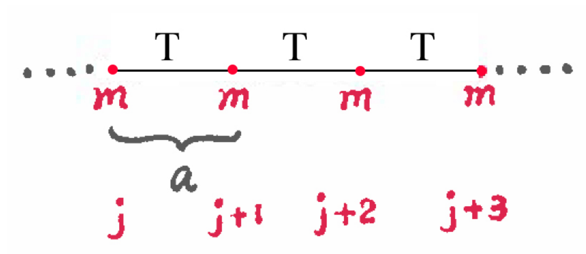
8.03 Lecture 16

We have discussed this system in lecture 8:



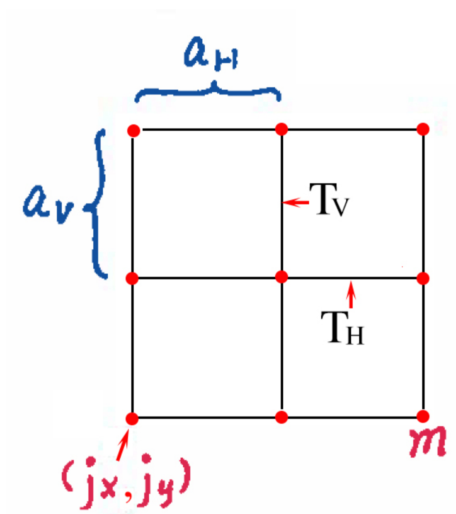
Mass can only move up and down in the $\hat{y}$ direction. We have solved it by "space translation symmetry." We obtained the dispersion relation:

$$\omega(k) = \frac{T}{ma} \sin\left(\frac{ka}{2}\right)$$

Where T is string tension, m is mass, a is the distance between masses at equilibrium. Eigenvectors (where j is the label of the mass):

$$e^{ikj \cdot a}$$

Today we are doing 2D and 3D system!! In general, we don't know how to solve those systems! :(But we know how to solve highly symmetric systems!! If we consider an intinitely long array of masses:



Where m is the mass, T_V, T_H are the tensions, and we have ideal strings. Particles can only move in the $\hat{z}$ direction. Good news: space translation symmetry! Eigenvectors:

$$e^{ik_x x} e^{ik_y y}$$

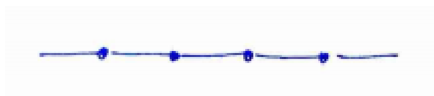
Where $x \equiv j_x a_H$ and $y = j_y a_V$ and (j_x, j_y) are indices.

$$\Rightarrow \quad \psi(x, y) = Ae^{ik_x x} e^{ik_y y} = Ae^{i\vec{k} \cdot \vec{r}}$$

We can use the expression above to get the dispersion relation:

$$\omega^2 = \frac{4T_H}{ma_h}\sin^2\left(\frac{k_x a_H}{2}\right) + \frac{4T_V}{ma_V}\sin^2\left(\frac{k_y a_V}{2}\right)$$

This is a dispersive medium because $\frac{\omega}{|\vec{k}|}$ is not a constant.
At fixed ω: If we consider a 1D bead-string system:



There are two solutions (or eigenvectors of S matrix) which gives angular frequency ω

$$e^{ikx} \text{ and } e^{-ikx}$$

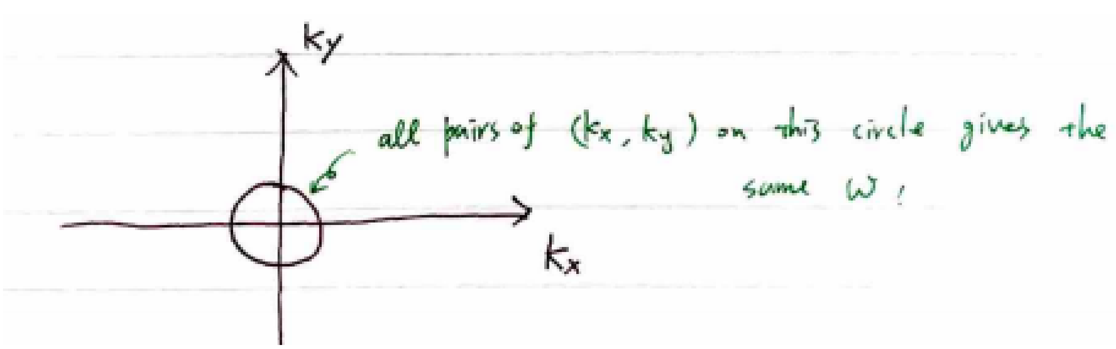
This is $\cos(kx)$ and $\sin(kx)$!!

