with the equation of hydrostatic equilibrium:

$$\frac{dP}{dz} = -g\rho$$

where g is approximately constant through the atmosphere, and is given by GM/R^2. We can use the polytropic equation of state to eliminate ρ from the equation of hydrostatic equilibrium:

$$\frac{dP}{dz} = -g\left(\frac{P}{K}\right)^{3/5}$$

Separating variables, we find:

$$P^{-3/5}dP = -g\left(\frac{1}{K}\right)^{3/5} dz$$

We then integrate the left-hand side from P_0 to P and the right hand side from 0 to z to find:

$$\frac{5}{2}\left(P^{2/5} - P_0^{2/5}\right) = -gK^{-3/5}z$$

Solving for $P(z)$ we have:

$$P(z) = \left[P_0^{2/5} - \frac{2}{5}gK^{-3/5}z\right]^{5/2} = P_0\left[1 - \frac{2}{5}\frac{g}{P_0^{2/5}K^{3/5}}z\right]^{5/2}$$

The pressure therefore, goes to zero at a finite height $z_{\max}$, where:

$$z_{\max} = \frac{5P_0^{2/5}K^{3/5}}{2g} = \frac{5K\rho_0^{2/3}}{2g} = \frac{5P_0}{2g\rho_0}$$

Problem 6 (Galaxy Rotation Curve)

It has been suggested that our Galaxy has a spherically symmetric dark-matter halo with a density distribution, $\rho_{\rm dark}(r)$, given by:

$$\rho_{\rm dark}(r) = \rho_0 \left(\frac{r_0}{r}\right)^2 \quad ,$$

where ρ_0 and r_0 are constants, and r is the radial distance from the center of the galaxy. For star orbits far out in the halo you can ignore the gravitational contribution of the ordinary matter in the Galaxy.

a. Compute the rotation curve of the Galaxy (at large distances), i.e., find $v(r)$ for circular orbits.

$$-\frac{GM(<r)}{r^2} = -\frac{v^2}{r} \qquad \text{(from } \; F = ma)$$

$$M(<r) = \int_0^r \rho_0 \left(\frac{r_0}{r}\right)^2 4\pi r^2 dr = 4\pi \rho_0 r_0^2 r$$

Note that, in general, $M \neq \rho \times$volume! You must integrate over $\rho(r)$. From these expressions we find:

$$v(r) = \sqrt{4\pi G \rho_0 r_0^2} = \text{constant}$$

b. Find the Oort B coefficient that would be expected at a radial distance r_0.
[Recall: $A = -\frac{1}{2} r_0 \left(\frac{d\omega}{dr}\right)_0$, and $B = A - \omega_0$]

First we find $\omega(r)$:

$$\omega(r) = \frac{v}{r} = \frac{\sqrt{4\pi G \rho_0 r_0^2}}{r}$$

$$\frac{d\omega}{dr} = -\frac{\sqrt{4\pi G \rho_0 r_0^2}}{r^2}$$

$$\left(\frac{d\omega}{dr}\right)_0 = -\frac{\sqrt{4\pi G \rho_0}}{r_0}$$

$$A = -\frac{1}{2} r_0 \left(\frac{d\omega}{dr}\right)_0 = \sqrt{\pi G \rho_0}$$

From above,

$$\omega_0 = \sqrt{4\pi G \rho_0}$$

Thus,

$$B = A - \omega_0 = \sqrt{\pi G \rho_0} - \sqrt{4\pi G \rho_0} = -\sqrt{\pi G \rho_0}$$

Problem 7 (Dimensional Analysis)

The objective of this problem is to find a scaling law for how stellar luminosity depends on the mass of a (main-sequence) star. Carry out a dimensional analysis of the equation for the temperature gradient (for radiative diffusion):

$$\frac{dT}{dr} = -\frac{3\kappa\rho L}{64\pi\sigma_{\rm SB} r^2 T^3} \quad ,$$

where L, ρ, and T are the luminosity, density, and temperature at radial distance r, and κ is the radiative opacity (units: cross sectional area per unit mass), and $\sigma_{\rm SB}$ is the Stefan-Boltzmann constant. Take the radiative opacity, κ to be:

$$\kappa(\rho, T) = \kappa_0 \frac{\rho}{T^{7/2}} \quad ,$$

where κ_0 is a constant.

Find an expression for L as a function of M and constants, only.

Helpful relation: A typical stellar interior temperature is given, also by dimensional analysis, to be $T \simeq GM\mu/(kR)$, where μ is the mean mass per gas particle.

