Massachusetts Institute of Technology

Department of Physics

Course:	8.20 —Special Relativity
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Problem Set 4

handed out January 23rd, 2021

Note: In most problems, I will work with the convention that a 3-vector is denoted by an arrow ($\vec{p}$), while a 4-vector would just be denoted as p (I have dropped the subscript term, p^μ, as this is the convention). In particular physics, we often set the speed of light to the unit-less value of 1 (c = 1). Masses are often referred to in terms of energy (i.e. 511 keV, 938 MeV, etc...)

Problem 1: Acceleration in Special Relativity [25 pts]

In class we determined that the momentum of a particle traveling at velocity $\vec{u}$ with respect to an observer is given by $\vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}}$.

(a) Find the force $\vec{F}$ by taking the derivative with respect to ordinary time.

(b) It is possible to also define a 4-vector for acceleration, just like we did for 4-velocity, by taking again the time derivative with respect to proper time

$$\alpha^\mu = \frac{d\eta^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2}$$

Find the components of α^μ.

(c) Express those components in terms of the force term you found in part (a).

(a) Remembering $\vec{F} = \frac{d\vec{p}}{dt}$, where t is ordinary time, we can carry out this operation with our SR-compatible momentum.

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2/c^2}}$$
$$\vec{F} = \frac{d\vec{p}}{dt} = m\frac{d}{dt}(\frac{\vec{u}}{\sqrt{1-u^2/c^2}})$$
$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{m}{\sqrt{1-u^2/c^2}}\frac{d\vec{u}}{dt} + \frac{m\vec{u}(-\frac{1}{2c^2}(-2\vec{u}\cdot\frac{d\vec{u}}{dt}))}{(1-u^2/c^2)^{3/2}}$$

Let $\vec{a} = \frac{d\vec{u}}{dt}$ denote the (ordinary) acceleration of the particle.

$$\boxed{\vec{F} = \frac{m}{\sqrt{1-u^2/c^2}}(\vec{a} + \frac{\vec{u}(\vec{u}\cdot\vec{a})}{c^2-u^2})}$$

This expression reduces to the familiar $\vec{F} = m\vec{a}$ if $\vec{u} = 0$.

(b) Using the same procedure as we did for the proper velocity, we take the derivative of the proper velocity with respect to proper time.

$$\eta^\mu = \frac{dx^\mu}{d\tau}$$
$$\frac{d\eta^\mu}{d\tau} = \frac{d\tau}{dt}\frac{d\eta^\mu}{dt}$$
$$\frac{d\eta^\mu}{d\tau} = \frac{d\tau}{dt}(\frac{d}{dt}\gamma c, \frac{d(\gamma\vec{u})}{dt})$$
$$\frac{d\eta^\mu}{d\tau} = \frac{d\tau}{dt}(c\frac{d\gamma}{dt}, \vec{u}\frac{d\gamma}{dt} + \gamma\frac{d\vec{u}}{dt})$$
$$\frac{d\eta^\mu}{d\tau} = \gamma(c\frac{d\gamma}{dt}, \vec{u}\frac{d\gamma}{dt} + \gamma\frac{d\vec{u}}{dt})$$

Since $\frac{d\gamma}{dt} = \gamma\frac{\vec{u}\cdot\vec{a}}{c^2-u^2}$, the expression expands to

$$\boxed{\frac{d\eta^\mu}{d\tau} = \gamma^2(c\frac{\vec{u}\cdot\vec{a}}{c^2-u^2}, \vec{a} + \frac{\vec{u}(\vec{u}\cdot\vec{a})}{c^2-u^2})}$$

(c) Not the prettiest expression. We can clean it up a bit by making use of the force expression we previously derived.

$$\frac{d\eta^\mu}{d\tau} = \gamma(c\gamma\frac{\vec{u}\cdot\vec{a}}{c^2-u^2}, \frac{\vec{F}}{m})$$
$$\boxed{\frac{d\eta^\mu}{d\tau} = \gamma(\frac{\vec{u}\cdot\vec{F}}{mc}, \frac{\vec{F}}{m})}$$

Problem 2: π^0 Decay [25 pts]

The π^0 is a heavy meson with a mass of 135 MeV/c^2 that decays almost immediately to two back-to-back photons (with a lifetime of $\tau = 8.4 \times 10^{-17}$ s).

