# Massachusetts Institute of Technology

## Department of Physics

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| Course: | 8.20 —Special Relativity |
| Term: | IAP 2021 |
| Instructor: | Markus Klute |

# Problem Set 1

handed out January 5th, 2021

## Problem 1: Estimates of $\beta$ [25 points]

Although always applicable, the effects of special relativity remain for the most part unseen in our day-to-day experiences. This is because special relativity depends on the velocity of the objects. Most macroscopic objects have very small velocities compared to the speed of light (defined as 299 792 458 meters/second *exactly*).

A useful quantity that can aid one in estimating the size of relativistic effects is the parameter $\vec{\beta} \equiv \frac{\vec{v}}{c}$, where $\vec{v}$ is the velocity vector and $c$ is the speed of light (Note that $\beta$ is itself a unit-less vector. For this problem, we will be finding the magnitude of $\vec{\beta}$). Estimate the magnitude of $\vec{\beta}$ for the following:

(a) You, on a brisk walk. (4 points)

(b) The speed of the Red Line subway between Harvard and Central. (4 points)

(c) The cruise speed of a Concorde supersonic passenger airliner, which could fly from New York to London in under 3 hours. (4 points)

(d) The orbital motion of a satellite in a geosynchronous orbit above the Earth's equator. (4 points)

(e) The orbital motion of Earth around the sun. (4 points)

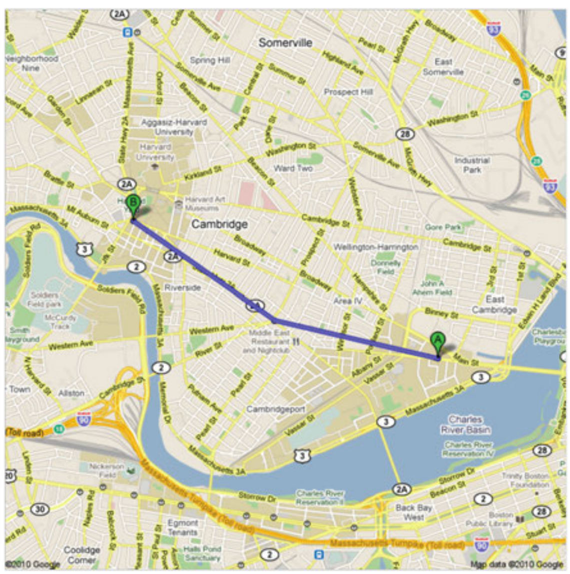
(f) The speed of Halley's Comet, the only short-period comet visible to the naked-eye from Earth, at the perihelion. (5 points)

- This problem essentially deals with two types of motion. In certain cases (walking, airliner, etc.) we deal with *linear* motion, in which case we seek to simply estimate $\beta$ via $\vec{v}/c$. The other category is associated with *circular* motion, in which case we will wish to look at the tangental velocity of the object as a fair approximation of $\beta$. For those cases, assuming circular motion, we find

$$
\begin{aligned}
|\vec{a}| = |\omega \times (\omega \times R)| = R\omega^2 &= \frac{v_\perp^2}{R} \\
|\vec{a}| &= \frac{GM}{R^2} \\
|\vec{\beta}| = \frac{|\vec{v}|}{c} &= \sqrt{\frac{GM}{Rc^2}}
\end{aligned}
$$

(a) Even on a good day, I rarely top 3 km/hour, so $\beta = \frac{3\times10^3 m}{3600s\cdot3\times10^8 m/s} \simeq 3\times10^{-9}$.

(b) Google maps claims 2 miles in 6 minutes (about 20 miles/hour), or about 9 meters per second. That yields $\beta \simeq 3\times10^{-8}$






(c) The Concorde supersonic passenger airliner cruises at the speed of 1341 mph (or 600 meters/second), about 1.7 times the speed of sound in air at sea level. This yields $\beta = 2\times10^{-6}$. Although small, this small beta can be probed by experimental means (see later the Hafele-Keating experiment).

(d) For this problem, we actually know the orbital period quite well, but we do not know apriori what orbit this satellite will use. But we can calculate it by finding the radius where centripedial and gravitational forces cancel. From above, we can re-write this as

$$v = \omega R = \sqrt{\frac{GM}{R}}$$
$$R = (\frac{GM}{\omega^2})^{\frac{1}{3}}$$

The angular speed $\omega$ is $\frac{2\pi}{86400s} = 7.27 \times 10^{-5}$ rad/s, yielding $R = 4.2 \times 10^7$ meters. The corresponding velocity at that radius is therefore 3 km/second. Therefore $\beta \sim 10^{-5}$.

