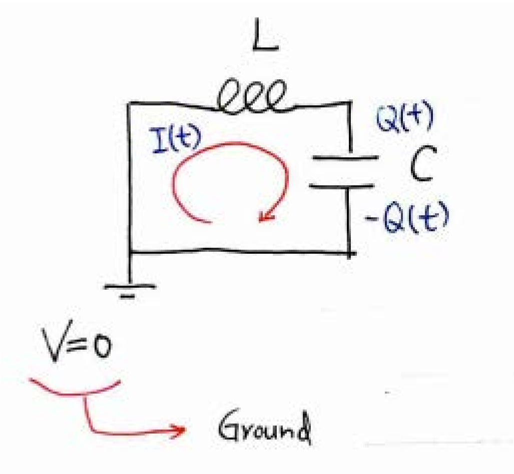
8.03 Lecture 2



Example: LC Circuit

"Coordinate system": Define clockwise current to be positive.

$$I(t) = \frac{dQ(t)}{dt}$$

$$\text{At} \quad t = 0 \quad I(0) = I_{Initial} \quad \text{and} \quad Q(0) = 0$$

*Voltage drop:

$$L\frac{dI}{dt}$$

Capacitor:

$$V = \frac{Q}{C}$$

$$L\frac{dI}{dt} + \frac{Q}{C} = 0 \quad \Rightarrow \quad \ddot{Q} + \frac{Q}{LC} = 0$$

$$Q(t) = A\cos(\omega_0 t + \phi)$$

$$Q(t) = \frac{I_{Initial}}{\omega_0}\sin\omega_0 t$$

We use the initial condidtions $I(0)$ and $Q(0)$ to get $\phi = -\frac{\pi}{2}$ and $A = \frac{I_{Initial}}{\omega_0}$ and we define $\omega_0 = \sqrt{\frac{1}{LC}}$
To compare with our mass and spring system:

$$\begin{array}{ccc} m & \leftrightarrow & \text{L} \\ k & \leftrightarrow & \text{1/C} \\ x & \leftrightarrow & \text{Q} \end{array}$$

In general, our equation of motion will look like:

$$\mu\frac{d^2x}{dt^2} = -kx$$

Where μ is a generalized mass, x is a generalized coordinate and k is a generalized spring constant.

In this system:

$$\text{“Kinetic energy”} = \frac{1}{2}\mu\left(\frac{dx}{dt}\right)^2$$
$$\text{“Potential energy”} = \frac{1}{2}kx^2$$
$$\text{“Total energy”} \Rightarrow E = \frac{1}{2}\mu\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2$$

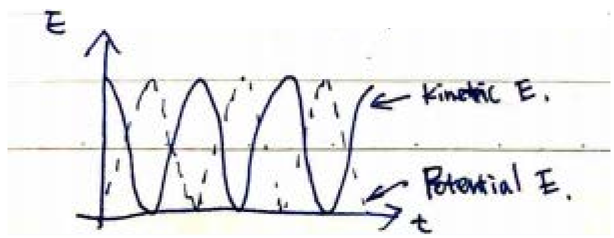
If we solve the equation: $\ddot{x} + \omega_0^2 x = 0$ where $\omega_0 = \sqrt{\frac{k}{\mu}}$

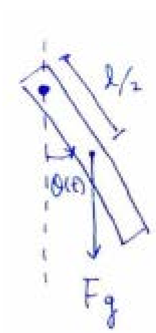
$$\begin{aligned} x(t) &= A\cos(\omega_0 t + \phi) \\ \frac{dx(t)}{dt} &= -A\omega_0 \sin(\omega_0 t + \phi) \end{aligned} \tag{1}$$

Plugging this into the total energy we get:

$$\begin{aligned} E &= \frac{1}{2}\mu A^2\omega_0^2 \sin^2(\omega_0 t + \phi) + \frac{1}{2}kA^2\cos^2(\omega_0 t + \phi) \\ &= \frac{1}{2}kA^2(\sin^2(\omega_0 t + \phi) + \cos^2(\omega_0 t + \phi)) \\ &= \frac{1}{2}kA^2 \quad \text{Constant!!!} \end{aligned} \tag{2}$$

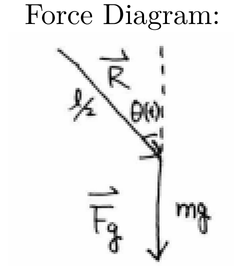
We see our total energy is proportional to A^2, the square of the amplitude of oscillation and k, the generalized spring constant. The total energy remains constant but sometimes it is kinetic and sometimes it is stored in the potential, as in the figure.





