* Review damped driven oscillator

There are two examples on "Transient Behavior"
(slide 3 and 4)

* Coupled Oscillators

In general, the motion of coupled systems can be extremely complicated.

DEMO: Double Pendulum

The first join: travels in a semicircular path

The second join: varied, unpredictable trajectories.

8.03: We focus on the small displacement

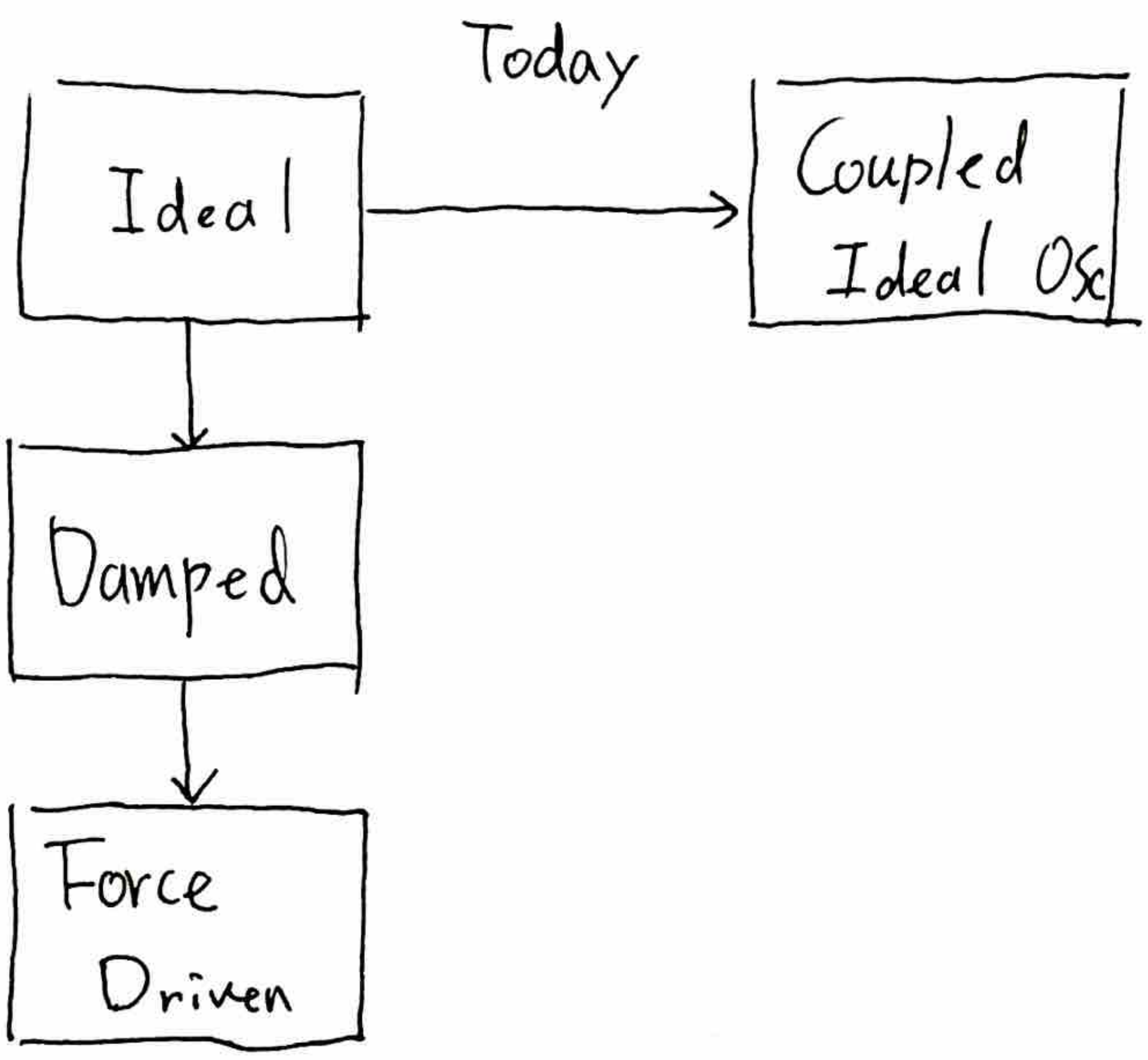
Two examples of coupled oscillators:

* DEMO: Coupled Pendulum [Couple different objects]

Talk to each other by the spring.

