

# Massachusetts Institute of Technology

## Department of Physics

Course: 8.20 —Special Relativity

Term: IAP 2021

Instructor: Markus Klute

### Problem Set 3

handed out January 19th, 2021

#### Problem 1: "This Ion Cannon goes to 11..." [20 pts]

A Separatist vessel is attempting to escape from a Republic cruiser (sorry, this is somewhat regular dinner conversation ...) in hot pursuit. The separatist ship is traveling at a velocity  $\frac{2c}{3}$  while the cruiser is traveling at a velocity of  $\frac{c}{5}$  (both with respect to some "stationary" observer located on a nearby ship).

- (a) The cruiser launches a blasting rocket which travels at  $\frac{c}{5}$  with respect to the cruiser ship. What speed does the rocket achieve and can it catch up to the vessel?
- (b) But wait! The blaster rocket itself fires another ballistic missile, also at a speed of  $\frac{c}{5}$  with respect to the rocket. Does *that* missile hit its target?
- (c) Analyze the above situation, but this time from the point of view of the cruiser. Does the rebel ship still manage to escape?

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- (a) For all these problems, we make use of Einstein's velocity addition rule. In part (a), we have a bullet which launches at speed  $\frac{c}{5}$  with respect to the Republic cruiser, also traveling at speed  $\frac{c}{5}$  (with respect to some observer). To that observer, he sees the bullet fly off at a speed...

$$v_{\text{new}} = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

$$v_{\text{new}} = \frac{\frac{c}{5} + \frac{c}{5}}{1 + \frac{c^2}{25c^2}} = \frac{5c}{13} < \frac{2c}{3}$$

No luck, the ship escapes.

- (b) Now the bullet launches another bullet traveling with a relative speed of  $\frac{c}{5}$  relative to the first bullet. So we do the exercise again...

$$v_{\text{super}} = \frac{v_2 + v_3}{1 + \frac{v_2 v_3}{c^2}}$$

$$v_{\text{super}} = \frac{\frac{5c}{13} + \frac{c}{5}}{1 + \frac{5c^2}{65c^2}} = \frac{19c}{35} < \frac{2c}{3}$$

- (c) If we analyze the situation from the point of view of the Republic cruiser, the cruiser is at rest and the first bullet travels away at a speed  $\frac{c}{5}$ . The second bullet has a speed of  $\frac{5c}{13}$  (from part (a)). The Separatist ship speed is given by...

$$v_{\text{ship}} = \frac{\frac{2c}{3} - \frac{c}{5}}{1 - \frac{2c^2}{15c^2}} = \frac{7c}{13} > \frac{5c}{13}$$

So, the Separatists get away. Bastards!

## Problem 2: The Ladder Paradox [20 pts]

Suppose a farmer living in Mr. Thompson's Wonderland<sup>1</sup> using his unicycle (see figure) wishes to store his ladder in his barn (which is much shorter than the ladder). Having studied relativity (but only for the 1st week) he reasons once the ladder is at high enough speed, it should Lorentz contract and easily fit within the door. Once the back of the ladder has cleared the door, he will shut the door, stop the ladder and lock it inside. A neighbor politely points out that from the ladder's vantage point, it is the barn that will shrink. Who is right?

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<sup>1</sup>Mr. Thompson's Wonderland refers to a 1959 paper by Terrell whereby the speed of light is much smaller and hence the effects of special relativity are readily apparent.

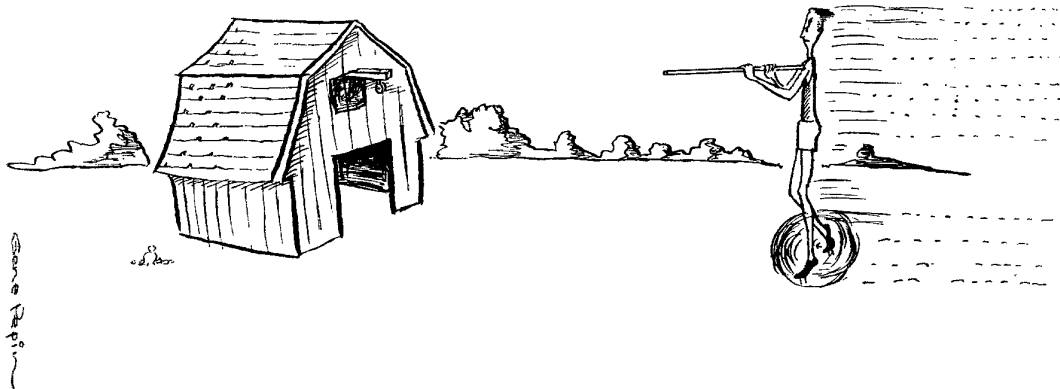


FIG. 1. Pole vaulter running toward open front door of barn with contracted pole.<sup>6</sup>

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- They are both right. That is because there are really two events taking place. Event A is that the back of the ladder passes the door while event B is the front of the ladder reaches the end of the barn. In the frame of the barn, event A precedes B, while from the ladder's perspective, B precedes A. Since events are not simultaneous, there is no problem.

