MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Physics Department
Earth, Atmospheric, and Planetary Sciences Department

Astronomy 8.282J-12.402J April 13, 2005

Quiz 2

Name Solutions (please print)
Last **First**

1. Work any **7** of the **10** problems - indicate clearly which 7 you want graded.
2. Spend 10 or 15 minutes reviewing the problems, and select 7.
3. Immediately after this you have two continuous hours to work the problems.
4. All problems are worth 14 points; everyone gets 2 points just for taking the exam.
5. Closed book exam; you may use two pages of notes, and a calculator.
6. Wherever possible, try to solve the problems using general analytic expressions. Plug in numbers only as a last step.
7. If you have any questions, e-mail the Instructor.
8. Turn exam in during lecture, Friday, April 15.

Time Started: Time Stopped:

Signature ______________________________

Problem	Grade	Grader
1		
2		
3		
4		
5		
6		
7		
8		
9		
10		
Total		

APPROXIMATE VALUES OF USEFUL CONSTANTS

Constant	cgs units		mks units	
c (speed of light)	3×10^{10}	cm/sec	3×10^{8}	m/sec
G (gravitation constant)	7×10^{-8}	dyne-cm^2/g^2	7×10^{-11}	N-m^2/kg^2
k (Boltzmann's constant)	1.4×10^{-16}	erg/K	1.4×10^{-23}	J/K
h (Planck's constant)	6.6×10^{-27}	erg-sec	6.6×10^{-34}	J-sec
$m_{\rm proton}$	1.6×10^{-24}	g	1.6×10^{-27}	kg
eV (electron Volt)	1.6×10^{-12}	erg	1.6×10^{-19}	J
$M_\odot$ (solar mass)	2×10^{33}	g	2×10^{30}	kg
$L_\odot$ (solar luminosity)	4×10^{33}	erg/sec	4×10^{26}	J/sec
$R_\odot$ (solar radius)	7×10^{10}	cm	7×10^{8}	m
σ (Stefan-Boltzmann cons)	6×10^{-5}	erg/cm^2-sec-K^4	6×10^{-8}	J/m^2-sec-K^4
Å(Angstrom)	10^{-8}	cm	10^{-10}	m
km (kilometer)	10^{5}	cm	10^{3}	m
pc (parsec)	3×10^{18}	cm	3×10^{16}	m
kpc (kiloparsec)	3×10^{21}	cm	3×10^{19}	m
Mpc (megaparsec)	3×10^{24}	cm	3×10^{22}	m
year	3×10^{7}	sec	3×10^{7}	sec
day	86400	sec	86400	sec
AU	1.5×10^{13}	cm	1.5×10^{11}	m
$1'$ (arc minute)	$1/3400$	rad	$1/3400$	rad
$1''$ (arc second)	$1/200,000$	rad	$1/200,000$	rad

Problem 1

A star in the Andromeda galaxy yields a *bolometric* flux at the Earth of $F = 1.0 \times 10^{-13}$ ergs cm^{-2} sec^{-1} (1.0×10^{-16} Watts m^{-2}). It has a B-V color index of -0.24. Take the distance to Andromeda to be 1 Mpc. Make use of the table below, where appropriate, to answer the following questions (interpolate between entries using any interpolation scheme that is reasonable). The reference flux for a *bolometric* magnitude of 0.0 is $F_0 = 2.5 \times 10^{-5}$ ergs cm^{-2} sec^{-1} (2.5×10^{-8} Watts m^{-2}).

a. Find the bolometric magnitude, $M_{\rm bol}$ of the star.

$$M_{\rm bol} = -2.5 \log\left(\frac{1 \times 10^{-13}}{2.5 \times 10^{-5}}\right) \simeq 21$$

b. What is the approximate effective temperature, $T_{\rm effective}$, of the star.

