PROFESSOR: We can go a step forward now. So we're trying to understand these operators. And the next claim is that if you multiply an arbitrary upward permutation operator times S, or you multiply on the other side, it's the same. And in fact, it's just S. On the other hand, if you multiply by a permutation operator A, it is still the same as multiplying A from the right with that permutation operator. But this time you get epsilon alpha 0 times A.

So this is a claim of the simple way in which permutation operators interact with your A and S. So here is, again, the same kind of argument. We'll say it first. So P alpha 0 acting on the list of permutation operators just rearranges the list. We just argue that taking the inverse of the permutation operators just rearranges the list. Multiplying by P alpha 0 the list of all permutation operators-- here you have all the permutation operators. Now you multiply by P alpha 0. And then the claim is that they just go somewhere but you get the same list at the end.

Now that, again, you can prove by the same argument. Show that two elements that are different here go to two elements that are different. And any element here can be written as P alpha 0 of something here. I think you can do that. And it's a repeat of this kind of argument we've been doing.

So P alpha 0 rearranges the list. So that's a first thing. So therefore, let's try one of these. So the first one, P alpha 0 on S. So all we can do is use the formula for S. So you have 1 over N factorial, the sum of. S is the sum over alpha of P alpha. But now we have P alpha 0.

And then we don't really do much at this stage. We say, look, by this claim that P alpha 0 rearranges all the operators, this is just another sum, P beta, of all the operators. This P alpha 0 times P alpha is another operator. But at the end of the job, when you sum over alpha, you get just the list rearranged. So here it is, the list, rearranged. So this is back to S.

For the A case-- let's do it. P alpha 0. We have to work a little more. A is always a little more subtle. 1 over N factorial alpha P alpha 0. But this time I have the epsilon alpha of the A. P alpha 0 P alpha. OK.

To make this clear, I'm going to put the sine factor two times. So we'll have sum over alpha. I'll put epsilon alpha. And then I'll put epsilon alpha 0. Then another epsilon alpha 0. Whatever epsilon alpha 0 is I can do this because I'm inserting just the 1 here. And then we have P

alpha 0 P alpha.

Now look at this operator and look at this sine factor. A few things are clear here. This epsilon alpha 0 has nothing to do with the sum. So this epsilon alpha 0 can go out of the sum. And then you have a sum-- so 1 over N factorial epsilon alpha 0. The sum of this epsilon alpha epsilon alpha 0 P alpha 0 P alpha.

But here, it's clear that if this is some P beta, this sine factor is, in fact, epsilon beta, because if P beta is made by these two permutations of the sine factor of that permutation, it's the product of these two sine factors. This is kind of clear because it's just a number of transpositions again. This is counting the number of transpositions mode two. This is counting the number of transpositions mode two. And this works like that.

If you have two even permutations, the product is an even permutation. And therefore, the two pluses give you a plus. If one is even and one is odd, the product is an odd permutation, and you have a 1 and a minus 1. This product is minus 1. So this is an epsilon beta P beta sum over beta. And that is the antisymmetrizer. So you've got epsilon alpha 0 in here, and the rest is, again, the antisymmetrizer.

So these two properties have been proven. I've proven them when P is from the left. The proof when P is from the right is very similar. Almost no changes.

OK. Little by little, these operators are a little complicated because they have lots of terms. And therefore, manipulating them requires some care. But now we're ready for the main claim about these operators. We can do the arithmetic with them, and finally say that A and S are orthogonal projectors.

So what does that mean? It means that S squared is equal to S, A squared is equal to A, and we'll also show that they're orthogonal to each other in that AS equal SA equals 0 as well. So if you first apply the symmetrization operator to a state and then you apply the antisymmetrization, you get 0.

So once you symmetrize, that symmetric state has no component along the antisymmetric state. They're complementary. So this is really what you want from these operators. They're doing the right job. So this is the last thing we need to prove about them. But by now, we really have worked quite well. So it's going to be simple.

The other statement, of course, of orthogonal projectors also includes the fact that they are Hermitian operators. So we already showed that they are Hermitian, so we're almost there. We just need to prove that these two things happen. So let's see how difficult or easy this is.

So first case, the S squared equal S. S squared is S times S, so I will write it as one S, here, acting on the other S. Now, the next step is just erasing this parenthesis over alpha. And now we have it there. The product of any projector times S is S.

So projector in alpha is just S. So we'll have 1 over N factorial, the sum over alpha of S. But S has no alpha index. So it goes out of the sum. It's equal to S times 1 over N factorial, the sum over alpha of 1. And the sum over alpha of 1 is a 1 added for each element of the permutation group, which is N factorial elements. Therefore, this sum is N factorial. And therefore, this cancels with that N factorial and we get S, as we wanted.

So S squared is S. It is a projector operator. That's good. Let's do the second. A squared is going to be 1 over N factorial sum epsilon alpha P alpha acting on an antisymmetric A. So this is the first A, and there's the second A. I should put the sum over alpha.

But we already calculated P alpha on A. This is sum over alpha epsilon alpha. And P alpha on A was calculated on that board. It's just another epsilon alpha times A. Well, epsilon alpha times epsilon alpha, whether is 1 or minus 1, the product is just plus 1. So you get 1 over N factorial. And again, the sum over alpha of A, the A goes out. Sum over alpha of 1, and we're back to A. So A squared is also A.