$$\cos(kx) = \frac{1}{2}(e^{ikx} + e^{-ikx})$$
$$\sin(kx) = \frac{1}{2i}(e^{ikx} - e^{-ikx})$$

We know from the discussion above, the eigenvector of $M^{-1}k$ matrix is sin or cos. Back to the two-dimensional case: If we fix the angular frequency to be ω. There are multiple values of k_x and k_y which can give the same ω (actually infinite number of choices). This is because k_x and k_y are continuous: can be any value before we introduce boundary conditions. If we lower k_x a bit we can increase k_y to compensate! Example: if I have dispersion relation of this form:

$$\omega^2 = 5\sin^2 k_x + 5\sin^2 k_y$$

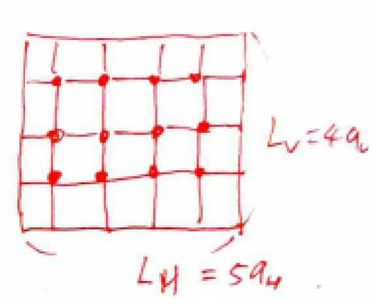
There are many possible pairs of k_x and k_y which gives the same ω!!!



Now we add the wall back in:

$$\psi(0, y, t) = \psi(L_H, y, t) = \psi(x, 0, t) = \psi(x, L_V, t) = 0$$

In this example: $L_H = 5a_H$ and $L_V = 4a_V$



There are now only four modes of the finite system with the same ω

$$Ae^{\pm ik_x x}e^{\pm ik_y y}$$

$$k_x = \frac{n_x \pi}{L_H} \qquad k_y = \frac{n_y \pi}{L_V}$$

$$L_H = 5a_H \qquad L_V = 4a_V$$

and n_x runs from 1 to 4 while n_y runs from 1 to 3. Linear combinations of

$$e^{+ik_x x}e^{+ik_y y}\ ,\ e^{+ik_x x}e^{-ik_y y}\ ,\ e^{-ik_x x}e^{+ik_y y}\ ,\ e^{-ik_x x}e^{-ik_y y}$$

gives $A \sin k_x x \sin k_y y$ which satisfy the boundary conditions.

$$\Rightarrow\ \psi_{(n_x,n_y)}(x,y,t) = A_{(n_x,n_y)} \sin\left(\frac{n_x \pi x}{L_H}\right) \sin\left(\frac{n_y \pi y}{L_V}\right)$$

Discrete case general solution:

$$\psi(x,y,t) = \sum_{n_x,n_y} A_{(n_x,n_y)} \sin\left(\frac{n_x \pi x}{L_H}\right) \sin\left(\frac{n_y \pi y}{L_V}\right)$$

Continuous case (assuming $T_H = T_V = T$) $a_H = a_V \to 0$

$$\omega^2 = \frac{4T}{ma}\frac{k_x^2 a^2}{4} + \frac{4T}{ma}\frac{k_y^2 a^2}{4}$$
$$= \frac{Ta}{m}(k_x^2 + k_y^2)$$

Define the surface mass density, $\rho = m/a^2$, and the surface tension, $T_s = T/a$

$$\omega^2 = \frac{T_s}{\rho_s}(k_x^2 + k_y^2) = \frac{T_s}{\rho_s}|\vec{k}|^2$$

Similar to one dimensional case. Continuous limit gives:

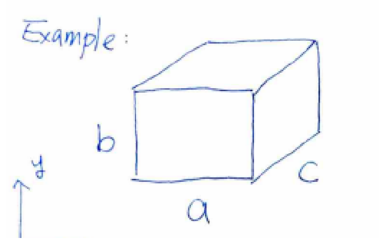
$$\frac{\partial^2}{\partial t^2}\psi(x,y,t) = v^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi(x,y,t)$$
$$= v^2 \nabla^2 \psi(x,y,t)$$
$$\Rightarrow\ \frac{\partial^2}{\partial t^2}\psi(x,y,t) = v^2 \nabla^2 \psi(x,y,t)$$

$$\psi \propto A \sin(k_x x)\sin(k_y y)\sin(\omega t + \phi)$$

Where $v = \sqrt{T_s/rho_s}$. Similarly in the 3D case:

$$\frac{\partial^2}{\partial t^2}\psi(x,y,z,t) = v^2\nabla^2\psi(x,y,z,t)$$

Continuous case: 3D sound wave. Example: sound wave in a box



$$k_x = \frac{n_x\pi}{a} \qquad k_y = \frac{n_y\pi}{b} \qquad k_z = \frac{n_z\pi}{c}$$