* DEMO: Wilberforce Pendulum [Couple degrees of freedom]

Caused by a slight coupling between rotation and vertical motion due to the geometry of the spring.
When the mass moves up and down, it cause the spring to unwind and gives the mass a twist.



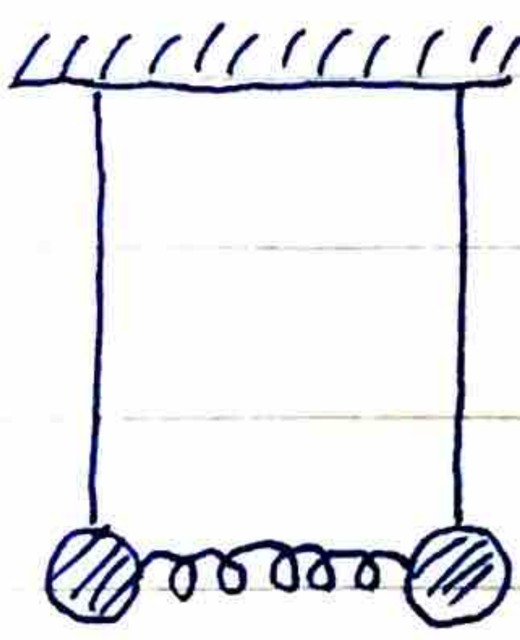
Already one oscillator ⇒ tedious to work out the math

If we want to study coupled oscillator →
it really becomes complicated

We want to learn Physics :

In order to make things simpler ⇒ consider ideal case.

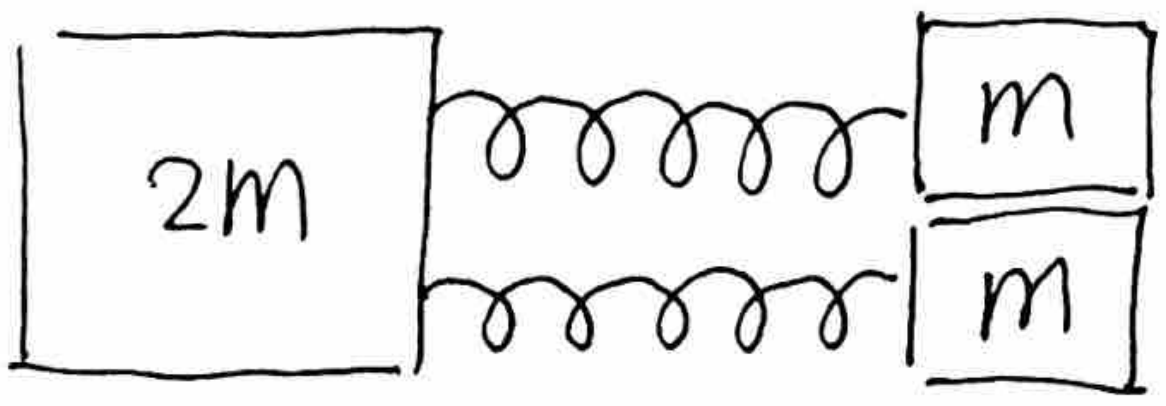
$\boxed{\text{DEMO}}$ Coupled pendulum. Wilberforce Pendulum



The motion seems to be very complicated !

We will see that this is an illusion !
If we look at it in the right way
⇒ we can see simple harmonic oscillator in the complicated system !

Let's consider an example:



There are many many kind of motion!

If you stare at this system long enough

⇒ you can identify a special kind of motion!