Finally, let's do one of those mixed products, three. A multiplied with S. So as usual, we write the first operator explicitly. Epsilon alpha P alpha on S. But P alpha on S, we know already, is just S. So 1 over N factorial sum over alpha epsilon alpha times S.

The S goes out. 1 over N factorial S sum over alpha of epsilon alpha. That's as far as we can simplify this. And then we recall, in any permutation group Sn, the number of even permutations is equal to the number of odd permutations. So there is equal number of pluses and equal number of minuses in that sum.

So this sum is equal to 0. And therefore, this result is 0. The operators are orthogonal, and they will do the job. So good. We have orthogonal operators. And therefore, we can do with these projectors what you would expect we should be able to do.

So here it is, the last statement. S takes V tensor N to Sym N of V. It's a projector to the

symmetric states. Why is that? Well, take a psi belonging to V tensor N. Then consider S on psi. That's the operator, S, acting on the state, psi. And that should be a symmetric state.

And to check that it's symmetric we apply a P alpha on S psi. But P alpha on S is S, so that's just S psi. So the state of psi is independent-- unchanged-- by the action of the P alpha. So the state S psi is, indeed-- this implies that S psi belongs to Sym N V, as claimed.

So the operator does what you wanted it to do. This property helps for the other one. So A is an operator that takes arbitrary states in the tensor product to anti N V. So indeed, if, again, you have some psi in V N, you can then form A psi and see where it lies. For that you apply P alpha on A psi.

But P alpha on A from that middle blackboard is epsilon alpha A psi. But that is precisely the definition of an antisymmetric state. So indeed, A takes states to antisymmetric states.

If you think of the vector space V tensor N here, there's some subspace here, Sym N V, and some subspace here, anti N V. And these two subspaces don't contain any common element. Maybe I should-- well, I don't know. Maybe I should really make them touch at one point. That point would be what vector?

AUDIENCE: 0?

PROFESSOR: 0. Yes. If you have a vector subspace it has to have the 0 vector. So here is the 0 vector. But they don't fill the whole thing. In general, they don't fill the whole thing. When you have two particle states, however, they do.

Remember when we had N equals 2, we had the symmetrizer was 1 plus P 2 1. And A was 1/2 1 minus P 2 1. In that case, the symmetrizer plus the antisymmetrizer was equal to 1. But this is only for two particles. If you have three particles, this will not work. Suppose I get three particles-- maybe I'll use this blackboard.

And the fact that S plus A is equal to the unit operator means something, of course. It means that still for N equals 2, means that any vector can be written as S plus A times the vector, because this is 1. And therefore, it's SV plus AV. So any vector where V belongs to the tensor product, because you have two particles, can be written as a symmetric state plus an antisymmetric state. That is the statement that the symmetric states plus the antisymmetric state span the space.

So that is true for N equals to 2. So in N equals to 2, the two states, the two spaces, span the space. On the other hand, for N equals to 3, remember we had-- and this is a good way to write some things explicitly. If you have N equals to 3, you have the permutation 1 2 3, which is the identity, the permutation 3 1 2, the permutation of 2 3 1. All these were even permutations.

This is original, and this can be built with two permutations. Then you have P 1 3 2, where you flip just 2 and 3. This has one transposition. You can cycle them. 3 2 1. P 2 1 3. These are the even, and here are the odd permutations.

So this was for three particles. Six operators, three even, three odd. So what is the symmetrizer? The symmetrizer, 1 over 3 factorial. And you should sum. So it's 1 over 3 factorial, which is 6. The 1 plus P 3 1 2 plus P 2 3 1. I keep adding. You should add all of them. Plus P 3 2 1 plus P 2 1 3. Boom. That's it. All the ones, you have to add them.

What is A is 1/6. You add the ones that are even. 1 plus P 3 1 2 plus P 2 1 3, and you subtract all the ones that are odd. So you minus P 1 3 2 minus P 3 2 1 minus P 2 1 3. So those are the two operators in case you wanted to see them.

But now notice that A plus S is equal to 1/3 1 plus P 3 1 2 plus P 2 3 1. And that's not the equal to the identity, as it was before. It's equal to something else. So you cannot say that the vector is equal to A plus S times the vector. That's not true here. Doesn't happen.

So there is more to the triple tensor product than the symmetric states and the antisymmetric states. That's a nice subject for a slightly more advanced course in quantum mechanics where you studying the Young tableaux of the permutation group. And while for two things, two objects that are represented with Young tableaux with little squares, you can form something that is called symmetric and antisymmetric, for three objects you can form a symmetric object, you can form an antisymmetric object, but you can form a mixed symmetry object.

And this Young tableaux represent the tensors that enter here. There are other projectors to this kind of object. These objects are not symmetric nor antisymmetric. They have partial symmetry and partial antisymmetry. And they take a while to get around to visualize them.

But that's why the permutation group is interesting and why we try to understand it better and better. So that's basically the mathematics that we need for understanding permutations. So we're going to do some of the physics that follows from this now.