$T_e \simeq 17250\ K$, based on a linear interpolation from the table; or $\sim$ 16850 K, based on a logarithmic interpolation. Let's adopt $T_e \simeq 17,000$ K.

c. Calculate the approximate radius of the star.

$$F = \frac{L}{4\pi d^2} = \frac{4\pi\sigma R^2 T^4}{4\pi d^2} \simeq 1 \times 10^{-13}$$

$$R = \sqrt{\frac{Fd^2}{\sigma T^4}} \simeq 6.4\ R_\odot$$

Table 1: Abbreviated Table of Main-Sequence Star Properties

Spectral Type	B-V (color)	Mass ($M_\odot$)	$T_{\rm effective}$ (°K)
O5	-0.45	40	35,000
B0	-0.31	17	21,000
B5	-0.17	7	13,500
A0	0.00	3.5	9,700
A5	+0.16	2.1	8,100
F0	+0.30	1.8	7,200
F5	+0.45	1.4	6,500

Problem 2

The binary system called "Cygnus X-1" consists of a black hole (Star 1; for this problem considered to be a point mass) orbiting a normal star (Star 2) with a period $P_{\rm orb} = 5.6$ days. Optical astronomers measure the orbital motion of the normal star via Doppler shifts and determine a "projected" velocity of $v_2 \sin i = 75$ km sec^{-1}. This can be combined with the orbital period to determine the mass function:

$$f(M) = 0.25\ M_\odot \quad .$$

The spectral type of the normal star is found to be B0. Other studies have led to a determination of the orbital inclination angle which turns out to be $i = 30°$.

Use the information above to determine the mass of the black hole. A table of stellar properties is given in Problem 1; use this to help you determine the mass of Star 2. If you have done the problem correctly you will end up with the equivalent of a cubic equation in the mass of the black hole, $M_{\rm BH}$. You can either solve this with Mathematica or some other similar computer program, or simply plug in a handful of trial solutions and "zero in" on an approximate answer (10% is good enough).

$$f(M) = \frac{M_B \sin i^3}{\left(1 + \frac{M_c}{M_B}\right)^2} \simeq 0.25\ M_\odot$$

$$\frac{M_B}{\left(1 + \frac{M_c}{M_B}\right)^2} \simeq 2\ M_\odot$$

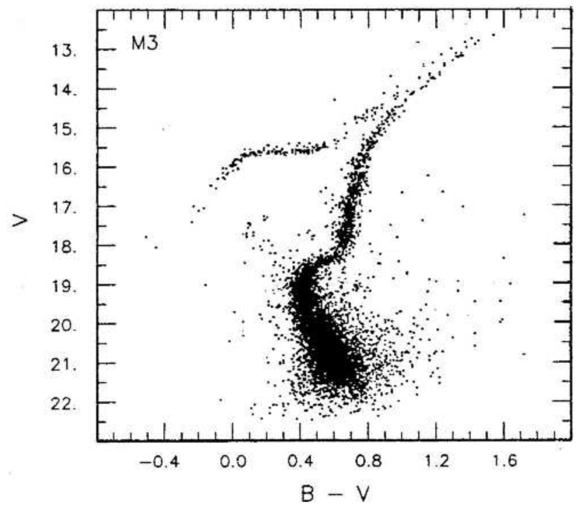
From the table, the companion-star mass is $M_c \simeq 17\ M_\odot$. So,

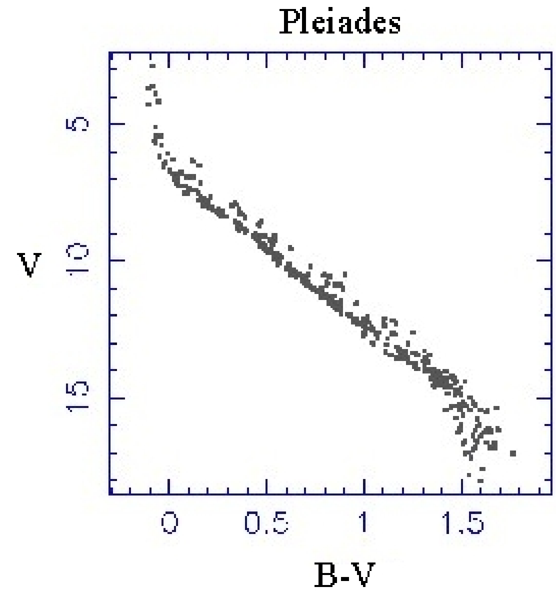
$$\frac{M_B}{\left(1 + \frac{17\ M_\odot}{M_B}\right)^2} \simeq 2\ M_\odot$$

After several trial-and-error guesses for M_B, we find that the left hand side of the equation equals $2\ M_\odot$, only when $M_B \simeq 11.8\ M_\odot$.