"Normal Mode"

Every part of the system is oscillating

at the same phase and the same frequency

We will later realize:

The most general motion is a superposition of the

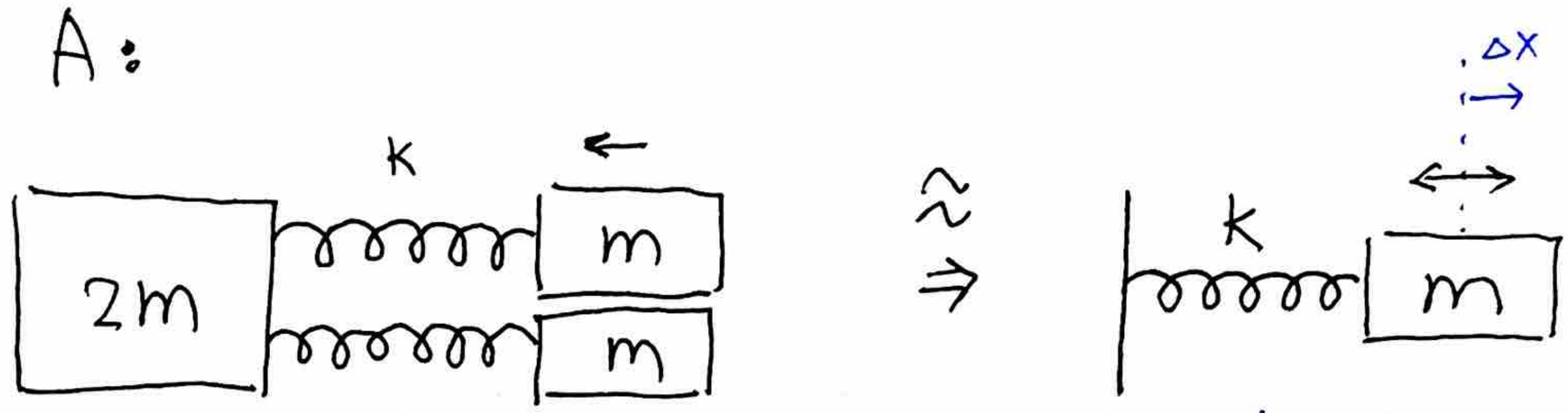
normal modes

→ We can understand the system systematically step-by-step.

In general, coupled oscillators are complicated.
But there are easier cases that we can solve things by logic.

Can you guess the normal modes of this example?

Mode A:

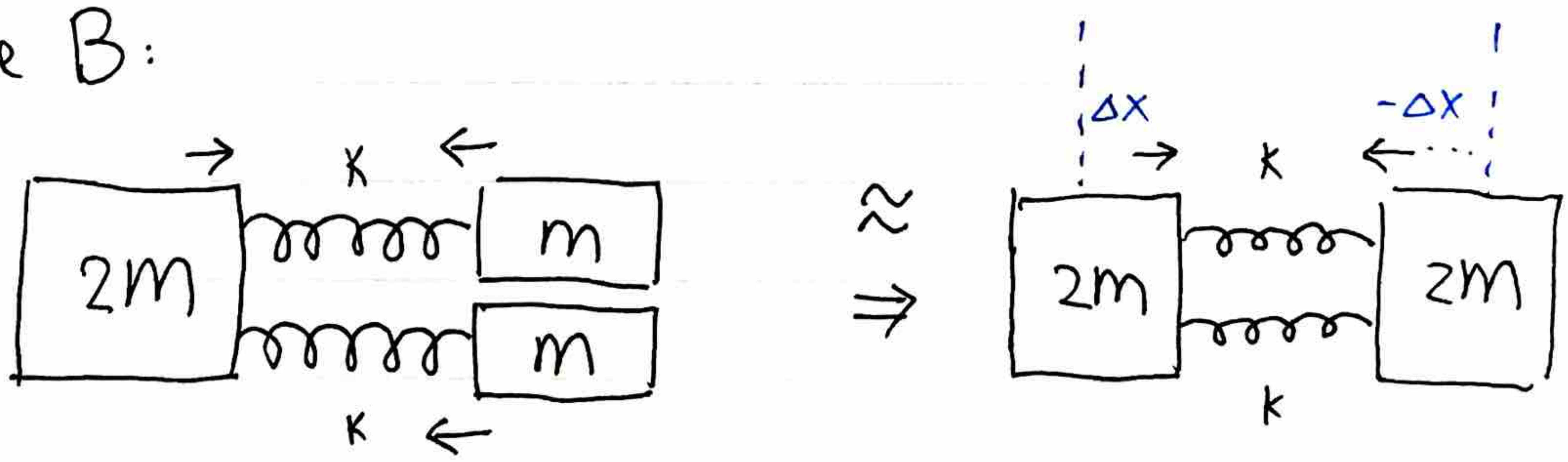


$F = -k\Delta x$

$\Rightarrow \omega_A^2 = \frac{k}{m}$

This one is "oscillating" with 0 amplitude :)

