Massachusetts Institute of Technology

Department of Physics

Course:	8.20 —Special Relativity
Term:	IAP 2020
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Problem Set 1

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Problem 1: Estimates of β [25 points]

Although always applicable, the effects of special relativity remain for the most part unseen in our day-to-day experiences. This is because special relativity depends on the velocity of the objects. Most macroscopic objects have very small velocities compared to the speed of light (defined as 299 792 458 meters/second *exactly*).

A useful quantity that can aid one in estimating the size of relativistic effects is the parameter $\vec{\beta} \equiv \frac{\vec{v}}{c}$, where $\vec{v}$ is the velocity vector and c is the speed of light (Note that β is itself a unit-less vector. For this problem, we will be finding the magnitude of $\vec{\beta}$). Estimate the magnitude of $\vec{\beta}$ for the following:

(a) You, on a brisk walk. (4 points)

(b) The speed of the Red Line subway between Harvard and Central. (4 points)

(c) The cruise speed of a Concorde supersonic passenger airliner, which could fly from New York to London in under 3 hours. (4 points)

(d) The orbital motion of a satellite in a geosynchronous orbit above the Earth's equator. (4 points)

(e) The orbital motion of Earth around the sun. (4 points)

(f) The speed of Halley's Comet, the only short-period comet visible to the naked-eye from Earth, at the perihelion. (5 points)

Problem 2: Central Forces [15 points]

In class, we learned that under Galilean transformations, it is not possible to discern between inertial frames (i.e. the laws of physics remain unchanged). This certainly holds true for central forces, where the force depends on the relative distance between object, rather than their absolute positions.

(a) Suppose you have a potential of the form $U(\vec{r_1}, \vec{r_2}) = U(|\vec{r_1} - \vec{r_2}|)$. Show that the force resulting from such a potential is invariant under Galilean transformations. (7 points)

(b) Suppose now you have a potential that depends on the absolute magnitude of $\vec{r_1}$ and of $\vec{r_2}$. Show that if you have a potential of the form $U(\vec{r_1}, \vec{r_2}) = U(|\vec{r_1}|^2 - |\vec{r_2}|^2)$, the force resulting from such a potential is not invariant under Galilean transformations. (8 points)

Problem 3: Elastic Collisions and Galilean Invariance [20 points]

Suppose that you have two objects, m_1 and m_2 which collide with one another. The observer measures that momentum is conserved in his frame. Now suppose another observer watches the same event, but is moving with velocity u with respect to the first observer.

(a) Under what condition(s) will the second observer also conclude that momentum is conserved? (10 points)

(b) Suppose the first observer also observes that the collision is elastic (i.e. kinetic energy is conserved). Under what condition(s) will the moving observer make the same conclusion? (10 points)

Problem 4: Invariance of the Wave Equation [25 points]

Waves play an important role in physics (we have a whole course, 8.03, devoted to them). Part of the motivation behind special relativity can be traced back to the wave equation.

(a) Suppose you have a generic function of space and time, f(x,t) [for simplicity, we consider the one dimensional case here]. Show that the equation satisfies the following differential equation:

$$\frac{\partial^2}{\partial x^2} f(x,t) - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} f(x,t) = 0 \tag{1}$$

then it has a solution of the form:

$$f(x,t) = g_1(x - ct) + g_2(x + ct) \tag{2}$$

where g_1 and g_2 are any two functions. (5 points)

(b) Draw a sample function that satisfies the above wave equation at time t = 0 and a later time. How do you interpret what is going on? (5 points)

(c) Show that the above wave equation is *not* invariant under Galilean transformations (i.e. going to a frame moving with velocity v). (10 points)

(d) Show that for part (c) I can find a new solution to the transformed equation by altering the speed of propagation such that $f(x', t') = g_1(x' - (c - v)t') + g_2(x' + (c + v)t')$ (5 points)

[Hint: Use the chain rule...

$$\frac{\partial f(u(x,t))}{\partial x} = \frac{df(u)}{du}\frac{\partial u(x,t)}{\partial x}$$

...to help tackle the problem.]

Problem 5: Review: Taylor Series [15 points]

Taylor expansion is a particularly useful tool in physics. For one, it allows physics to estimate the order of effects, especially when the effect is expected to be very small. Often times a calculation can be greatly simplified by carrying out calculations only to first or second order.

Here is a poor physicists approach to Taylor series. I can expand any continuously differentiable function about a point (call that x_0) by looking at the derivatives of that function:

$$f(x + x_0) \simeq f(x_0) + \frac{1}{1!}\frac{\partial}{\partial x}f(x)|_{x=x_0}(x - x_0) + \frac{1}{2!}\frac{\partial^2}{\partial x^2}f(x)|_{x=x_0}(x - x_0)^2 + \cdots \quad (3)$$

Often one wants to expand around the zero point, which allows one to re-write the above a bit more compactly.

$$f(x) \simeq \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} x^n \quad (4)$$

Why does this matter for special relativity? Relativistic corrections are often of order β or β^2 and often difficult to detect directly since, as seen in problem 1, β is very small for most of the macroscopic object.

(a) Write the Taylor expansion of the following functions for small values of x ($|x| \ll 1$), to the second order. (5 points)
- $f(x) = \sin(x)$
- $f(x) = \cos(x)$
- $f(x) = (1 - x^2)^n$
- $f(x) = (1 - x^2)^{-1/2}$

(b) Let $x = x' - vt'$ and repeat problem (a) for small values of t' ($|t'| \ll 1$). (5 points)

(c) Another important quantity that we will see a lot in this class is the boost factor (traditionally written as γ), where it is defined as $\gamma \equiv 1/\sqrt{1-\beta^2} \equiv 1/\sqrt{1-(\frac{v}{c})^2}$. Taylor expand this function to the second order in β. Using the values of β calculated for Problem #1, find the corresponding value for γ approximated to the second order in β. (5 points)