Problem 3

Following are HR diagrams for the globular cluster, M3, and the open cluster - the Pleiades.





a. Use these HR diagrams to find the ratio of the distances between M3 and the Pleiades. At B-V = 0.5, for example, the main-sequence magnitudes from the two diagrams are:

$$V_{\rm M3} \simeq 20, \quad \text{and} \quad V_{\rm Pl} \simeq 9.5$$

But, we can write:

$$V = -2.5 \log \left(\frac{L}{4\pi d^2 F_{\rm ref}} \right)$$

We assume that stars on the main sequence with B-V = 0.5 all have the same L. Thus,

$$V_{\rm M3} - V_{\rm Pl} = 2.5 \log \left(\frac{d_{\rm M3}^2}{d_{\rm Pl}^2} \right) = 5 \log \left(\frac{d_{\rm M3}}{d_{\rm Pl}} \right) \simeq 10.5$$

Thus, we find $d_{\rm M3} = 125 \times d_{\rm Pl}$.

b. If we take the absolute visual magnitude of a star with color B-V=0 to be $M_V = 0$, find the actual distance to the Pleiades.

$V = M_V +$ distance modulus, and I estimate $V \simeq 6.5$ for B-V = 0.

Thus, the distance modulus is ~ 6.5, which implies a distance of ~ 200 pc.

c. Explain why the two HR diagrams look so different.

The Pleiades is sufficiently young that most of the main-sequence is still in existence. By contrast, the track of stars running nearly vertically upward in the HR diagram for M3 indicates that all but the least massive stars (i.e., with $M \lesssim 1 M_\odot$) have evolved away.

d. How much intrinsically more luminous are stars at the top of the HR diagram for M3 than those at the bottom?

$$V_{\rm min} \simeq 22, \quad \text{and} \quad V_{\rm max} \simeq 13; \quad \Delta V \simeq 9$$

$$V_{\rm min} - V_{\rm max} = -2.5 \log \left(\frac{L_{\rm min}}{L_{\rm max}} \right) \quad \Rightarrow \quad \frac{L_{\rm max}}{L_{\rm min}} \simeq 4000$$

Problem 4

The image below is of the galaxy NGC 4565. Assume that this is a typical spiral galaxy seen nearly edge on.

Credit: Russell Croman

a. Identify as many generic features of this galaxy as you can. You may write on the white space provided and draw arrows to the appropriate places on the figure.

bulge
disk
dust lanes
no globular clusters visible
halo could be identified, though not visible

b. Indicate a likely size scale beside the drawing. How thick would you guess this galaxy is half way out from the center?

$\sim$ 40 kpc in diameter
$\sim$ 1 kpc in thickness

c. Indicate approximately where the Sun would be located if this were the Milky Way.

About half way out in the disk, near the midplane

Problem 5

The disk of a galaxy can be modeled as a uniform slab of material of mass density, ρ, that is of (full) thickness, $2H$, in the $\hat{z}$ direction, and is effectively infinite in the $\hat{x}$ and $\hat{y}$ directions. Assume that the mass density for $z > H$ is zero.

a. Compute the effective gravity, $\vec{g}$, at an arbitrary distance, z, inside and above the disk. Sketch $\vec{g}(z)$ for all z (i.e., for $+$ and $-$ values of z). Hint: make use of Gauss' law for gravity $\int \vec{g} \cdot d\vec{A} = -4\pi GM$.

$$\int \vec{g} \cdot d\vec{A} = -4\pi GM$$

For a cylindrical "pillbox" of end area, A:

$$-g(z)2A = -4\pi G\rho 2Az \quad \text{(inside the disk)}$$

$$-g(z)2A = -4\pi G\rho 2AH \quad \text{(outside the disk)}$$

or

$$g(z) = 4\pi G\rho z \quad \text{(inside)}$$
$$g(z) = 4\pi G\rho H \quad \text{(outside)}$$

pointing toward the midplane

b. Find the speed, v_z that a star must have, starting at the middle of the disk, to get above height H, i.e., just outside of the mass distribution. Express your answer in terms of ρ, G, and H.