Mode B:

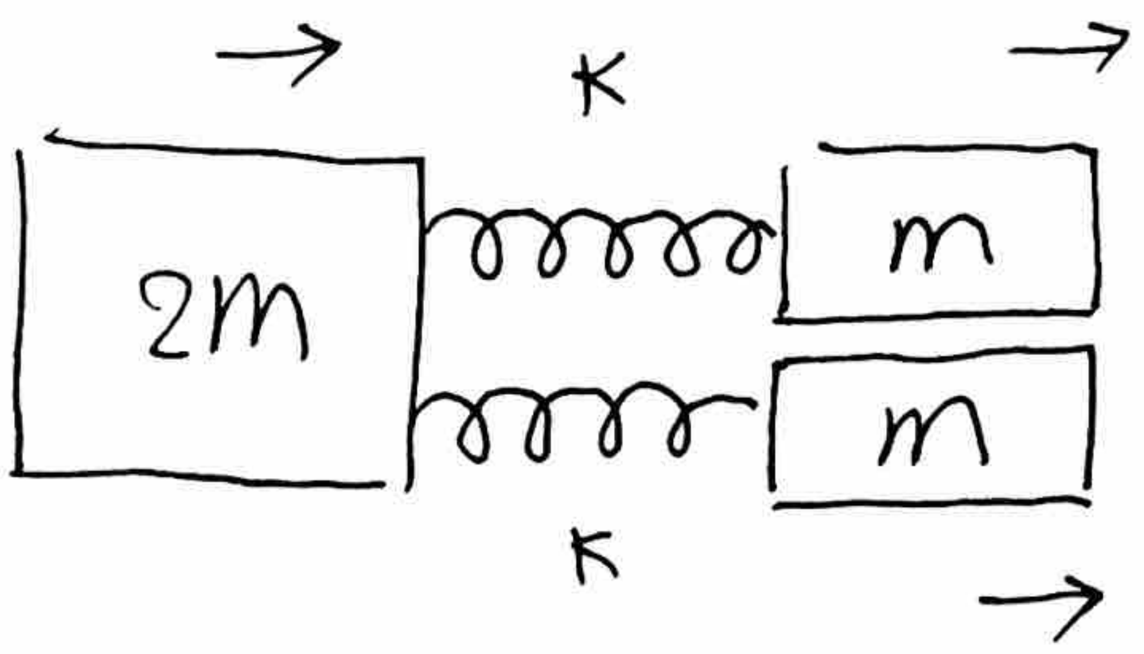


$F = -2k\,2\Delta x = -4k\Delta x$

$\omega_B^2 = \frac{4k}{2m} = \frac{2k}{m}$

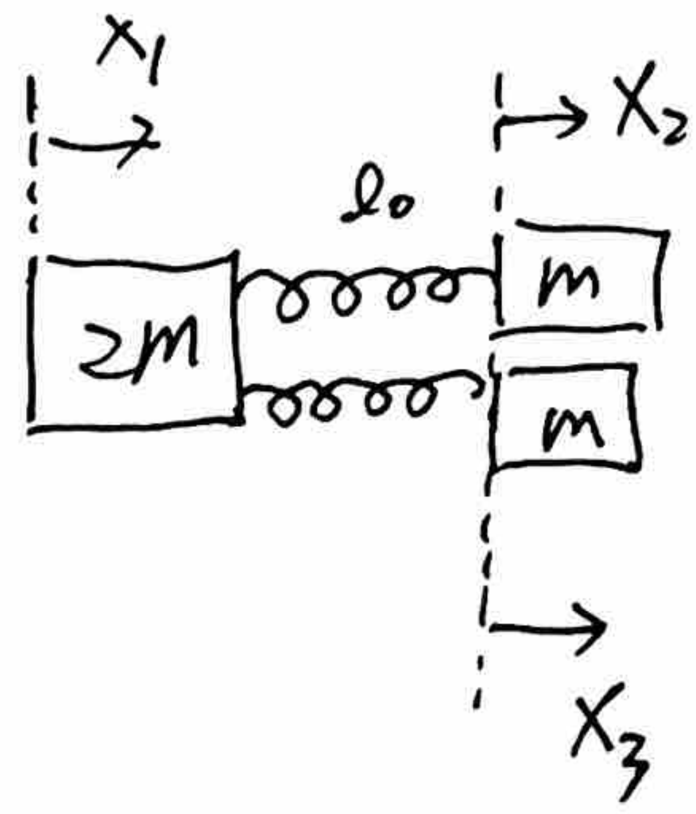
Mode C ?

