PROFESSOR: We spoke last time of the existence of symmetric states. And for that we were referring to states psi S that belonged to the M particle Hilbert space. And V is the vector space that applies to the states of the M particles.

And we constructed some states. So within the construction, we postulated, there will be some states that [AUDIO OUT] that meant that any permutation of [AUDIO OUT] state will be for the symmetric state or all alpha. So that meant the state was symmetric.

Whatever you apply to it, any permutation, the state would be invariant. So these are special states, if they exist, in VN. Then there would be anti-symmetric states. And we concluded that an anti-symmetric state-- with an A for anti-symmetric-- also stayed in this tensor product.

That state would react differently to the permutation operators. It would change up to a sine epsilon alpha times psi A. It would be an eigen state of all those permutation operators. But with eigenvalue epsilon and alpha where epsilon alpha was equal to plus 1 if P alpha is an even permutation. Or minus 1 if P alpha is an odd permutation.

So whether this permutation is even or odd, we also discussed depends on whether it's built with an even or odd number of transpositions. With transpositions being permutations in which one state is flipped for another state. Within the end states, you pick two, and these two are flipped, that's a transposition.

All permutations can be built through transpositions-- with transpositions-- and therefore you can tell from a permutation whether it's an even or an odd one depending of whether it's built with even number of transpositions or odd number of transpositions. Now, the number of transpositions you need to build the permutation is not fixed. If we say it's even, means it's even mod 2.

So you might have a permutation is built with 2 transpositions. And also somebody else can write it as 4 transpositions and 6 transpositions. It just doesn't matter. So a few facts that we learned about these things are that all the P alphas are unitary operators.

And we also learned that all transpositions are Hermitian operators. Now, transpositions, of course, are permutations. So they're Hermitian, and they're unitary as well. And finally, we learned that the number of even permutations is equal to the number of odd permutations in any permutation group of n objects.

So these were some of the facts we learned last time already. We have now that symmetric states form a subspace. If you have two symmetric states, you can multiply a state by a number, it will still be symmetric. If you add two symmetric states, will still be symmetric. So symmetric states form a subspace of the full vector space V tensor N.

And anti-symmetric states also form a subspace of the vector space, V tensor N. So let's write these facts. So symmetric states form the subspace, and it's called sym N of V of V tensor N. Anti-symmetric states form the subspace anti N V of V tensor N.

All right. So these are the states. But we have not learned how to build them, how to find them, and even more, what to do with them. So main thing is if this form subspaces, there should be a way to write the projector that takes from your big space down to the subspace. So here is the claim that we have.

Here are two operators. We'll call this S, a symmetrizer And S will be billed as 1 over N factorial, the sum over alpha of all the permutations, P alpha. That's the definition.

And then we'll have an anti-symmetrizer, A. This is also 1 over N factorial, sum over alpha. Epsilon alpha P alpha. Where epsilon alpha, again, is that sine factor for each permutation.

So here are two operators, and we're going to try to prove that these operators are orthogonal projectors that take you to the subspace's symmetric states and anti-symmetric states. So that is our first goal. Understanding that these operators do the job. And then we'll see what we can do with them.

So there are several things that we have to understand about these operators. If they are projectors, they should square to themselves. So S times S should be equal to S. Remember, that's the main equation of a projector. A projector is P squared equals P.

And the second thing, they should be Hermitian. So let's try each of them. So the first claim is that S and A are Hermitian. In particular, S dagger equal S. And A dagger equal A.

So if you want to prove something like that, it's not completely obvious at first sight that those statements are true. Because we saw, for example, that transpositions are Hermitian operators. But the general permutation operator is not Hermitian. It's unitary.

So it's not so obvious. You cannot just say each operator is Hermitian, and it just works out. It's a little more complicated than that. But it's not extremely more complicated than that.

So let's think of the following statement. I claim that if you have the list of all the permutation operators. Put the list in front of you. All of them. And you apply Hermitian conjugation to that whole list, you get another list of operators. And it will be just the same list scrambled. But you will get the same list. It will contain all of them.

And that is kind of obvious if you think about it a little more. You have here, for example, the list of all the permutations. And here, you apply Hermitian conjugation, HC or dagger. And you get another list. And this list is the same as this one although reordered. In a sense, it permutes the permutation operators, if you wish.

And the reason is clear. If you have two operators here, and you apply-- they are different-- and you apply Hermitian conjugation, it should give two different operators here. Because if they were equal, you could apply again Hermitian operator conjugation. And you would say, oh, they're equal. But you assume they were different. So two different operators go to two different operators here.

Moreover, any operator here-- you can call it O-- is really equal to O dagger. O dagger. Hermitian conjugation twice gives you back. So any operator here is the Hermitian conjugation. Hermitian conjugate of some operator there. So every operator here will end up at the end of an arrow coming here. Maybe those arrows, we could put them like they go in all directions.

But the fact is that when you take Hermitian conjugation of all the list, you get all the list back. Therefore, that takes care of the symmetrizer. When I take the Hermitian conjugation of P alpha, I might get another P beta. But at the end of the day, the whole sum will become again the whole sum.

So that proves the fact that this map is 1 to 1 The map Hermitian conjugation is 1 to 1. 1 and surjective meaning that every element is reached as well. Means that S dagger is equal to S. Because the list doesn't change.

Here is a slightly more subtle point. This time, the list is going to change. But then maybe when this changes to another operator, maybe the epsilon is not the right epsilon. You would have to worry about that.

But that's no worry either. That also works out clearly. Because of the following thing. If a permutation P alpha is even, it's billed with an even number of transpositions. If you take the Hermitian conjugation of an even number of transpositions, you'll get an even number of transpositions in the other order.

That's what Hermitian conjugation is. Therefore, Hermitian conjugation doesn't change the fact that the permutation is even or is odd. So if P alpha has a sine factor E of P alpha. Let's write it more explicitly. And P alpha dagger has as a sine factor E of P alpha dagger. These two are the same.

If P alpha is even, P alpha dagger is even. If P alpha is odd, P alpha is odd. Therefore, when you take the Hermitian conjugate of this thing, you may map. You think the Hermitian conjugate of this sum. You have here a dagger would be 1 over N factorial, sum over alpha, epsilon alpha, P alpha dagger, which is another permutation.

But this permutation. For this permutation, this sine factor is the correct sine factor. Because the sine factor for P alpha dagger is the same as the sine factor of E alpha. Since we get the whole sum by the statement before, A is also Hermitian.

So OK. Not completely obvious, but that's a fact. If you think more abstractly, this is something you could know on a group. If you have a group that has all kinds of elements. Many of you have written papers in groups in your essays. Then if you take every group element, and you replace it by its inverse, you get the same list of group elements. It's just scrambled.

And remember, since this permutations are unitary, taking Hermitian conjugation is the same thing as taking an inverse. So when you take a group, and you take the inverse of every element, you get back the list of the elements of the group. And that's what's happening here.