$$\phi = \int_0^z g(z)dz = 2\pi G\rho z^2 \quad \text{(inside)}$$

In order for a star to reach $z = H$ from the midplane, its kinetic energy must exceed the potential energy which is equal to the result found in part (a) evaluated at $z = H$, and multiplied by the mass of the star:

$$\frac{1}{2}mv^2 \gtrsim 2\pi G\rho H^2 m$$

or,

$$v^2 \gtrsim 4\pi G\rho H^2$$

Problem 6

A galaxy is found to have a rotation curve, $v(r)$, given by

$$v(r) = \frac{\left(\frac{r}{r_0}\right)}{\left(1+\frac{r}{r_0}\right)^{3/2}}\ v_0 \quad ,$$

where r is the radial distance from the center of the galaxy, r_0 is a constant with the dimension of length, and v_0 is another constant with the dimension of speed. The rotation curve is defined as the orbital speed of test stars in circular orbit at radius r.

a. Find an expression for $\omega(r)$, where ω is the angular velocity. What Oort A coefficient would an astronomer living in this Galaxy at $r = r_0$ measure? [Recall: $A = -\frac{1}{2} r_0 \left(\frac{d\omega}{dr}\right)_0$]

$$\omega = v/r \quad \Rightarrow \quad \omega(r) = \frac{v_0}{r_0} \frac{1}{1 + r/r_0)^{3/2}}$$

$$A = \frac{3}{4} \frac{v_0}{r_0} \frac{1}{(1 + r/r_0)^{5/2}}$$

which, when evaluated at $r = r_0$ (from the definition of A), yields:

$$A = \frac{3}{2^{9/2}} \frac{v_0}{r_0}$$

b. Find an expression for the mass, $M(< r)$, contained in this galaxy inside of radius r. Assume a spherically symmetric mass distribution.

From F = ma, we have:

$$\frac{GM(< r)}{r^2} = \frac{v^2}{r} \quad \Rightarrow \quad M(< r) = \frac{v^2 r}{G}$$

$$M(< r) = \frac{r^3 v_0^2}{r_0^2 (1 + r/r_0)^3 G}$$

Problem 7

Short answer questions:

a. The density of stars in a particular globular star cluster is 10^6 pc^{-3}. Take the stars to have the same radius as the Sun, and to have an average speed of 10 km sec^{-1}. Find the mean free path for collisions among stars. Find the corresponding mean time between collisions. (Assume that the stars move in straight-line paths, i.e., are not deflected by gravitational interactions.)

$$\ell \simeq \frac{1}{n\sigma} = \frac{1}{10^6 \mathrm{pc}^{-3} \pi R^2}$$

$$\ell \simeq \frac{1}{3 \times 10^{-50} \mathrm{cm}^{-3} \times 1.5 \times 10^{22} \mathrm{cm}^2} \simeq 2 \times 10^{27} \text{ cm}$$

$$\tau_{\mathrm{coll}} \simeq \frac{2 \times 10^{27} \text{ cm}}{10^6 \text{ cm/sec}} \simeq 2 \times 10^{21} \text{ sec} \simeq 6 \times 10^{13} \text{ years}$$

b. A white dwarf star composed entirely of carbon ($_6C^{12}$) reaches a mass of 1.4 $M_\odot$ and all the carbon burns rapidly to magnesium ($_{12}Mg^{24}$). Compute the energy released in this reaction (you should consult the table of Atomic Mass Excess in Problem 10). Compare the nuclear energy released with the gravitational binding energy, U, of the white dwarf. For U you can use $U \simeq GM^2/R$, and choose some reasonable value for R. Is there sufficient nuclear energy to disrupt the white dwarf, i.e., to blow it apart?

