PROFESSOR: Last time we discussed the differential equation. I'll be posting notes very soon. Probably this afternoon, at some time. And last time, we solved the differential equation, we found the energy eigenstates, and then turned into an algebraic analysis in which we factorized the Hamiltonian.

Which meant, essentially, that you could write the Hamiltonian-- up to an overall constant that doesn't complicate matters-- as the product of an a dagger a. And that was very useful to show, for example, that any energy eigenstate would have to have energy greater than h omega over 2. We call this a dagger a the number operator n, which is a Hermitian operator.

Recall that the dagger of a product of operators is the reverse order product of the daggered operators. So the dagger of a dagger a is itself.

And then a was related to x and p, and so was a dagger. Recall that x and p are Hermitian. And there are overall constants here that were wrote last time, but now they're not that urgent. And a and a dagger, the commutator is equal to one. That was very useful.

Finally, we also show that while the energy of any state would have to be greater than h omega over 2, if you had a state that is killed by a hat, it would have the lowest allowed energy-- which is h omega over 2. And, therefore, that is the ground state.

And we looked at this differential equation, and we found this Gaussian wave function. And it's a first order differential equation. And, therefore, it has just one solution. And, therefore, there is just one ground state, and it's a bound state. And, of course, you wouldn't expect more than one ground state, because there's no degeneracies in the bound state spectrum of a one-dimensional potential.

So we found one ground state was phi 0, and it's killed by a-- which means that it's killed by n-hat, because a is to the right in n-hat. So the a finds phi 0 and just kills it.

Now, the other thing to note is that the Hamiltonian is really, pretty much, the same thing as the number operator multiplied by something with units of energy. The number operator has no units, because a and a dagger have no units. And that's very useful. So it's like a dimensionless version of the energy.

And, certainly, if you have an eigenstate of h it must be an eigenstate of n. And the eigenvalue of n-- if we call it capital n. Therefore, you can imagine this equation acting on an eigenstate-- which happens to be an eigenstate of n or of h. On the left-hand side you would read the energy, and on the right-hand side you would read the eigenvalue of the n operator.

So that gives you a very nice simple expression. You see that the energy is the number plus a 1/2 multiplied by h-bar [INAUDIBLE].

So that's pretty much the content of what we reached last time. And now we have to complete the solution. And the plan for today is to complete the solution, familiarize ourselves with these operators, learn how to work with a harmonic oscillator with them. And then we'll leave the harmonic oscillator for the time being-- let you do some exercises with it-- but turn to scattering states. So the second part of today's lecture we'll be talking about scattering states.

OK. So when we look at this thing and you have a number operator-- which encodes the Hamiltonian-- it's a good idea to try to understand how it interacts with the other operators that you have here. And a good question, whenever you have operators, is the commutator. So you can ask, what is the commutator of n with a?

And this commutator is going to show up. But it's basically that kind of thing. If you have a and a dagger, you ask, what is the commutator? If you have n, you ask, what is the commutator with the other thing?

So n with a would be the commutator of a dagger with a-- a like that. And sometimes I will not write the hats to write things more quickly. Now, in this commutator, you can move the a out, and you have a dagger a a. And a dagger a is minus 1. Because aa dagger is 1. So this is minus a.

So that's pretty nice. It's simple. How about n with a dagger? Well, this would be a dagger a with a dagger. A dagger with a dagger commute. So this a dagger can go out, and you get aa dagger. And that's 1, so you get a dagger.

So it's a nice kind of computation relation. You would have commutation reserve x with p given a constant. Now n with a gives a number times a. N commutated with a dagger gives a number times a dagger. And those numbers are pretty significant, so I'll write this again. N with a is minus a. And n with a dagger is plus a dagger.

This is part of the reason-- as we will see soon-- that the name of a-- which we call destruction operator, because it destroys the vacuum-- it's sometimes called lowering operator, because it comes with a negative sign here. And we'll see a better reason for that name. A dagger is sometimes called the creation operator or the raising operator, because it increases some number, as you will see. And here it's reflected by these plots.

But we need a little more than that. We need a little more commutators than this. So for example, if I would have the commutator of a with a dagger to the k-- you can imagine this. You have to become very used and very comfortable with these commutation relations. And sometimes the only way to do that is to just do examples. So I'm doing this with a k here.

Maybe-- when you review this lecture-- you should do it with k equals 2 or with k equals 3, and do it a few times. Until you're comfortable with these things, and you know what identities you've been using. If this was a little quick, then go more slowly and make absolutely sure you know how to do those commutators.

In here, I'm going to say what happens. You have an a and you have to move it across a string of a-hats. Now, moving an a cross an a-hat-- because of the commutator-- gives you a factor of 1, but it destroys the a and the a-hat. As you move the a across the a-hats-- because this is a with all the a-hats, here, minus the a-hats times the a there.

So if you could just move the a all the way across-- then you cancel with this. What you get is what happens when you're moving it to across. And you're moving across a string of those. So each time you try to move on a across an a-hat, you get this factor of 1 and you kill the a and you kill one a dagger.

So this answer will not have an a, and it will have one less a dagger. So a dagger to the k, minus 1. And then I would argue-- and you should do it more slowly-- that you have to go across k of those. And each time you get a factor of 1, and you lose the a and the a dagger. So at the end you get a k.

You should realize that this is not all that different from the kind of commutators you had. Like, with x to the n. This was very similar-- it might be a good time to review how that was done-- in which that pretty much gives you an x to the n minus 1, times a factor of n, because p is a derivative. You could almost think of a as the derivative with respect to a dagger. And then this commutator would be 1.

So this is true. And there is also-- if you want-- an a dagger with a to the n or a to the k. This would give you-- if you had just one of them you would get a minus sign, because a dagger with a is that. But the same thing holds, you're going to get one less a-hat.

So a-hat to the k minus 1. A factor of k-- because k times you're going to move an a dagger across an a. And a minus because you're getting a dagger commutator with a, as opposed to a commutator with a dagger-- which is 1.

So these are two very nice and useful equations that you should be comfortable with. Now, this implies that you can do more with an n operator. So n with a-hat to the k, this time will be minus k a-hat to the k. It doesn't change the number of a-hats, because you're now making commutators with a dagger a. So each time you have this commuted with one a-hat, the a dagger and the a give you 1, but you have another a back. So the power is the same.

The sign comes from this sign.

AUDIENCE: Shouldn't the n there have a hat?

PROFESSOR: Yes, it should have a hat. I'm sorry. Yes. And, similarly, n a-hat dagger to the k. This is k a-hat dagger to the k. So what happened before, that n-hat operator leaves the a same but puts a number-- leaves the a dagger the same and puts a number. Here, you see it happening again.

N with a collection-- with a string of a-hats-- gives you the same string, but the number. And with a collection of a daggers gives you the same collection of a daggers with a number. And the number happens to be the number of a's or the number of a daggers. So that's the reason it's called the number operator, because the eigenvalues are the number of creation operators or the number of destruction operators.

I was a little glib by calling it the eigenvalue. But it almost looks like an eigenvalue equation, which have an operator, another operator, and a number times the second operator. It is not exactly an eigenvalue equation, though, because with eigenvalues you would just have this acting on the second one.

But the fact that this case appear here are the reason these are number operators. So it was a little quick for many of you. Some of you may have seen this before. It was a little slow, but the important thing is after a couple of days from now, or by Friday, you find all this very straightforward.