Mode C :



$F = 0$

$\omega_c = 0$



Mode A:

$$\begin{cases} x_1 = 0 \\ x_2 = A\cos(\omega_A t + \phi_A) \\ x_3 = -A\cos(\omega_A t + \phi_A) \end{cases}$$

Mode B:

$$\begin{cases} x_1 = B\cos(\omega_B t + \phi_B) \\ x_2 = -B\cos(\omega_B t + \phi_B) \\ x_3 = -B\cos(\omega_B t + \phi_B) \end{cases}$$

Mode C:

$$x_1 = x_2 = x_3 = C + vt$$

General Solution:

$A, B, C, \phi_A, \phi_B, v$ are constants to be determined by initial condition

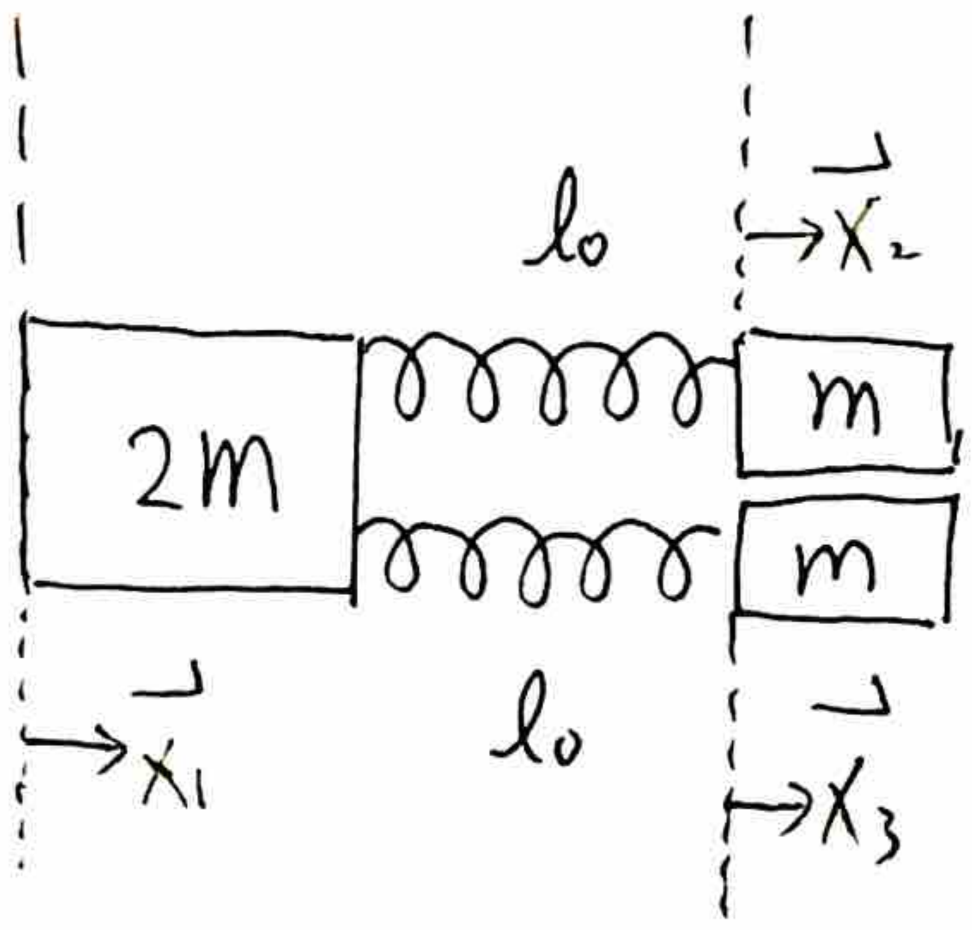
$$x_1 = 0 + B\cos(\omega_B t + \phi_B) + C + vt$$

$$x_2 = A\cos(\omega_A t + \phi_A) + \left(-B\cos(\omega_B t + \phi_B)\right) + C + vt$$

$$x_3 = -A\cos(\omega_A t + \phi_A) + \left(-B\cos(\omega_B t + \phi_B)\right) + C + vt$$

3 second order differential eqs, 6 unknown : This is the Full Solution!