(e) The Earth's radial orbit is 1 AU or $1.5 \times 10^{11}$ meters, while the Sun has a mass of $2 \times 10^{30}$ kg. Using the above formula, we find $\beta = 10^{-4}$

(f) Halley's comet has a large eccentricity of 0.96. At aphelion, it is about 35 AU from the Sun, well beyond the orbit of the Neptune. In comparison, it is only 0.58 AU from the Sun at perihelion, not far away from the orbit of Mercury. Its speed was measured to be 54.55 km/s at perihelion on February 9, 1986, yielding $\beta = 1.8 \times 10^{-4}$.

Comet Halley is expected to return to perihelion on July 27, 2061. Most of you will likely be there to witness its return!

## Problem 2: Central Forces [15 points]

In class, we learned that under Galilean transformations, it is not possible to discern between inertial frames (i.e. the laws of physics remain unchanged). This certainly holds true for central forces, where the force depends on the relative distance between object, rather than their absolute positions.

(a) Suppose you have a potential of the form $U(\vec{r_1}, \vec{r_2}) = U(|\vec{r_1} - \vec{r_2}|)$. Show that the force resulting from such a potential is invariant under Galilean transformations. (7 points)

(b) Suppose now you have a potential that depends on the absolute magnitude of $\vec{r_1}$ and of $\vec{r_2}$. Show that if you have a potential of the form $U(\vec{r_1}, \vec{r_2}) = U(|\vec{r_1}|^2 - |\vec{r_2}|^2)$, the force resulting from such a potential is not invariant under Galilean transformations. (8 points)

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(a) We would expect that central forces (such as gravity) should be consistent with Galilean relativity, but let us demonstrate that is the case. Let's work in the Cartesian system and define the force as follows:

$$F_i = m\frac{d^2 x_i}{dt^2} = -\frac{\partial}{\partial x_i} U(|\vec{r}_2 - \vec{r}_1|)$$

The left side of that equation is clearly invariant under Galilean transformations. Remembering $t' = t$ (absolute time), we have

$$m\frac{d^2 x_i'}{dt'^2} = m\frac{d^2}{dt^2}(x_i - v_i t) = m\frac{d^2 x_i}{dt^2}$$

OK, what about the right-hand side? Let's simplify things a bit by defining the distance vector $r_{12}$

$$r_{12} = (|\vec{r}_2 - \vec{r}_1|) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

and

$$\frac{\partial r_{12}}{\partial x_i} = \frac{x_{2,i} - x_{1,i}}{r_{12}}$$

It is clear that transposing the position $\vec{r}'_{1,2} = \vec{r}_{1,2} + \vec{v}t$ leaves $\vec{r}_2 - \vec{r}_1$ unchanged, i.e. $r'_{12} = r_{12}$. Likewise, it is true that the derivative of $r_{12}$ remains unchanged, i.e.

$$\frac{\partial r'_{12}}{\partial x'_i} = \frac{x'_{2,i} - x'_{1,i}}{r'_{12}} = \frac{x_{2,i} - x_{1,i}}{r_{12}}$$

Finally, one can show

$$-\frac{\partial}{\partial x_i}U(|\vec{r_2}-\vec{r_1}|) = -\frac{d}{dr_{12}}U(|\vec{r_2}-\vec{r_1}|)\frac{\partial r_{12}}{\partial x_i} = -\frac{d}{dr_{12}}U(|\vec{r_2}-\vec{r_1}|)\frac{x_{2,i}-x_{1,i}}{r_{12}}$$

$$-\frac{\partial}{\partial x'_i}U(|\vec{r'}_2-\vec{r'}_1|) = -\frac{d}{dr'_{12}}U(|\vec{r'}_2-\vec{r'}_1|)\frac{\partial r'_{12}}{\partial x'_i} = -\frac{d}{dr_{12}}U(|\vec{r_2}-\vec{r_1}|)\frac{x_{2,i}-x_{1,i}}{r_{12}}$$

Which implies

$$-\frac{\partial}{\partial x'_i}U(|\vec{r'}_2-\vec{r'}_1|) = -\frac{\partial}{\partial x_i}U(|\vec{r_2}-\vec{r_1}|)$$

The laws of motion therefore remain unchanged due to the Galilean transformation about an arbitrary direction in velocity.