(a) What are the energies of the two photons emitted in the center-of-mass frame of the π^0 when it decays?

(b) Suppose one of the two photons makes an angle θ with respect to the x-axis in the center of mass frame. What is the minimum energy the π^0 must have in order for both photons to be boosted in the forward direction (i.e. make an angle less than 90° from the positive x-axis)? This is convenient if your detector doesn't fully encompass the region surrounding your pion.

(c) Suppose with your detector (read as, lab frame) you measure both photons and each makes a ±45° angle with respect to the beam axis. From this information, tell me how far the π^0 moved from when it was created to when it decayed.

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(a) Photons have zero rest mass, hence their energies are given by $E = pc$. Since in the rest mass their momenta must be equal and opposite, so must their energies. With zero rest mass, that means each photon carries away exactly $m_\pi c^2/2$ of energy.

(b) We use the Lorentz transform to boost from one frame (center-of-mass) to the other (lab). We don't know what the boost factors are yet, but we have a means to calculate them. Let me refer to the lab frame as S' and the center-of-mass frame as S. I will also label my photons as E_1 and E_2.

$$p'_{1,x} = \gamma(p_{1,x} + \beta E_1)$$
$$p'_{2,x} = \gamma(p_{2,x} + \beta E_2)$$

But recall that $E_1 = E_2 = E$. Furthermore, the momenta are equal and opposite to one another (and equal to the energy).

$$p'_{1,x} = E\gamma(\cos\theta + \beta)$$
$$p'_{2,x} = E\gamma(-\cos\theta + \beta)$$

Suppose the quantity $(\cos\theta + \beta)$ is positive so that $p_{1,x}$ is positive (i.e. pointed along the beam direction). For the second photon to also be boosted forward, $(-\cos\theta + \beta)$ must be greater than zero, which means $\boxed{\beta > \cos\theta}$. Since we ask for the energy of the parent π^0 in the lab frame, we can use the following

$$\begin{aligned} E_\pi^{\text{lab}} &= \gamma m_\pi \\ E_\pi^{\text{lab}} &= \frac{m_\pi}{\sqrt{1-\beta^2}} \\ E_\pi^{\text{lab}} &\geq \frac{m_\pi}{\sqrt{1-\cos\theta^2}} \\ &\boxed{E_\pi^{\text{lab}} \geq \frac{m_\pi}{|\sin\theta|}} \end{aligned}$$

(c) In this scenario, the two photons make the same opening angle with the beam axis. This can only happen when, in the rest frame, the two photons are emitted at angles of $\pi/2, 3\pi/2$. To see this explicitly, set $p_{1,x} = p_{2,x}$

$$\begin{aligned} p'_{1,x} &= E\gamma(\cos\theta + \beta) \\ p'_{2,x} &= E\gamma(-\cos\theta + \beta) \\ p'_{1,x} = p'_{2,x} \to E\gamma(\cos\theta + \beta) &= E\gamma(-\cos\theta + \beta) \\ \cos\theta &= -\cos\theta = 0 \end{aligned}$$

This implies a constraint on β

$$\begin{aligned} p'_{1,x} = p'_{2,x} = E\gamma\beta &= \frac{1}{2} m_\pi \gamma\beta \\ p'_{1,x} = p'_1 \cos\theta' &= \frac{1}{2} m_\pi \gamma\beta \\ E'_1 \cos\theta' &= \frac{1}{2} m_\pi \gamma\beta \end{aligned}$$

What is E'_1? That's given by the other boost formula

$$\begin{aligned} E'_1 &= \gamma(E + \beta\cos\theta) \\ E'_1 &= \gamma E = \frac{1}{2}\gamma m_\pi \end{aligned}$$