Slide 8



Force diagram analysis gives:

$$2m\ddot{x}_1 = k(x_2 - x_1) + k(x_3 - x_1)$$
$$m\ddot{x}_2 = k(x_1 - x_2)$$
$$m\ddot{x}_3 = k(x_1 - x_3)$$

We can reorganize it:

$$\begin{cases} 2m\ddot{x}_1 = -2kx_1 + kx_2 + kx_3 \\ m\ddot{x}_2 = kx_1 - kx_2 + 0x_3 \\ m\ddot{x}_3 = kx_1 + 0x_2 - kx_3 \end{cases}$$

Now our job is to solve the equations

It is possible to solve it directly, but here we will use matrix as a tool to help us.

$\Rightarrow$ Convert everything to matrices!!

I can rewrite the coupled equation as:

$$M\ddot{X} = -KX$$

$$M = \begin{pmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{pmatrix} \qquad X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$$

$$K = \begin{pmatrix} 2K & -K & -K \\ -K & K & 0 \\ -K & 0 & K \end{pmatrix}$$

Go to complex notation:

Use the definition of normal mode:

$$x_j = Re(Z_j) \qquad Z = \underbrace{e^{i(\omega t+\phi)}}_{} \underbrace{\begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}}_{A}$$

Oscillating at (angular) frequency ω and phase ϕ

$$M\ddot{Z} = -KZ \qquad \ddot{Z} = -\omega^2 Z$$

$$\Rightarrow M\omega^2 Z = KZ$$

cancel $e^{-i\omega t+\phi}$

$$\Rightarrow M\omega^2 A = KA$$

$\times M^{-1}$

$$\Rightarrow \omega^2 A = M^{-1}KA$$

$$\Rightarrow (M^{-1}K - \omega^2 I)A = 0 \qquad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

To have solution: $\det\left[MK^{-1} - \omega^2 I\right] = 0$

$$M^{-1} = \begin{pmatrix} \frac{1}{2m} & 0 & 0 \\ 0 & \frac{1}{m} & 0 \\ 0 & 0 & \frac{1}{m} \end{pmatrix}$$

Check

$$M^{-1}M = I$$

$$\left(M^{-1}K - \omega^2 I \right) = \begin{pmatrix} \left(\frac{k}{m} - \omega^2\right) & \frac{-k}{2m} & \frac{-k}{2m} \\ \frac{-k}{m} & \frac{k}{m} - \omega^2 & 0 \\ -\frac{k}{m} & 0 & \frac{k}{m} - \omega^2 \end{pmatrix}$$

Define $\omega_0^2 = \frac{k}{m}$

$$\Rightarrow \det \begin{pmatrix} \omega_0^2 - \omega^2 & -\frac{1}{2}\omega_0^2 & -\frac{1}{2}\omega_0^2 \\ -\omega_0^2 & \omega_0^2 - \omega^2 & 0 \\ -\omega_0^2 & 0 & \omega_0^2 - \omega^2 \end{pmatrix} = 0$$

$$(\omega_0^2 - \omega^2)^3 - \frac{1}{2}\omega_0^4(\omega_0^2 - \omega^2) - \frac{1}{2}\omega_0^4(\omega_0^2 - \omega^2) = 0$$

$$(\omega_0^2 - \omega^2)\left(\cancel{\omega_0^4} - 2\omega_0^2\omega^2 + \omega^4 - \cancel{\omega_0^4} \right) = 0$$

$$(\omega_0^2 - \omega^2)\, \omega^2 \left(\omega^2 - 2\omega_0^2 \right) = 0$$

$$\Rightarrow \omega = \omega_0 , \sqrt{2}\,\omega_0 , 0$$ (take the absolute value)

$$= \sqrt{\frac{k}{m}} , \sqrt{\frac{2k}{m}} , 0$$

Got the same result !!