Guess

$$\vec{\psi} \propto \sin(k_x x)\sin(k_y y)\sin(k_x x)\sin(\omega t + \phi)$$

Plug into wave equation:

$$\begin{aligned}\omega^2 &= v^2(k_x^2 + k_y^2 + k_z^2)\\ &= v^2\left(\left(\frac{n_x\pi}{a}\right)^2 + \left(\frac{n_y\pi}{b}\right)^2 + \left(\frac{n_z\pi}{c}\right)^2\right)\end{aligned}$$

Where n_x,n_y, and n_z are integers.

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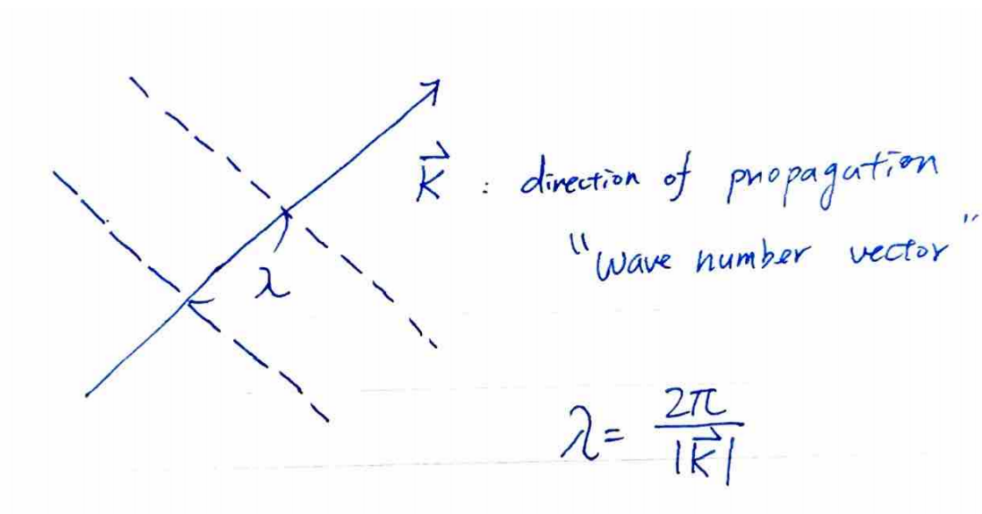
2 and 3D progressive wave:
Simple example: "plane waves"

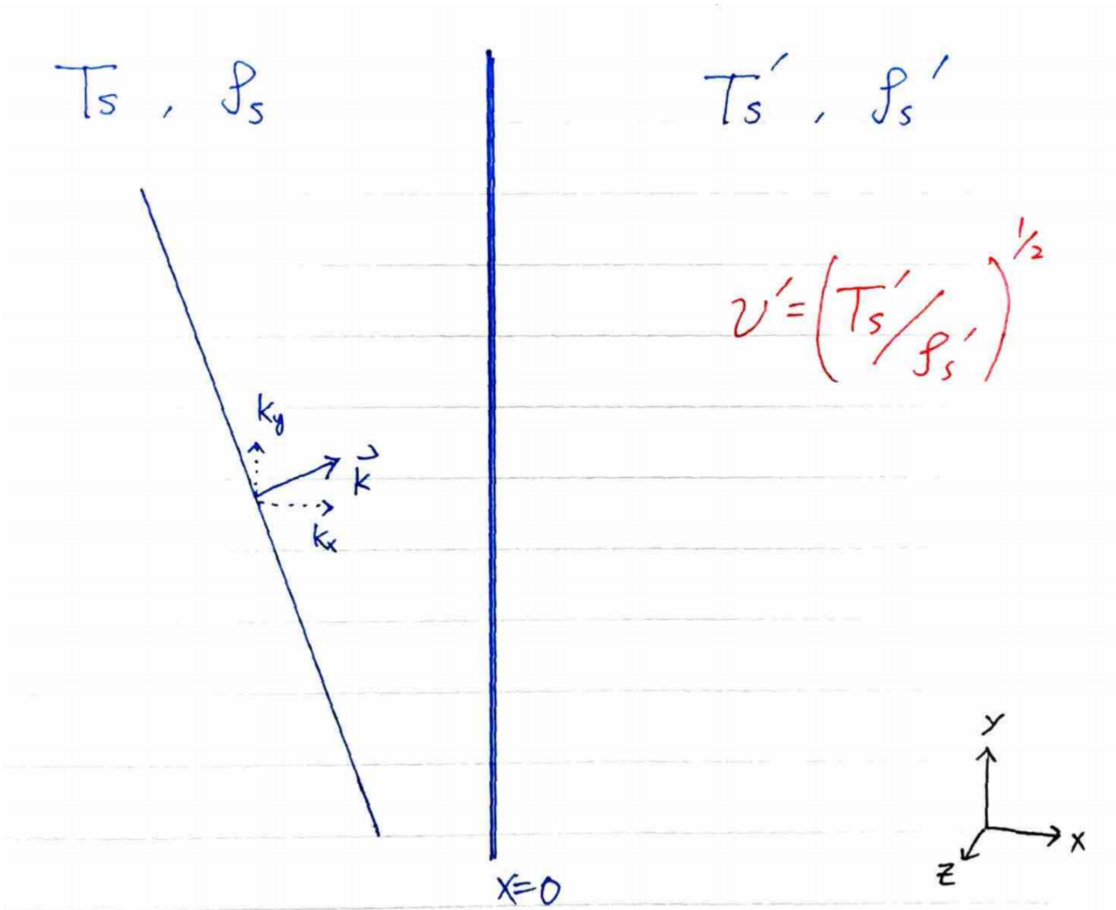
$$\psi(\vec{r},t) = Ae^{i(\vec{k}\cdot\vec{r}-\omega t)}$$