(b) A potential of the form $U(\vec{r_1},\vec{r_2}) = U(|\vec{r_1}|^2-|\vec{r_2}|^2)$ poses serious issues because it depends on the absolute position, and hence is not invariant under linear translations (that is, it depends on where I am in absolute space, yikes!). To show this explicitly, let's re-write the quantity $|\vec{r_1}|^2-|\vec{r_2}|^2$ in a slightly different way. Let

$$\vec{r_+} = \vec{r_1}+\vec{r_2}$$
$$\vec{r_-} = \vec{r_1}-\vec{r_2}$$

So we have

$$|\vec{r'}_1|^2-|\vec{r'}_2|^2 = \vec{r_+}\cdot\vec{r_-}$$

Although the quantity $\vec{r_-}$ is not altered by a shift in position (because it is a difference, just as $r_{12}$, $\vec{r_+}$ clearly *is* altered (it depends on the coordinate system chosen and thus requires an absolute definition of space... clearly a violation of Galilean relativity. In fact, it is easy to show that under the shift by $\vec{v}t$.

$$|\vec{r'}_1|^2-|\vec{r'_2}|^2 = \vec{r_+}\cdot\vec{r_-}-2\vec{v}\cdot(\vec{r_-})t$$

and therefore

$$U(|\vec{r'}_1|^2-|\vec{r'_2}|^2) \neq U(|\vec{r_1}|^2-|\vec{r_2}|^2)$$

The potential is certainly not invariant. What about the derivative? The conclusions are similar. Let us look, for example at the derivative with respect to $x_1$. Let $z=|r_1|^2-|r_2|^2$.

$$\frac{\partial U(z)}{\partial x_1} = \frac{\partial U(z)}{\partial z}\frac{\partial z}{\partial x_1} = \frac{\partial U(z)}{\partial z}(2x_1)$$

and

$$\frac{\partial U(z')}{\partial x'_1} = \frac{\partial U(z')}{\partial z'}\frac{\partial z'}{\partial x'_1} = \frac{\partial U(z')}{\partial z'}(2x'_1)$$

Since $x'_1 \neq x_1$, the derivative (and hence the force) is not equivalent either.

## Problem 3: Elastic Collisions and Galilean Invariance [20 points]

Suppose that you have two objects, $m_1$ and $m_2$ which collide with one another. The observer measures that momentum is conserved in his frame. Now suppose another observer watches the same event, but is moving with velocity $u$ with respect to the first observer.

(a) Under what condition(s) will the second observer also conclude that momentum is conserved? (10 points)

(b) Suppose the first observer also observes that the collision is elastic (i.e. kinetic energy is conserved). Under what condition(s) will the moving observer make the same conclusion? (10 points)

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(a) Begin by writing the momenta in one frame

$$p_{1,i} + p_{2,i} = p_{1,f} + p_{2,f}$$

where the subscript $i$ and $f$ stand for initial and final. Since $p = mv$ we can re-write this as

$$m_{1,i}v_{1,i} + m_{2,i}v_{2,i} = m_{1,f}v_{1,f} + m_{2,f}v_{2,f}$$

where $v = \frac{d}{dt}x$. Note that we have not assumed that the particles retained the same mass after their collision. When we perform a Galilean transformation, we let $x' = x - ut$, where $u$ is constant. Thus, we find

$$\frac{dx'}{dt} = v - u$$
$$m_{1,i}(v_{1,i} - u) + m_{2,i}(v_{2,i} - u) = m_{1,f}(v_{1,f} - u) + m_{2,f}(v_{2,f} - u)$$
$$m_{1,i}v_{1,i} + m_{2,i}v_{2,i} - (m_{1,f}v_{1,f} + m_{2,f}v_{2,f}) = u(m_{1,i} + m_{2,i} - m_{1,f} - m_{2,f})$$