Since θ' is measured to be 45°, we now find what β is

$$\begin{aligned} E'_1 \cos\theta' &= \frac{1}{2} m_\pi \gamma\beta \\ \frac{1}{2}\gamma m_\pi \cos\theta' &= \frac{1}{2} m_\pi \gamma\beta \\ \beta = \cos\theta' &= \frac{1}{\sqrt{2}} \end{aligned}$$

By the same token, $\gamma = \sqrt{2}$. But what are we really after? We want to know how far the π^0 traveled before it decayed. The lifetime of the pion in the lab frame is given by $\gamma\tau$ (because of time dilation), while it's speed is given by βc. So the total length (on average) traveled by the pion before it decayed is given by $\boxed{d = \gamma\beta c\tau}$ or $\boxed{25\text{ nm}}$.

Problem 3: Review: Mendelstam Variables [25 pts]

High energy physicists try as best they can to express various quantities (energy, momentum, cross-sections, etc.) in terms of invariant quantities. This is not mere aesthetics; it is far easier to make calculations if those calculations are independent of what frame one is working in. One such tool are Mendelstam variables, which describe the energy-momentum exchange when 2 particles collide with one another.

Consider the (inelastic) collision of two particles (1 and 2) to yield two different particles (3 and 4), each with a different mass $m_{i=1,2,3,4}$. The Mendelstam variables are defined as follows:

$$s \equiv (p_1 + p_2)^2/c^2$$
$$t \equiv (p_1 - p_3)^2/c^2$$
$$u \equiv (p_1 - p_4)^2/c^2$$

(a) Calculate the quantity $s + t + u$.

(b) Find the lab-frame energy of particle 1 in terms of Mendelstam variables (Suppose we are working with a fixed-target experiment, where lab frame implies particle 2 is at rest.)

(c) Finally, find the total center-of-mass energy $(E_1 + E_2)$ in terms of Mendelstam variables.

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(a) Expanding s, t and u in terms of their 4-vector components, we have

$$(s + t + u)c^2 = (p_1 + p_2)^2 + (p_1 - p_3)^2 + (p_1 - p_4)^2$$

$$(s + t + u)c^2 = p_1^2 + p_2^2 + 2p_1 \cdot p_2 + p_1^2 + p_3^2 - 2p_1 \cdot p_3 + p_1^2 + p_4^2 - 2p_1 \cdot p_4$$

$$(s + t + u)c^2 = 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1 \cdot p_2 - 2p_1 \cdot p_3 - 2p_1 \cdot p_4$$

$$(s + t + u)c^2 = 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1 \cdot (p_2 - p_3 - p_4)$$

Since $p_1 = p_3 + p_4 - p_2$, we find

$$(s+t+u)c^2 = 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1 \cdot (-p_1)$$

$$(s+t+u)c^2 = 3p_1^2 + p_2^2 + p_3^2 + p_4^2 + 2p_1 \cdot (-p_1)$$

$$(s+t+u)c^2 = 3p_1^2 + p_2^2 + p_3^2 + p_4^2 - 2p_1^2$$

$$(s+t+u)c^2 = p_1^2 + p_2^2 + p_3^2 + p_4^2$$

$$\boxed{(s+t+u) = m_1^2 + m_2^2 + m_3^2 + m_4^2}$$

(b) You found this in Problem #1 already. The technique is certainly the same

$$\begin{aligned} s &= (p_1 + p_2)^2/c^2 \\ sc^2 &= m_1^2c^2 + m_2^2c^2 + 2p_1 \cdot p_2 \\ sc^2 &= m_1^2c^2 + m_2^2c^2 + 2(E_2E_1/c^2 - \vec{p}_1 \cdot \vec{p}_2) \\ sc^2 &= m_1^2c^2 + m_2^2c^2 + 2E_1m_2 \\ &\boxed{E_1 = \frac{(s - m_1^2 - m_2^2)c^2}{2m_2}} \end{aligned}$$

