8.251 – Homework 2

B. Zwiebach   Spring 2007

Due Wednesday, February 21.

1. (10 points) Problem 3.1

2. (10 points) Problem 3.2

3. (10 points) Lorentz invariants and their densities.

   We want to understand how the Lorentz invariance of charge (all Lorentz frames agree on the value of the charge in a system) implies that charge *density* is the zeroth component of a Lorentz vector.

   Consider two Lorentz frames S and S' with S' boosted along the $+x$ direction of S with velocity v. In frame S we have cubic box of volume L^3 with edges, all of length L, aligned with the axes of the coordinate system. The box is at rest and is filled by a material, also at rest, that has uniform charge density ρ_0. In this frame the current density vanishes $\vec{j} = \vec{0}$.

   Use the Lorentz invariance of charge to calculate the charge density and current density $(c\rho', \vec{j}\,')$ that describe the material in the box according to frame S'. Verify that the values $(c\rho_0, \vec{0})$ and $(c\rho', \vec{j}\,')$ behave as four-vectors $j^\mu \equiv (c\rho_0, \vec{0})$ and $j'^\mu \equiv (c\rho', \vec{j}\,')$, in the sense that the quantities $(c\rho', \vec{j}\,')$ are obtained from the rest frame values $(c\rho_0, \vec{0})$ by a Lorentz boost. Write a formula for the current j^μ in terms of ρ_0 and the four-velocity of the box, and check that your result works both for frames S and S'.

4. (10 points) Lorentz transformation of electric fields.

   Consider two Lorentz frames S and S' with S' boosted along the $+x$ direction of S with velocity v. We have a large parallel plate capacitor at rest in S, with uniform but opposite surface charge densities $\pm\sigma_0$ on the plates. The capacitor can be aligned in two ways:

   (i) With the plates along planes of constant y, so that the resulting electric field $E_\perp$ points in the y direction, a direction transverse to the boost.

   (ii) With the plates along planes of constant x, so that the resulting electric field $E_\parallel$ points in the x direction, the direction along the boost.

   Use the Lorentz invariance of the charge to calculate the electric fields seen by the observer S' in both cases (i) and (ii). Thus derive and state the transformation rules that relate the fields $(E_\perp, E_\parallel)$ seen by S and the fields $(E'_\perp, E'_\parallel)$ seen by S'. Use the tensor transformation law $F'^{\mu\nu} = L^\mu{}_\rho L^\nu{}_\sigma F^{\rho\sigma}$ to confirm your result. For this purpose take $E_x = E_\parallel$ and $E_y = E_\perp$ and assume no other field exists in the S frame.

5. (10 points) Problem 3.8

6. (20 points) Problem 3.9.