The left-hand side of the equation is zero because of momentum conservation observed in the original at rest frame. What about the right-hand side? In general this would imply non-momentum conservation unless total mass was conserved as well. That is,

$$(m_{1,i} + m_{2,i}) = (m_{1,f} + m_{2,f})$$

(b) In terms of kinetic energy, the first observer also can make the following claim

$$\frac{1}{2}m_{1,i}v_{1,i}^2 + \frac{1}{2}m_{2,i}v_{2,i}^2 = \frac{1}{2}m_{1,f}v_{1,f}^2 + \frac{1}{2}m_{2,f}v_{2,f}^2$$

To simplify the notation, let's see what happens when we perform a Galilean transformation to a kinetic energy term of one particle

$$\frac{1}{2}mv^2 \rightarrow \frac{1}{2}m(v-u)^2 = \frac{1}{2}mv^2 + \frac{1}{2}mu^2 - 2mvu$$

Applying this to all terms in the above equation, we find

$$\begin{aligned}(\frac{1}{2}m_{1,i}v_{1,i}^2 + \frac{1}{2}m_{2,i}v_{2,i}^2 - \frac{1}{2}m_{1,f}v_{1,f}^2 - \frac{1}{2}m_{2,f}v_{2,f}^2)\\ -u(m_{1,i}v_{1,i} + m_{2,i}v_{2,i} - m_{1,f}v_{1,f} - m_{2,f}v_{2,f})\\ +\frac{1}{2}u^2(m_{1,i} + m_{2,i} - m_{1,f} - m_{2,f})\\ = 0\end{aligned}$$

The first term is zero because kinetic energy was conserved. The second term is zero because momentum was conserved. Therefore

$$\frac{1}{2}u^2(m_{1,i} + m_{2,i} - m_{1,f} - m_{2,f}) = 0$$

and thus if the second observer also concludes that kinetic energy was conserved, the total mass of the system is conserved. Note that even if we did not have an elastic collision, the conservation of momentum and mass lead to the fact that the change in kinetic energy is maintained between the stationary and moving frame.

Note that this works for $n$ masses, not just two.

## Problem 4: Invariance of the Wave Equation [25 points]

Waves play an important role in physics (we have a whole course, 8.03, devoted to them). Part of the motivation behind special relativity can be traced back to the wave equation.

(a) Suppose you have a generic function of space and time, f(x,t) [for simplicity, we consider the one dimensional case here]. Show that the equation satisfies the following differential equation:

$$\frac{\partial^2}{\partial x^2} f(x,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} f(x,t) = 0 \tag{1}$$

then it has a solution of the form:

$$f(x,t) = g_1(x - ct) + g_2(x + ct) \tag{2}$$

where $g_1$ and $g_2$ are any two functions. (5 points)

(b) Draw a sample function that satisfies the above wave equation at time t = 0 and a later time. How do you interpret what is going on? (5 points)

(c) Show that the above wave equation is *not* invariant under Galilean transformations (i.e. going to a frame moving with velocity $v$). (10 points)

(d) Show that for part (c) I can find a new solution to the transformed equation by altering the speed of propagation such that $f(x',t') = g_1(x' - (c-v)t') + g_2(x' + (c+v)t')$ (5 points)

[Hint: Use the chain rule...

$$\frac{\partial f(u(x,t))}{\partial x} = \frac{df(u)}{du}\frac{\partial u(x,t)}{\partial x}$$

...to help tackle the problem.]

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(a) Let $u = u(x,t)$ designate the argument inside of the arbitrary function $f(x,t)$ such that $f(x,t) = f(u(x,t))$. Now, let us find the partial derivatives with respect to position and time such that we can later substitute into the wave equation.

$$\frac{\partial f(x,t)}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x}$$

$$\frac{\partial^2 f(x,t)}{\partial x^2} = \frac{\partial}{\partial x}(\frac{df}{du}\frac{\partial u}{\partial x}) = \frac{d^2 f}{du^2}(\frac{\partial u}{\partial x})^2 + \frac{df}{du}\frac{\partial^2 u}{\partial x^2}$$

Likewise for the time derivative, we find...

$$\frac{\partial f(x,t)}{\partial t} = \frac{df}{du}\frac{\partial u}{\partial t}$$
$$\frac{\partial^2 f(x,t)}{\partial t^2} = \frac{\partial}{\partial t}(\frac{df}{du}\frac{\partial u}{\partial t}) = \frac{d^2 f}{du^2}(\frac{\partial u}{\partial t})^2 + \frac{df}{du}\frac{\partial^2 u}{\partial t^2}$$

