PROFESSOR: This definition in which the uncertainty of the permission operator Q in the state psi. It's always important to have a state associated with measuring the uncertainty. Because the uncertainty will be different in different states. So the state should always be there. Sometimes we write it, sometimes we get a little tired of writing it and we don't write it. But it's always implicit.

So here it is. From the analogous discussion of random variables, we were led to this definition, in which we would have the expectation value of the square of the operator minus the square of the expectation value. This was always-- well, this is always a positive quantity. Because, as claim 1 goes, it can be rewritten as the expectation value of the square of the difference between the operator and its expectation value.

This may seem a little strange. You're subtracting from an operator a number, but we know that numbers can be thought as operators as well. Operator of minus a number acting on a state is well defined. The operator acts on the state, the number multiplies a state. So this is well defined. And claim 1 is proven by direct computation.

You certainly indeed prove. You can expand what is inside the expectation value, so it's Q hat squared. And then the double product of this Q hat and this number. Now, the number and Q hat commute, so it is really the double product. If you have A plus B times A plus B, you have AB plus BA, but if they commute it's 2AB, so this is minus 2 Q hat Q. Like that.

And then, the last term is the number squared, so it's plus Q squared. And sometimes I don't put the hats as well. And all this is the expectation value of the sum of all these things. The expectation value of a sum of things is the expectation value of the first plus the expectation value of the second, plus the expectation value of the next. So we can go ahead and do this, and this is therefore expectation value of Q squared minus the expectation value of this whole thing.

But now the expectation value of a number times an operator, the number can go out. And this is a number, and this is a number. So it's minus 2 expectation value of Q, number went out. And then you're left with expectation value of another Q. And the expectation value of a number is just the number, because then you're left within the world of psi star psi, which is equal to 1. So here is plus Q hat squared.

And these two terms, the second and the third, are the same really. They are both equal to expectation value of Q squared. They cancel a little bit, and they give you this. So indeed, this is equal to expectation value of Q squared minus expectation value of Q squared.

So claim 1 is true. And claim 1 shows in particular that this number, delta Q squared, in the expectation value of a square of something, is positive. We'll see more clearly in a second when we have claim number 2. And claim number 2 is easily proven. That's another expression for uncertainty.

For claim number 2, we will start with the expectation value of Q minus Q squared, like this, which is the integral dx psi star of x and t, Q minus expectation value of Q, Q minus expectation value of Q, on psi.

The expectation value of this thing squared is psi star, the operator, and this. And now, think of this as an operator acting on all of that. This is a Hermitian operator. Because Q hat is Hermitian, and expectation value of Q is real. So actually this real number multiplying something can be moved from the wave function to the starred wave function without any cost.

So even though you might not think of a real number as a Hermitian operator, it is. And therefore this whole thing is Hermitian. So it can be written as dx. And now you have this whole operator, Q minus Q hat, acting on psi of x and t. And conjugate. Remember, the operator, the Hermitian operator, moves to act on psi, and the whole thing [INAUDIBLE]. And then we have here the other term left over.

But now, you see that you have whatever that state is and the state complex conjugated. So that is equal to this integral. This is the integral dx of the norm squared of Q hat minus Q hat psi of x and t squared, which means that thing, that's its complex conjugate.

So this completes our verification that these claims are true, and allow us to do the last step on this analysis, which is to show that if you have an eigenstate of Q, if a state psi is an eigenstate of Q, there is no uncertainty. This goes along with our measurement postulate that says an eigenstate of Q, you measure Q and you get the eigenvalue of Q and there's no uncertainty. In particular, we'll do it here I think.

If psi is an eigenstate of Q, so you'll have Q psi equal lambda psi, where lambda is the eigenvalue.

Now, this is a nice thing. It's stating that the state psi is an eigenstate of Q and this is the eigenvalue, but there is a little bit more than can be said. And it is. It should not surprise you that the eigenvalue happens to be the expectation value of Q on the state psi. Why? Because you can take this equation and integrate dx times psi star. If you bring that in into both sides of the equation then you have Q psi equals integral dx psi star psi, and the lambda goes up.

Since my assumption whenever you do expectation values, your states are normalized, this is just lambda. And by definition, this is the expectation value of Q. So lambda happens to be equal to the expectation value of Q, so sometimes we can say that this equation really implies that Q hat psi is equal to expectation value of Q psi times psi.

It looks a little strange in this form. Very few people write it in this form, but it's important to recognize that the eigenvalue is nothing else but the expectation value of the operator of that state. But if you recognize that, you realize that the state satisfies precisely Q hat minus Q on psi is equal to 0. Therefore, if Q hat minus Q on psi is equal to 0, delta Q is equal to 0. By claim 2. Q hat minus Q expectation value kills the state, and therefore this is 0. OK then.

The other way is also true. If delta Q is equal to 0, by claim 2, this integral is 0. And since it's the sum of squares that are always positive, this state must be 0 by claim 2. And you get that Q minus Q hat psi is equal to 0. And this means that psi is an eigenstate of Q.

So the other way around it also works. So the final conclusion is delta Q is equal to 0 is completely equivalent of-- I'll put in the psi. Psi is an eigenstate of Q.

So this is the main conclusion. Also, we learned some computational tricks. Remember you have to compute an expectation value of a number, uncertainty, you have these various formulas you can use. You could use the first definition. Sometimes it may be the simplest. In particular, if the expectation value of Q is simple, it's the easiest way.

So for example, you can have a Gaussian wave function, and people ask you, what is delta of x of the Gaussian wave function? Well, on this Gaussian wave function, you could say that delta x squared is the expectation value of x squared minus the expectation value of x squared.

What is the expectation value of x? Well, it would seem reasonable that the expectation value of x is 0. It's a Gaussian centered at the origin. And it's true. For a Gaussian it would be 0, the expectation value of x. So this term is 0. You can also see 0 because of the integral. You're integrating x against psi squared. Psi squared is even, x is odd with respect to x going to minus x. So that integral is going to be 0. So in this case, the uncertainty is just the calculation of the expectation value of x squared, and that's easily done. It's a Gaussian integral.

The other good thing about this is that even though we have not proven the uncertainty principle in all generality. We've only [? multivated ?] it. It's precise with this definition. So when you have the delta x, delta p is greater than or equal to h bar over 2, these things are computed with those definitions. And then it's precise. It's a mathematically rigorous result. It's not just hand waving. The hand waving is good. But the precise result is more powerful.