A dimensional analysis of the above equation first yields:

$$\frac{T}{R} \sim c_1 \frac{\kappa\rho L}{R^2 T^3}$$

where c_1 is a constant constructed from the constants in the original dT/dr equation. By using the expression given for κ, we find:

$$\frac{T}{R} \sim c_2 \frac{\rho^2 L}{R^2 T^{13/2}}$$

where $c_2 = c_1\kappa_0$. The expression for L is then:

$$L \sim c_2^{-1} R T^{15/2} \rho^{-2} \sim 16 c_2^{-1} R T^{15/2} M^{-2} R^6$$

where we have approximated $\rho \sim M/(4R^3)$. Next, we make use of the given relation among $T, M,$ and R from the 'helpful relation'.

$$L \sim c_3 M^{11/2} R^{-1/2}$$

where $c_3 = 16 c_2^{-1} (G\mu/k)^{15/2}$. This is as far as one can go with the information given in the problem. If you happened to recall that $R \propto M$ for stars on the main sequence, then you could reach the following proportionality:

$$L \sim M^5$$

Problem 8 (Helium Burning Star)

A pure He star with mass equal to $1\,M_\odot$ has a radius of $0.2\,R_\odot$, a central density, $\rho_c = 10^4$ g cm^{-3} (or 10^7 kg m^{-3}), and a central temperature of $T_c = 1.32 \times 10^8$ K. It is in thermal equilibrium, which means that the nuclear luminosity generated in the interior equals the luminosity radiated from the surface. Consider all the He burning to take place in the central 10%, by mass, of the star, and no burning to occur outside this region. Further, assume that the density and temperature throughout the nuclear burning region are constant at ρ_c and T_c. The energy released per gram per second from He burning is given by the expression:

$$\mathcal{E}(\rho, T) \simeq 4 \times 10^{11} \rho^2 T_8^{-3} \exp[-42.9/T_8] \ \text{ ergs gm}^{-1} \text{ sec}^{-1} \quad .$$

For all mks units (including ρ) simply take the leading coefficient to be 40 rather than 4×10^{11}.

a. Find the luminosity of this He star.

$$L = \int \mathcal{E}(\rho, T) \rho 4\pi r^2 dr$$

However, there is no need to do an integral since you are told that T and ρ are constant through the burning region. Thus,

$$L = \mathcal{E}(\rho, T) \int \rho 4\pi r^2 dr \simeq \mathcal{E}(\rho, T) \times 0.1 \ M_\odot$$

Utilizing the expression for $\mathcal{E}$ given in the problem, we have:

$$L \simeq 4 \times 10^{11} \rho^2 T_8^{-3} \exp[-42.9/T_8] \times 0.1 \ M_\odot \ \text{ ergs sec}^{-1} \quad .$$

Or,

$$L \simeq 4 \times 10^{11} (10^4)^2 (1.32)^{-3} \exp[-42.9/1.32] \times 0.1 \ M_\odot \simeq 2.7 \times 10^{37} \ \text{ ergs sec}^{-1} \quad .$$

Note that the definition of T_8 was unfortunately *not* given in the problem; it actually means $T/10^8\,K$. No points were taken off if you used the full temperature instead of T_8.

b. Find the effective (i.e., surface) temperature of the He star.

Use the relation $L = 4\pi\sigma R^2 T^4$ to compute T_{eff}:

$$T = \left(\frac{L}{4\pi\sigma R^2} \right)^{1/4} \simeq 117{,}000 \text{ K}$$

c. The reaction that powers this star is $3\,{}_2He^4 \longrightarrow {}_6C^{12} + \gamma$. The atomic mass excess of a ${}_2He^4$ atom is 2.42 MeV and that of a ${}_6C^{12}$ atom is 0 MeV. Find the time for all the helium in the core of the star (i.e., the central 10% by mass) to be burned to carbon.

For each C atom that is formed, 3×2.42 MeV are released. This corresponds to 5.8×10^{17} ergs per gram of 'burned' material. The mass to be converted from He to C is $0.1\ M_\odot$. Thus, a total of ($0.1 \times 2 \times 10^{33}$ grams $\times\ 5.8 \times 10^{17}$ ergs per gram) ergs $= 1.15 \times 10^{50}$ ergs of energy will be released over the He core burning phase. At a release rate of $L = 2.7 \times 10^{37}$ ergs sec^{-1}, this process will last for 4.3×10^{12} seconds, or 1.3×10^5 years.

Problem 9 (Short Answer Questions)

a. Suppose air molecules have a collision cross section of 10^{-16} cm^2. If the (number) density of air molecules is 10^{19} cm^{-3}, what is the collision mean free path?

$$\ell = \frac{1}{n\sigma} = \frac{1}{10^{19}10^{-16}} = 10^{-3} \text{ cm}$$

b. What is the physical mechanism by which 21-cm radiation is produced?

Hyperfine transition in neutral hydrogen atoms in the $n = 1$ state. The interaction is between the intrinsic magnetic moment of the proton and that of the electron.

c. What fraction of the rest mass energy is released (in the form of radiation) when a mass ΔM is dropped from infinity onto the surface of a neutron star with $M = 1\, M_\odot$ and $R = 10$ km?

$$\Delta E = \frac{GM\Delta m}{R}$$

The fractional rest energy lost is $\Delta E/\Delta mc^2$, or

$$\frac{\Delta E}{\Delta mc^2} = \frac{GM}{Rc^2} \simeq 0.15$$

d. What is the slope of a $\log N(> F)$ vs. $\log F$ curve for a homogeneous distribution of objects, each of luminosity, L, where F is the flux at the observer, and N is the number of objects observed per square degree on the sky?

The number of objects detected goes as the cube of the distance for objects with flux greater than a certain minimum flux. At the same time the flux falls off with the inverse square of the distance. Thus, the slope of the $\log N(> F)$ vs. $\log F$ curve is -3/2.