This can be used to describe EM waves, sound waves, or waves on membranes. If there is no other medium, this wave will continue forever.
Let's continue a 2D membrane stretched in the $z = 0$ plane with surface mass density ρ_s and surface tension T_s

$$\omega^2 = v^2(k_x^2 + k_y^2)$$

and waves will travel at speed $v = \sqrt{\dfrac{T_s}{\rho_s}}$. To add some excitement:



We place a second membrane on the other side, and our wave approaches this membrane. What will happen? One would usually expect an incident wave to produce a reflected and transmitted wave.

$$\psi_L = A\underbrace{e^{i(\vec{k}\cdot\vec{r}-\omega t)}}_{\text{Incident}} + \sum_\alpha R_\alpha A\underbrace{e^{i(\vec{k}_\alpha\cdot\vec{r}-\omega t)}}_{\text{Reflected}} \qquad (x \le 0)$$

$$\psi_R = \sum_\beta T_\beta A\underbrace{e^{i(\vec{k}_\beta\cdot\vec{r}-\omega t)}}_{\text{Transmitted}} \qquad (x \ge 0)$$

Where $\sum_\alpha$ and $\sum_\beta$ sum over all possible $\vec{k}_\alpha$ and $\vec{k}_\beta$ which give angular frequency ω

$$|k_\alpha|^2 = \omega^2 \frac{\rho_s}{T_s} = \frac{\omega^2}{v^2} \quad , \quad |k_\beta|^2 = \omega^2 \frac{\rho_s'}{T_s'} = \frac{\omega^2}{v'^2}$$

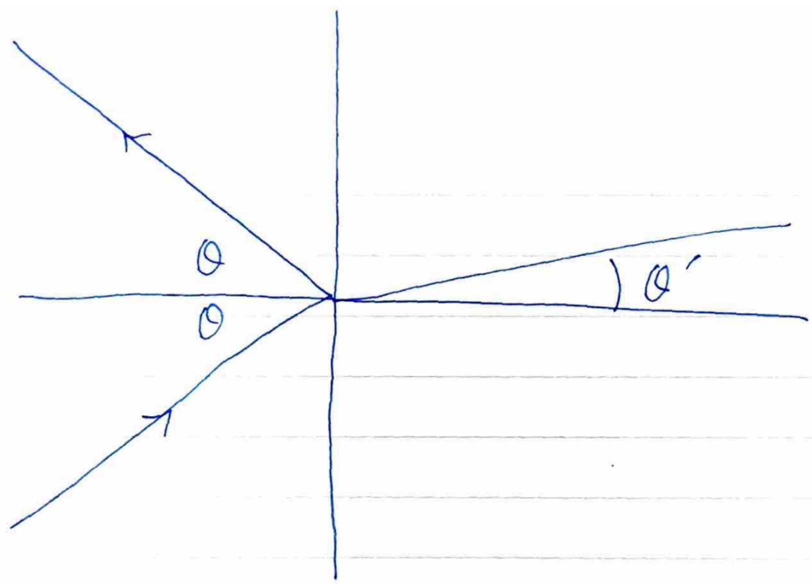
To calculate R_α and T_β as well as $\vec{k}_\alpha$ and $\vec{k}_\beta$ we need boundary conditions!
At $x = z = 0$ the membrane cannot break so we need $\psi_L = \psi_R$