But what about when the door is slammed shut and the ladder is brought to rest? Which frame wins? In this case, the farmer halts the ladder, but it takes time for the signal to reach the front of the ladder! Remember, there are no rigid objects in special relativity... it takes time for the signal to stop to propagate to the front. So the front of the ladder just keeps going and will punch through the wall when all is done (or send the farmer flying backwards).

### Problem 3: The Paradox of the Fast Walker [20 pts]

This is a slight twist from the ladder paradox problem, offered by Wolfgang Rindler (Am. J. Phys. 29, 365 (1961)). A fast walker is walking over a metal grid whose's spacing is exact the length of his foot. From his point of view, the crate spacing is Lorentz contracted, so he should easily be able to walk over the grid. But from the grid's frame of reference, it is the foot which is Lorentz contracted, and he will surely fall in. Explain what happens.

- For a good explanation of the effect, see Section 6.4 from Leo Sartori's *Understanding relativity : a simplified approach to Einstein's theories*. The book can be viewed online from MIT library. Go to <https://lib.mit.edu/search/bento?q=understanding+relativity> and click on "view online" for the first entry under "Books and media". The main takeaway here is that there is no rigid body in special relativity.

### Problem 4: Velocity Addition in Matrix Form [20 pts]

The Lorentz transformations allow one to correctly transform from one inertial frame to the next. In Lorentz transformations, spatial and time coordinates are intertwined, where one is no longer independent of another. As a reminder, the Lorentz transformation to a frame moving with velocity  $v$  along the x-axis is given as the following:

$$\begin{aligned} ct' &= \gamma(ct - \beta x) \\ x' &= \gamma(x - \beta ct) \\ y' &= y \\ z' &= z \end{aligned}$$

where  $\beta = v/c$  and  $\gamma = \frac{1}{\sqrt{1-(v/c)^2}}$ . These transformations can be represented more compactly in matrix notation.

- (a) Begin by writing down the *Galilean* transformation from a frame  $S$  to a frame  $S'$ , where  $S'$  is moving with a constant velocity  $v$  along the x-axis, in matrix notation. Use  $ct$  for the time component.
- (b) Now write down the Lorentz transformation for the same situation as above.
- (c) Use the property  $\gamma^2 - \gamma^2\beta^2 = 1$  to re-write the above matrix in terms of  $\cosh \phi$  and  $\sinh \phi$ , where  $\tanh \phi = \beta$ .
- (d) Allow  $\Lambda(\phi(\beta))$  to represent the matrix that you have written above. Show that  $\Lambda(\phi(\beta_1))\Lambda(\phi(\beta_2)) = \Lambda(\phi(\beta_1)+\phi(\beta_2))$ . Is this consistent with the velocity addition rule?
- (e) From the above, show that the inverse of  $\Lambda(\phi(\beta))$  is given by  $\Lambda(\phi(-\beta))$ .

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- (a) Let us recall what Galilean transformations are like...

$$\begin{aligned} ct' &= ct \\ x' &= x - vt \\ y' &= y \\ z' &= z \end{aligned}$$

In matrix form, this becomes...

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\beta & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

(b) For Lorentz transforms, the situation is much more symmetric...

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

(c) Here we use the property that  $\cosh^2 \phi - \sinh^2 \phi = 1$ . Let  $\gamma = \cosh \phi$  and  $\sinh \phi = \gamma\beta$ . The previous condition is fulfilled. Note also that  $\sinh \phi / \cosh \phi = \tanh \phi = \beta$ . So we have...

