BARTON ZWIEBACH: Scattering in dimension. And we will consider a world that is just one dimensional, x. And, in fact, there's an infinite barrier at x equals 0. Infinite barrier, nothing goes beyond there. On the other hand, in here, up to some distance R, there could be a potential V of x. So we will have a potential V of x, it will have the following properties-- it will be identically 0 for x greater than R, which is the range of the potential; it will be some function V of x for x in between R and 0; and it will be infinity for x less than or equal than 0.

This is your potential, it's a potential of range R-- range of the potential. And the experiment that we think about is somebody at x equal plus infinity throwing waves into this potential. And this observer can only get back a reflected wave, and from that reflected wave, the observer wants to deduce the type of the potential that you have there.

And that's absolutely the way physics goes in particle physics. In LHC, you throw protons or electrons together and you just catch what flies out of the collision with all the detectors and read that they then deduce what happened to the collision, what potential was there, what forces were there, was there a new particle? It's all found by looking at what comes out and flies away. So there's enormous amount of information on the potential from the data that comes out of you throwing in some particles in and waiting to see what comes back to you.

So we will always have this infinite wall. And this infinite wall at x equals 0 means that x less than 0 is never relevant. And this is analogous to R, the variable R in radial coordinates for which the radial distance is never negative. So in fact, what we'll do here has immediate applications when we will consider-- not in this course-- scattering in three dimensions.

So to begin this, we'll solve the simplest case where you have no potential whatsoever. Now no potential means still the barrier at x equals 0, the infinite barrier, but in between 0 and R, nothing is happening. So you have just a case of no potential. Is the case where you have the barrier here and x is over there and up to R, nothing's happening, it's just the wall. That's all there is, just one wall. So this is V-- no potential is V is equal to V of x is equal 0 for x greater than 0. And it's infinity for x less or equals than 0.

So in this case, let's assume we have an incident wave. An incident wave must be propagating in this way, so an incident wave is an e to the minus ikx. And if you have an outgoing wave, it would be some sort of e to the ikx. These are the only things that can be there. They correspond to energy eigenstates, this is the de Broglie wave function of a particle with momentum, in one direction or in the other direction.

But let's combine them in a way to produce a simple solution. So this solution, phi of x, will be the solution. Will be a combination that's similar-- e to the ikx and e to the minus ikx, and I should make the wave function vanish at x equals 0. At x equals 0, both exponentials are equal to 1, so if I want them to cancel, I should put a minus.

So in order to simplify this the best possible way, we can put a 1 over to 2i's over there so that we have a sine function, and the sine function is particularly nice. So we'll have e to the-- output like this-- e to the minus ikx plus e to the ikx over 2i, and this is just the sine function side of kx, which you would admit, it's a V solution over here, a sine of kx. On the other hand, I can think of this solution as having an incoming wave, which is minus e to the minus ikx over 2i, and an outgoing wave of e to the ikx over 2i.

So this is the representation of the solution when nothing is happening, and the good thing about this solution is that it tells us what we should write-- gives us an idea of what we should write when something really is happening. So now let's consider how we would write the general experiment in which you send in a wave but this time, there is really a potential.

So let's consider now, if no potential was there and now yes, potential-- so no potential here, so what does it mean yes potential? Well, it means you have this and you have some potential there up to some distance R, and then it flattens out, and something happens. So in order to compare, we'll take an incoming wave, the same as the one where there was no potential. So let's take an incoming wave, which is of the form e to the minus ikx times minus over 2i. But I must say here, I must write something more-- I must say that x is greater than R, otherwise this is not the solution.

You see, in the region where the potential really exists, where the-- goes up and down, you don't know the solution. It would take solving the Schrodinger equation. You know the solution where the potential is 0, so yes, this incoming wave is the solution of the Schrodinger equation in this potential when x is greater than R.

And how about the outgoing wave? Well, we would like to write it like that. So we'll say 1 over 2i e to the ikx is also an outgoing wave, and we have no hope of solving it here, finding what's happening here unless we solve a complicated equation, but then let's look outside-- we're still looking outside. But that cannot be the outgoing wave. This is the same as the other one and there is a potential, so something must be different. On the other hand, if you think about it, very little can be different because you must have a solution with 0 potential and-- you know these plane waves going out are the only things that exist.

And now you decide, oh, if that's the case, I cannot put another function of x in there because that's not a solution. The best I can do is multiply by a number, because maybe there's very little outgoing wave or there is not, but then I think of another thing-- remember if you had e to the A, e to the minus ikx plus B e to the ikx, well, the probability current was proportional to A squared minus B squared.

And this time, however, you have-- you're sending in a wave and you're getting back a wave and this is a stationary state-- we're trying to get energy eigenstate, solutions of some energy just like this energy. And the only way it can happen is if they carry the same amount of probability-- probability cannot be accumulating here, nor it can be depleted there as well, so the currents associated to the two waves must be the same. And the currents are proportional to those numbers that multiply these things squared, so in fact, A squared must be equal to B squared, and therefore we cannot have like a 1/3 here, it would just ruin everything.

So the only thing I can have is a phase. It's only thing-- cannot depend on x, because that was an unsolved equation. Cannot be a number that is less than 1 or bigger than 1. The only thing we can put here is a phase.

So we'll put an e to the 2i delta. And this delta will depend on k or will depend on the energy, and it will depend on what your potential is, but all the information of this thing is in this delta that depends on k. And you say, well, that's very little, you just have one phase, one number that you could calculate and see, but remember, if you have a delta of k, you could measure it for all values of k by sending particles of different energies and get now a whole function. And with a whole function delta of k, you have some probability of getting important information about the potential.

So we'll have a phase there, e to the i delta of k. And let's summarize here, it's due to current conservation-- the current of this wave and the current of the outgoing waves should be the same. And also note that no extra x dependents is allowed. So this will produce the J incident will be equal to J reflected.

Now you could say, OK, very good, so there's delta, there's a phase-- should I define it from 0 to 2 pi? From minus pi to pi? It's kind of natural to define it from minus pi to pi, and you could look at what it is, but as we will see from another theorem, Levinson's theorem, it will be convenient to just simply say, OK, you fixed the phase delta at k equals 0. At 0 energy scattering, you read what is your delta-- unless you increase the energy, the phase will change.

So if you have a phase, for example, on a circle, and the phase starts to grow and to grow and to grow and to grow and to go here, well, should you call this pi and this minus pi? No. You probably should just-- if it keeps growing with energy-- and it might happen, that the phase keeps growing with energy-- well, pi, 2 pi, 3 pi, 4 pi, just keep the phase continuous. So keeping the phase continuous is probably the best way to think about the phase. You start at some value of the phase and then track it continuously. There is always a problem with phases and angles, they can be pi or minus pi's the same angle, but try for continuity in defining the phase when we'll face that problem.

So let's write the solution. We have this, so the total solution. We call the solution with no potential phi, this one we'll call psi of x will be 1 over 2i, the first term-- e to the ikx plus 2i delta minus e to the minus ikx. It's convenient to pull out of i delta to make the two terms have opposite arguments, so e to the ikx plus i delta, and-- or e to the ikx plus delta parenthesis minus e to the minus ikx plus delta. So this is e to the i delta times sine of kx plus delta.

So that's this full scattered wave, not the full reflected-- well, that word again. This is the full wave that you have for x greater than R. So let's write it here-- psi of x. It's not the reflected wave nor that it covers everything. We include-- it's for x greater than R, but we include the incoming and outgoing things, because both are defined for x greater than R, so the total wave is this one. And you notice that if the phase shift is 0, you are nicely back to the wave function phi that we found before.