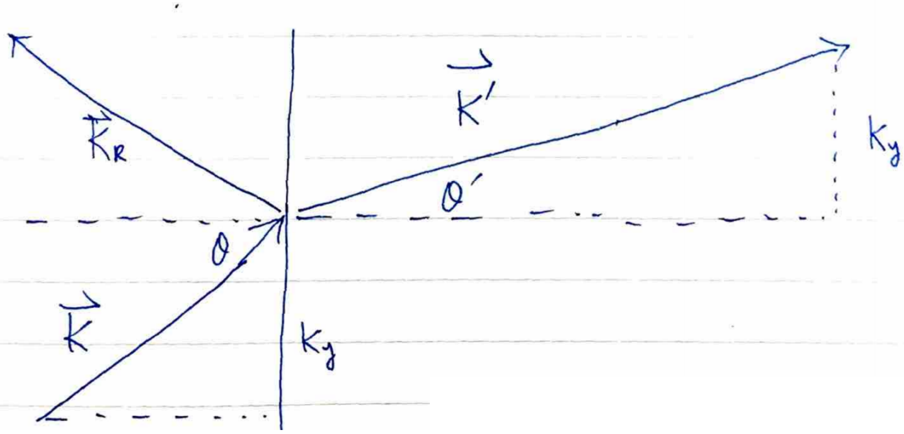
$$\psi(0, y, 0, t) = Ae^{i(k_y y - \omega t)} + \sum_\alpha R_\alpha A e^{i(k_{\alpha y} y - \omega t)} = \sum_\beta T_\beta A e^{i(k_{\beta y} y - \omega t)}$$

Where the equality is established with the boundary condition. This can only be true when $k_{\alpha y} = k_{\beta y} = k_y$. Only when

$$k_{\alpha x} = -\sqrt{\omega^2/v^2 - k_y^2} = -k_x \quad \text{and} \quad k_{\beta x} = \sqrt{\omega^2/v'^2 - k_y^2}$$

We can satisfy the boundary condition.



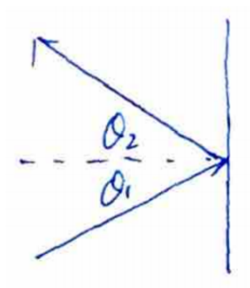


We have $|\vec{k}|\sin\theta = |\vec{k}'|\sin\theta'$

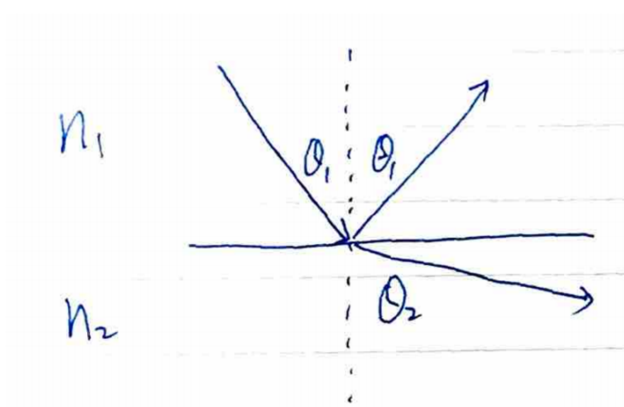
$$\begin{aligned} n &= \frac{c}{v} = \frac{c}{\omega}|k| \\ n' &= \frac{c}{v'} = \frac{c}{\omega}|\vec{k}'| \\ \Rightarrow \; n\sin\theta &= n'\sin\theta' \end{aligned}$$

Snell's Law! We have just proved the two MOST IMPORTANT LAWS of geometrical optics!!!

(1.) Reflection: $\theta_1 = \theta_2$



(2.) Snell's Law: $n_1 \sin\theta_1 = n_2 \sin\theta_2$ where n is a refraction index



(3.) It works for water, glass, sound, and light waves!
(4.) If we continue to increase θ_1 then

$$\frac{n_1}{n_2}\sin\theta_1 > 1$$

There is no transmission!