Now, we are told that the argument $u(x,t)$ has the form $x \pm ct$. This implies...

$$\frac{\partial u}{\partial x} = 1$$
$$\frac{\partial u}{\partial t} = \pm c$$
$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} = 0$$

Plugging into the original wave equation, we find...

$$\begin{aligned} \frac{\partial^2}{\partial x^2} f(x,t) - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} f(x,t) &= 0 \\ = \frac{d^2 f}{du^2}(\frac{\partial u}{\partial x})^2 - \frac{1}{c^2}\frac{d^2 f}{du^2}(\frac{\partial u}{\partial t})^2 \\ = \frac{d^2 f}{du^2}(1 - \frac{1}{c^2}(\pm c)^2) \\ = 0 \end{aligned}$$

That means as long as I can write $f(x,t)$ as $g_1(x-ct) + g_2(x+ct)$, it satisfies the wave equation.

(b) I can draw any smooth function here and, if it satisfies the wave equation, it means that function can "travel" left or right and maintain its shape. If it is of the form $x - ct$ then it moves along the *positive* x axis, while if it is of the form $x + ct$ it moves along the negative x-axis as time rolls forward.

(c) Problem (c) is a bit tricker because now $x'$ and $t'$ are themselves functions of $x$ and $t$. But we can still invoke the chain rule to help us again. First, let us solve what the wave equation would look like after Galilean transformations. Let us remember what those transformations are

$$x' = x - vt$$
$$t' = t$$

Taking the various derivatives of these quantities reveals the following

$$\frac{\partial x'}{\partial x} = 1; \frac{\partial t'}{\partial x} = 0$$
$$\frac{\partial t'}{\partial t} = 1; \frac{\partial x'}{\partial t} = -v$$

So now we can write out the wave equation terms in terms of $x'$ and $t'$.

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial f}{\partial t'}\frac{\partial t'}{\partial x}$$
$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial x'}$$

and...

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial f}{\partial x'}\frac{\partial x'}{\partial t}$$
$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial t'} - v\frac{\partial f}{\partial x'}$$

We will need to apply this method again to find the second order derivatives of these functions. Performing the operation again, we find...

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 f}{\partial x'^2}$$
$$\frac{\partial^2 f}{\partial t^2} = \frac{\partial^2 f}{\partial t'^2} - 2v\frac{\partial^2 f}{\partial x'\partial t'} + v^2\frac{\partial^2 f}{\partial x'^2}$$

Now we can put the whole thing together to re-express the wave equation in terms of $x'$ and $t'$...

$$\frac{\partial^2 f}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2 f}{\partial t'^2} + 2\frac{v}{c^2}\frac{\partial^2 f}{\partial x'\partial t'} - \frac{v^2}{c^2}\frac{\partial^2 f}{\partial x'^2} = 0$$

(d) OK, now let $f(x', t') = g(u(x' - wt'))$. I am going to use the same technique as in part (a) to find what $w$ should be to satisfy the above wave equation. Substituting for $f(x', t')$ for $g(u(x' - wt'))$, I find...

$$\frac{d^2 f}{du^2}(1 - \frac{w^2}{c^2} - 2\frac{v}{c^2}w - \frac{v^2}{c^2}) = 0$$
$$\frac{d^2 f}{du^2}(c^2 - (w + v)^2) = 0$$
$$\rightarrow w = \pm c - v$$

Pushing all the math aside for a moment, this makes a good deal of physical sense. Waves (for example, sound waves or water waves) propagate with respect to the speed of the medium. Under Galilean transformations, going to a moving frame shifts the velocity of propagation as well.

This will not hold true for light, where the speed of propagation is independent of the motion of the source.

## Problem 5: Review: Taylor Series [15 points]

Taylor expansion is a particularly useful tool in physics. For one, it allows physics to estimate the order of effects, especially when the effect is expected to be very small. Often times a calculation can be greatly simplified by carrying out calculations only to first or second order.