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh \phi & -\sinh \phi & 0 & 0 \\ -\sinh \phi & \cosh \phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

With  $\phi = \tanh^{-1} \beta$

(d) Let us perform two boost in a row, each with velocity  $\beta_1$  and  $\beta_2$  with angles  $\phi_1$  and  $\phi_2$ , respectively.

$$\begin{pmatrix} ct'' \\ x'' \\ y'' \\ z'' \end{pmatrix} = \begin{pmatrix} \cosh \phi_2 & -\sinh \phi_2 & 0 & 0 \\ -\sinh \phi_2 & \cosh \phi_2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cosh \phi_1 & -\sinh \phi_1 & 0 & 0 \\ -\sinh \phi_1 & \cosh \phi_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

$$\begin{pmatrix} (\cosh \phi_2 \cosh \phi_1 + \sinh \phi_2 \sinh \phi_1) & -(\cosh \phi_2 \sinh \phi_1 + \cosh \phi_1 \sinh \phi_2) & 0 & 0 \\ -(\cosh \phi_1 \sinh \phi_2 + \cosh \phi_2 \sinh \phi_1) & (\cosh \phi_2 \cosh \phi_1 + \sinh \phi_2 \sinh \phi_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Now we can use the properties of cosh and sinh to simplify the above expression.

$$\begin{aligned} \cosh(A + B) &= \cosh A \cosh B + \sinh A \sinh B \\ \sinh(A + B) &= \cosh A \sinh B + \sinh A \cosh B \end{aligned}$$

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh(\phi_2 + \phi_1) & -\sinh(\phi_2 + \phi_1) & 0 & 0 \\ -\sinh(\phi_2 + \phi_1) & \cosh(\phi_2 + \phi_1) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

Hence,  $\Lambda(\phi(\beta_1))\Lambda(\phi(\beta_2)) = \Lambda(\phi(\beta_1) + \phi(\beta_2))$ . Is this consistent with the velocity addition rule? Recall that  $\tanh \phi = \beta$ . We can use the angle addition rules for  $\tanh$  to help us.

$$\tanh(A + B) = \frac{\tanh A + \tanh B}{1 + \tanh A \tanh B}$$

Therefore, we have...

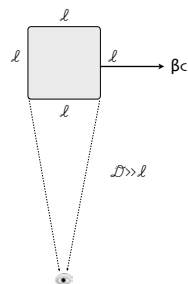
$$\begin{aligned} \tanh(\phi_2 + \phi_1) &= \frac{\tanh \phi_2 + \tanh \phi_1}{1 + \tanh \phi_2 \tanh \phi_1} \\ &= \frac{\beta_2 + \beta_1}{1 + \beta_2 \beta_1} \end{aligned}$$

This is indeed consistent the Einstein velocity addition rule.

- (e) Since  $\Lambda(\phi(\beta_1))\Lambda(\phi(\beta_2)) = \Lambda(\phi(\beta_1) + \phi(\beta_2))$ . If  $\beta_2 = -\beta_1$ , then  $\phi(\beta_2) = -\phi(\beta_1)$ . Therefore,  $\Lambda$  becomes unitary. Hence  $\Lambda(\phi(-\beta)) = \Lambda^{-1}$ , or, in matrix form...

$$\Lambda(\phi(-\beta)) = \Lambda^{-1} = \begin{pmatrix} \cosh \phi & \sinh \phi & 0 & 0 \\ \sinh \phi & \cosh \phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

### Problem 5: Rotating Cube [20 pts]



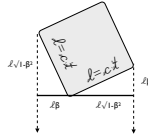
Suppose a cube whose sides are length  $l$  (in the cube's proper frame) travels to the right at velocity  $\beta c$  with respect to the lab frame. An observer is located in the lab frame and watches the cube pass by. We say the "left" and "right" faces of the

cube are perpendicular to the direction of motion, while the "front" and "back" faces are parallel. The cube's distance of closest approach is much, much greater than the length of the cube.

Because the light-path from corners of the "back" and "front" faces of the cube to the observer differ, the cube will appear to be rotating. What is the angle by which the cube appears to be rotated?

[ *Hint:* Everything we perceive are merely projections of three-dimensional objects on the two-dimensional plane of our retina. With this in mind, how would we perceive a rotated cube to be different from one that is not? How would the finite speed of light and other relativistic effects contribute to trick us to perceive the cube as rotated? ]

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Because  $D \gg l$ , we can assume lights from all points on the cube to the observer are parallel.

The time it takes for the light to travel from the back left corners to the front left edge is determined by the speed of light,  $l = ct$ . In that time, the object will move to the right by an amount  $\beta ct$ , and the delayed light will make the observer think that he or she sees a left face with that length. On the other hand, the front face is Lorentz contracted. Combining these two visual effects, it would appear to the observer that the cube has been rotated. The angles form as illustrated in the figure from a top-down view. The angle of rotation is given by

$$\tan \theta = \frac{l\beta}{l\sqrt{1-\beta^2}}$$

...or...

$$\tan \theta = \gamma\beta$$



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