(c) And, as shown earlier (this time without setting c=1)

$$\begin{aligned} s &= (p_1 + p_2)^2/c^2 \\ sc^2 &= (E_1/c + E_2/c)^2 \\ sc^4 &= (E_1 + E_2)^2 \end{aligned}$$

$$\boxed{E_{\text{total}}^{\text{CM}} = \sqrt{s}c^2}$$

Problem 4: Collider versus Linac [25 pts]

Suppose you were determined to discover a new particle (say, the Higgs) that required a very high energy center-of-mass energy,$\sqrt{s}$ to create[1]. You decide you will create this elusive particle by slamming two identical particles of mass m (say, two protons) against each other. You have two choices on how to build your machine. You can build either (a) a collider, which slams the two particles in a head-to-head collision or (b) a linac (linear accelerator) which slams one particle against a fixed stationary target of material.

[1] If you are confused as to what $\sqrt{s}$ is, look at Task 3 in this problem set.

(a) For a given kinetic energy K (which is either given solely to the proton in the linac or split evenly among the two protons in the collider) which option is the better choice to reach your targeted center-of-mass energy?

(b) For what other reasons might you chose the other, less energetic, option?

Figure 1: Photograph of the Fermi National Accelerator Facility. One can see both fixed target beamlines (linacs) and the main proton-anti-proton collider.

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(a) A collider is essentially a center-of-mass frame. Let $s = (p_1 + p_2)^2$ represent the addition of two 4-vectors. In the center-of-mass frame, we find...

$$s^{\text{cm}} = (p_1 + p_2)^2$$
$$s^{\text{cm}} = ((E_1 + E_2, \vec{p}_1 + \vec{p}_2)^2)$$
$$s = (E_1 + E_2)^2$$
$$\sqrt{s^{\text{cm}}} = E_1 + E_2$$

Let $E_{1,2} = K/2 + m$ where K is the kinetic energy imparted on each particle and m is the mass.

$$\sqrt{s^{\text{cm}}} = K + 2m = 2m(1 + \frac{K}{2m})$$

In the lab frame, we expand out the multiplication of the 4-vectors to obtain a simpler expression.

$$\begin{aligned} s &= (p_1 + p_2)^2 \\ s &= p_1^2 + p_2^2 + 2p_1 \cdot p_2 \\ s &= m^2 + m^2 + 2mE^{\text{lab}} \\ s &= 2m^2 + 2mE^{\text{lab}} \end{aligned}$$

Now let $E^{\text{lab}} = K + m$, since it gets all the kinetic energy of the system. We find

$$\begin{aligned} s^{\text{lab}} &= 2m^2 + 2m(K + m) \\ s^{\text{lab}} &= 4m^2 + 2mK \\ s^{\text{lab}} &= 4m^2(1 + \frac{K}{2m}) \\ \sqrt{s^{\text{lab}}} &= 2m\sqrt{1 + \frac{K}{2m}} \end{aligned}$$

Since we are told K is fixed, let us take the ratio of the collider experiment versus the linear accelerator experiment

$$R = \frac{\sqrt{s^{cm}}}{\sqrt{s^{\text{lab}}}} = \frac{1 + \frac{K}{2m}}{\sqrt{1 + \frac{K}{2m}}}$$

$$R = \sqrt{1 + \frac{K}{2m}} \geq 1$$

So, for a collider experiment, you always have more center-of-mass energy than an equivalent collision where one of the particles is at rest. If the probability of a reaction happening grows with center-of-mass energy, you are better off (energetically) to collide your beams.

(b) Well, colliding beams is hard! You have two protons, each only 1 fm across zipping past each other. The chances of hitting one another is really small, which is why beams need to be very, very intense (each burst has billions of protons each). That is the only way one can guarantee that a collision will occasionally happen.

A fixed target scheme has a dense target to hit (think Avogardo's number, 10^{23} targets or so)... The chances of a collision are far greater.