Atomic mass excess of $_6C^{12} \equiv 0$
Atomic mass excess of $_{12}Mg^{24} = -13.93$ MeV
Each reaction therefore liberates 13.93 MeV
The number of Mg nuclei that can be made is: $(1.4\ M_\odot/24 \text{ amu}) \simeq 7 \times 10^{55}$
The total nuclear energy released $= 10^{57}$ MeV $= 1.5 \times 10^{51}$ ergs
The gravitational binding energy, U:

$$U \simeq \frac{GM^2}{R} \simeq \frac{6.7 \times 10^{-8} \times 7.8 \times 10^{66}}{5 \times 10^8} \simeq 1 \times 10^{51} \text{ergs}$$

Thus, the two energies are rather comparable, with probably sufficient nuclear energy to blow the star apart.

Problem 8

The equation of state for cold (non-relativistic) matter may be approximated as:

$$P = a\rho^{5/3} - b\rho^{4/3}$$

where P is the pressure, ρ the density, and a and b are fixed constants. Use a dimensional analysis of the equation of hydrostatic equilibrium to estimate the "radius-mass" relation for planets and low-mass white dwarfs whose material follows this equation of state. Specifically, find $R(M)$ in terms of G and the constants a and b. You should set all constants of order unity (e.g., 4, π, 3, etc.) to 1.0. [Hint: solve for $R(M)$ rather than $M(R)$]. You can check your answer by showing that for higher masses, $R \propto M^{-1/3}$, while for the lower-masses $R \propto M^{+1/3}$.

$$\frac{dP}{dr} = -g\rho$$

$$\frac{a\rho^{5/3} - b\rho^{4/3}}{R} \sim \left(\frac{GM}{R^2}\right)\left(\frac{M}{R^3}\right)$$

$$\frac{aM^{5/3}}{R^6} - \frac{bM^{4/3}}{R^5} \sim \left(\frac{GM^2}{R^5}\right)$$

$$GM^2 \sim \frac{aM^{5/3}}{R} - bM^{4/3}$$

$$R \sim \frac{aM^{5/3}}{GM^2 + bM^{4/3}} \simeq \frac{aM^{1/3}}{GM^{2/3} + b}$$

For small masses, $R \propto M^{1/3}$ as for rocky planets, while for larger masses, $R \propto M^{-1/3}$ as for white dwarfs where the degenerate electrons are not yet relativistic.

Problem 9

Once a star like the Sun starts to ascend the giant branch its luminosity, to a good approximation, is given by:

$$L = \frac{10^5 L_\odot}{M_\odot^6} M_{\rm core}^6 \quad ,$$

where the symbol $\odot$ stands for the solar value, and $M_{\rm core}$ is the mass of the He core of the star. Further, assume that as more hydrogen is burned to helium – and becomes added to the core – the conversion efficiency between rest mass and energy is:

$$\Delta E = 0.007 \ \Delta M_{\rm core} c^2 \quad .$$

a. Use these two expressions to write down a differential equation, in time, for $M_{\rm core}$.

$$L \equiv \frac{\Delta E}{\Delta t} = \frac{0.007 \Delta M_{\rm core} c^2}{\Delta t} = \frac{10^5 L_\odot}{M_\odot^6} M_{\rm core}^6$$

b. Solve the differential equation for the core mass, $M_{\rm core}(t)$, as a function of time. To make the problem easier, do not evaluate either $L_\odot$ or $M_\odot$ until the next step.

$$\frac{dM_{\rm core}}{M_{\rm core}^6} = \frac{10^5 L_\odot}{M_\odot^6} \frac{dt}{0.007 c^2}$$

Integration yields:

$$\frac{1}{M_{\rm core,i}^5} - \frac{1}{M_{\rm core,f}^5} = \frac{5 \times 10^5 L_\odot t}{0.007 M_\odot^6 c^2} \quad ,$$

where the subcripts i and f stand for "initial" and "final" values, respectively.

c. Find the time for the star to ascend the giant branch when its core mass increases from $M_{\rm core} = 0.2 \ M_\odot$ to $M_{\rm core} = 0.5 \ M_\odot$.

$$t = \frac{0.007 M_\odot c^2}{5 \times 10^5 L_\odot} \left[\frac{1}{0.2^5} - \frac{1}{0.5^5} \right]$$