Break ?

DEMO

To get the relative amplitude of a normal mode:

Plug in the normal mode frequency you got in the equation $[M^{-1}k - \omega^2 I] A = 0$

For instance: take $\omega = \omega_B = \sqrt{\frac{2k}{m}}$

$$\Rightarrow \quad \begin{aligned} 0 &= 2kA_1 + kA_2 + kA_3 \\ 0 &= kA_1 + kA_2 + 0A_3 \\ 0 &= kA_1 + 0A_2 + kA_3 \end{aligned}$$

$$\Rightarrow \quad A_1 = -A_2 = -A_3$$

$X = \mathrm{Re}(Z) \Rightarrow$

$$\begin{aligned} X_1 &= B\cos(\omega_B t + \phi_B) \\ X_2 &= -B\cos(\omega_B t + \phi_B) \\ X_3 &= -B\cos(\omega_B t + \phi_B) \end{aligned}$$

Or

$$X = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} = B \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} \cos(\omega_B t + \phi_B)$$

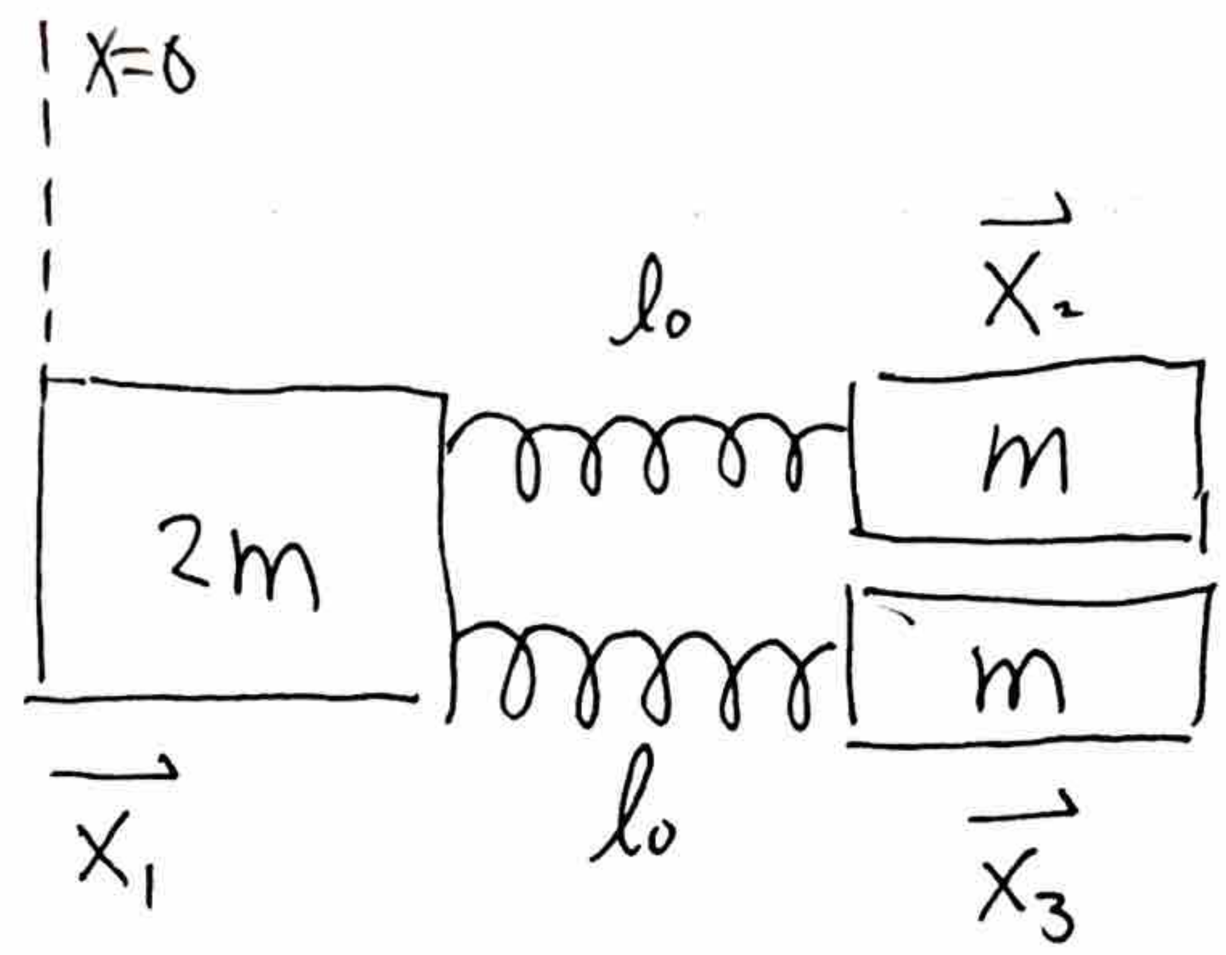
It turns out that this is a simple harmonic oscillation!!