Here is a poor physicists approach to Taylor series. I can expand any continuously differentiable function about a point (call that $x_0$) by looking at the derivatives of that function:

$$f(x + x_0) \simeq f(x_0) + \frac{1}{1!}\frac{\partial}{\partial x}f(x)|_{x=x_0}(x - x_0) + \frac{1}{2!}\frac{\partial^2}{\partial x^2}f(x)|_{x=x_0}(x - x_0)^2 + \cdots \quad (3)$$

Often one wants to expand around the zero point, which allows one to re-write the above a bit more compactly.

$$f(x) \simeq \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n \quad (4)$$

Why does this matter for special relativity? Relativistic corrections are often of order $\beta$ or $\beta^2$ and often difficult to detect directly since, as seen in problem 1, $\beta$ is very small for most of the macroscopic object.

(a) Write the Taylor expansion of the following functions for small values of x ($|x| \ll 1$), to the second order. (5 points)
- $f(x) = \sin(x)$
- $f(x) = \cos(x)$
- $f(x) = (1 - x^2)^n$
- $f(x) = (1 - x^2)^{-1/2}$

(b) Let $x = x' - vt'$ and repeat problem (a) for small values of $t'$ ($|t'| \ll 1$). (5 points)

(c) Another important quantity that we will see a lot in this class is the boost factor (traditionally written as $\gamma$), where it is defined as $\gamma \equiv 1/\sqrt{1-\beta^2} \equiv 1/\sqrt{1-(\frac{v}{c})^2}$. Taylor expand this function to the second order in $\beta$. Using the values of $\beta$ calculated for Problem #1, find the corresponding value for $\gamma$ approximated to the second order in $\beta$. (5 points)

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(a) In each of these cases, we wish to expand to second order. Using the formula above, we find that a second-order expansion is given by...

Table 1: default

| Function | f(0) | f'(0) | f''(0) | Result |
|---|---|---|---|---|
| $\sin x$ | sin(x=0) $= 0$ | $\cos(x=0) = 1$ | $-\sin(x=0) = 0$ | x |
| $\cos x$ | cos(x=0) $= 1$ | $-\sin(x=0) = 0$ | $-\cos(x=0) = -1$ | $1 - \frac{1}{2}x^2$ |
| $(1-x^2)^n$ | $(1-x^2)^n = 1$ | $n(1-x^2)^{(n-1)}(-2x) = 0$ | $n(n-1)(1-x^2)^{(n-1)}(-2x)^2$ $+n(1-x^2)^{(n-1)}(-2) = -2n$ | $1 - nx^2$ |
| $(1-x^2)^{-1/2}$ | 1 | 0 | 1 | $1 + \frac{1}{2}x^2$ |

Table 2: default

| Quantity | $\beta$ | $\gamma$ |
|---|---|---|
| (a) Walking | $3 \times 10^{-9}$ | $1 + 4.5 \times 10^{-18}$ |
| (b) Train | $3 \times 10^{-8}$ | $1 + 4.5 \times 10^{-16}$ |
| (c) Concorde | $2 \times 10^{-6}$ | $1 + 2 \times 10^{-12}$ |
| (d) Satellite | $1 \times 10^{-5}$ | $1 + 5 \times 10^{-11}$ |
| (e) Earth | $1 \times 10^{-4}$ | $1 + 5 \times 10^{-9}$ |
| (f) Comet Halley | $1.8 \times 10^{-4}$ | $1 + 1.62 \times 10^{-8}$ |

$$f(x) \simeq \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n$$

$$f(x) \simeq f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \mathcal{O}(x^3)$$

(b) The answers are: • $\sin(x' - vt') \approx \sin(x') - vt'\cos(x') - \frac{1}{2}v^2t'^2\sin(x')$

• $\cos(x' - vt') \approx \cos(x') + vt'\sin(x') - \frac{1}{2}v^2t'^2\cos(x')$

• $(1 - (x' - vt')^2)^n \approx (1 - x'^2)^n + 2nvx't'(1 - x'^2)^{(n-1)} + nv^2((2n-1)x'^2 - 1)t^2(1 - x'^2)^{(n-2)}$

• $(1 - (x' - vt')^2)^{-1/2} \approx \frac{1}{\sqrt{1-x'^2}} - \frac{vx't'}{(1-x'^2)^{3/2}} + \frac{(2x'^2+1)v^2t'^2}{2(1-x'^2)^{5/2}}$

The most important part of the answers is the order of $vt'$. Also it's fine to assume $x' = 0$.

(c) From the last exercise, we know we can expand $\gamma$ as $\simeq 1 + \beta^2/2$. The corrections to gamma follow accordingly...