Finally, plug in values for $M_\odot$ and $L_\odot$ to find:

$$t \simeq 6 \times 10^8 \ \ \text{yr}$$

Problem 10

This problem relates to the principal nuclear burning chain that powers the Sun, the $p-p$ chain.

a. Write down the 3 nuclear reactions in the $p-p$ chain.

$$_1H^1 + {}_1H^1 \Rightarrow {}_1H^2 + e^+ + \nu$$
$$_1H^2 + {}_1H^1 \Rightarrow {}_2H^3 + \gamma$$
$$_2H^3 + {}_2H^3 \Rightarrow {}_2H^4 + 2\ {}_1H^1$$

b. Use the table on the *following page* to compute the energy released from either reaction involving $_2He^3$ in part (a) – 3 significant figures are sufficient. (The table gives the atomic mass excesses, expressed in MeV.)

$$13.13 + 7.29 \Rightarrow 14.93 + \epsilon, \quad \text{for 2nd equation}$$
$$\epsilon \simeq 5.49 \ \ \text{MeV}$$

or

$$2 \times 14.93 \Rightarrow 2.42 + 2 \times 7.29 + \epsilon, \quad \text{for 3rd equation}$$
$$\epsilon \simeq 12.9 \ \ \text{MeV}$$

c. Compute how much energy is released, in total, from the conversion of 4 hydrogen nuclei into 1 helium nucleus (you may ignore the electrons). Hint: you may bypass intermediate reactions.

$$4 \times 7.29 \Rightarrow 2.42 + \epsilon$$
$$\epsilon \simeq 26.7 \ \ \text{MeV}$$

Table 4-1 from “Principles of Stellar Evolution and Nucleosynthesis” by Donald Clayton, published by McGraw-Hill.

Table 4-1 Atomic mass excesses†

Z	*Element*	*A*	*M − A, Mev*
0	*n*	1	8.07144
1	H	1	7.28899
	D	2	13.13591
	T	3	14.94995
	H	4	28.22000
		5	31.09000
2	He	3	14.93134
		4	2.42475
		5	11.45400
		6	17.59820
		7	26.03000
		8	32.00000
3	Li	5	11.67900
		6	14.08840
		7	14.90730
		8	20.94620
		9	24.96500
4	Be	6	18.37560
		7	15.76890
		8	4.94420
		9	11.35050
		10	12.60700
		11	20.18100
5	B	7	27.99000
		8	22.92310
		9	12.41860
		10	12.05220
		11	8.66768
		12	13.37020
		13	16.56160
6	C	9	28.99000
		10	15.65800
		11	10.64840
		12	0
		13	3.12460
		14	3.01982
		15	9.87320
7	N	12	17.36400
		13	5.34520
		14	2.86373
		15	0.10040
		16	5.68510
		17	7.87100
8	O	14	8.00800
		15	2.85990
		16	−4.73655
		17	−0.80770
		18	−0.78243
		19	3.33270
		20	3.79900
9	F	16	10.90400
		17	1.95190
		18	0.87240
		19	−1.48600
		20	−0.01190
		21	−0.04600
10	Ne	18	5.31930
		19	1.75200
		20	−7.04150
		21	−5.72990
		22	−8.02490
		23	−5.14830
		24	−5.94900
11	Na	20	8.28000
		21	−2.18500
		22	−5.18220
		23	−9.52830
		24	−8.41840
		25	−9.35600
		26	−7.69000
12	Mg	22	−0.14000
		23	−5.47240
		24	−13.93330
		25	−13.19070
		26	−16.21420
		27	−14.58260
		28	−15.02000
13	Al	24	0.1000
		25	−8.9310
		26	−12.2108
		27	−17.1961
		28	−16.8554
		29	−18.2180
		30	−17.1500
14	Si	26	−7.1320
		27	−12.3860
		28	−21.4899
		29	−21.8936
		30	−24.4394
		31	−22.9620
		32	−24.2000
15	P	28	−7.6600
		29	−16.9450
		30	−20.1970
		31	−24.4376
		32	−24.3027