take $\omega = \omega_A = \sqrt{\frac{k}{m}}$

$$\Rightarrow \quad A_1 = 0, \quad A_2 = -A_3$$

$$\begin{aligned} X_1 &= 0 \\ X_2 &= A\cos(\omega_A t + \phi_A) \\ X_3 &= -A\cos(\omega_A t + \phi_A) \end{aligned}$$

(Alternative way)



Force diagram analysis gives:

$$2m\ddot{X}_1 = k(X_2 - X_1 - l_0) + k(X_3 - X_1 - l_0)$$
$$m\ddot{X}_2 = k(X_1 - X_2 + l_0)$$
$$m\ddot{X}_3 = k(X_1 - X_3 + l_0)$$

Define $X_2' = X_2 - l_0$
$X_3' = X_3 - l_0$

$$2m\ddot{X}_1 = k(X_2' - X_1) + k(X_3' - X_1)$$
$$m\ddot{X}_2' = k(X_1 - X_2')$$
$$m\ddot{X}_3' = k(X_1 - X_3')$$

Reorganize it:

$$\begin{cases} 2m\ddot{X}_1 = -2kX_1 + kX_2' + kX_3' \\ m\ddot{X}_2' = kX_1 - kX_2' + 0X_3' \\ m\ddot{X}_3' = kX_1 + 0X_2' - kX_3' \end{cases}$$

Now use the definition of normal mode:

$$X_1 = \mathrm{Re}\left(A_1 e^{i(\omega t + \phi)}\right)$$
$$X_2' = \mathrm{Re}\left(A_2 e^{i(\omega t + \phi)}\right)$$
$$X_3' = \mathrm{Re}\left(A_3 e^{i(\omega t + \phi)}\right)$$

Same ω, ϕ

(Alternative Way)

$$-2m\omega^2 A_1 = -2kA_1 + kA_2 + kA_3$$
$$-m\omega^2 A_2 = kA_1 - kA_2 + 0A_3$$
$$-m\omega^2 A_3 = kA_1 + 0A_2 - kA_3$$

$$0 = (-2k + 2m\omega^2)A_1 + kA_2 + kA_3$$
$$0 = kA_1 + (m\omega^2 - k)A_2 + 0A_3$$
$$0 = kA_1 + 0A_2 + (m\omega^2 - k)A_3$$

We can rewrite it in the form of matrix

$$\begin{pmatrix} (2m\omega^2 - 2k) & k & k \\ k & (m\omega^2 - k) & 0 \\ k & 0 & (m\omega^2 - k) \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix} = 0$$

To have solution : $\det\left(\quad \right) = 0$

Determinant of the matrix $= 0$

$$(2m\omega^2 - 2k)(m\omega^2 - k)^2 - 2k^2(m\omega^2 - k) = 0$$

$$(m\omega^2 - k)\left[(2m\omega^2 - 2k)(m\omega^2 - k) - 2k^2\right] = 0$$

$$\Downarrow$$

$$2m^2\omega^4 - 4mk\omega^2$$

$$(m\omega^2 - k)\,\omega^2\,(2m^2\omega^2 - 4km) = 0$$

$$\omega = \sqrt{\frac{2k}{m}},\ \sqrt{\frac{k}{m}},\ 0$$

Got the same result!