Let's look at this example:
Newton's Law: $\vec{\tau} = I\vec{\alpha}$
Origin: $\theta = 0 \Rightarrow$ Pointing downward
Define anti-clockwise rotation to be positive.
Initial Condition: At $t = 0 \Rightarrow$
$\theta(0) = \theta_{Initial} \quad \dot{\theta}(0) = 0$

Force Diagram:



$$\vec{\tau} = \vec{R} \times \vec{F}$$
$$\tau = -mg\frac{l}{2}\sin\theta(t)$$

Since this system is rotating in a single plane we can drop the vector sign over τ

From Newton's Law:

$$\tau = I\alpha(t) = I\ddot{\theta}(t) = \frac{-mgl}{2}\sin\theta(t)$$
$$\ddot{\theta}(t) = \frac{-mgl}{2I}\sin\theta(t) \qquad I = \frac{1}{3}ml^2$$
$$\ddot{\theta}(t) = -\frac{3g}{2l}\sin\theta(t) = -\omega_0^2\sin\theta(t)$$

Where $\omega_0 = \sqrt{\frac{3g}{2l}}$
Now again: we have mathematically described the physical situation. This equation of motion contains everything we need to know. We need to solve this equation.
However, life is hard! We don't know how to solve $\ddot{\theta} = -\omega_0^2\sin\theta$
Not the end of the world. We can either solve it by a computer or we can consider a special case: small angle limit.

$$\theta(t) \to 0 \Rightarrow \sin\theta(t) \approx \theta$$

$$\begin{array}{rcccc} \theta(t) = & 1^\circ & \Rightarrow & \frac{\sin\theta}{\theta} & = 99.99\ \% \\ & 5^\circ & \Rightarrow & & =99.9\ \% \\ & 10^\circ & \Rightarrow & & =99.5\ \% \end{array}$$

The approximation is quite good!
The equation of motion becomes:

$$\ddot{\theta}(t) = -\omega_0^2\theta(t)$$

We have solved this equation of motion in previous lectures! Recall the solution:

$$\theta(t) = A\cos(\omega_0 t + \phi)$$

Implementing the initial conditions we conclude:

$$\begin{aligned} & 0 = -\omega_0 A\sin\phi \Rightarrow \phi = 0 \\ & \theta_{Initial} = A \\ \Rightarrow\quad & \theta(t) = \theta_{Initial}\cos(\omega_0 t) \end{aligned}$$

Now we will add a drag force:

$$\tau_{DRAG}(t) = -b\dot{\theta}(t)$$

We chose this form, not only because it is the most realistic but because it is solvable, but also it is not totally unrealistic. If we choose more complicated forms of the drag force, we would have to solve it by a computer.

*EQUATION OF MOTION:

$$\ddot{\theta}(t) = \frac{\tau(t)}{I} = \frac{\tau_g(t) + \tau_{DRAG}(t)}{I}$$
$$= \frac{-mg\frac{l}{2}\sin\theta(t) - b\dot{\theta}(t)}{\frac{1}{3}ml^2}$$

Implementing the small angle approximation, $\sin\theta(t) \approx \theta(t)$

$$\approx -\frac{3g}{2l}\theta(t) - \frac{3b}{ml^2}\dot{\theta}(t)$$
$$\text{Define} \quad \omega_0^2 = \frac{3g}{2l} \quad \Gamma = \frac{3b}{ml^2}$$

Again: The reason we define ω_0 and Γ is to simplify things and make our lives easier. We can write our equation of motion:

$$\ddot{\theta}(t) + \Gamma\dot{\theta}(t) + \omega_0^2\theta(t) = 0$$

Now we want to solve this equation. Use complex notation!

$$\theta(t) = \mathrm{Re}[z(t)] \quad z(t) = e^{i\alpha t}$$
$$\ddot{z}(t) + \Gamma\dot{z}(t) + \omega_0^2 z(t) = 0$$
$$(-\alpha^2 + i\Gamma\alpha + \omega_0^2)e^{i\alpha t} = 0$$
$$e^{i\alpha t} \text{ is never zero.}$$
$$\Rightarrow \alpha^2 - i\Gamma\alpha - \omega_0^2 = 0$$
$$\Rightarrow \alpha = \frac{i\Gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\Gamma^2}{4}}$$

Now there are three possibilities for the last term.

1. "Underdamped oscillation" where $\omega_0^2 > \frac{\Gamma^2}{4}$. The drag force is small.

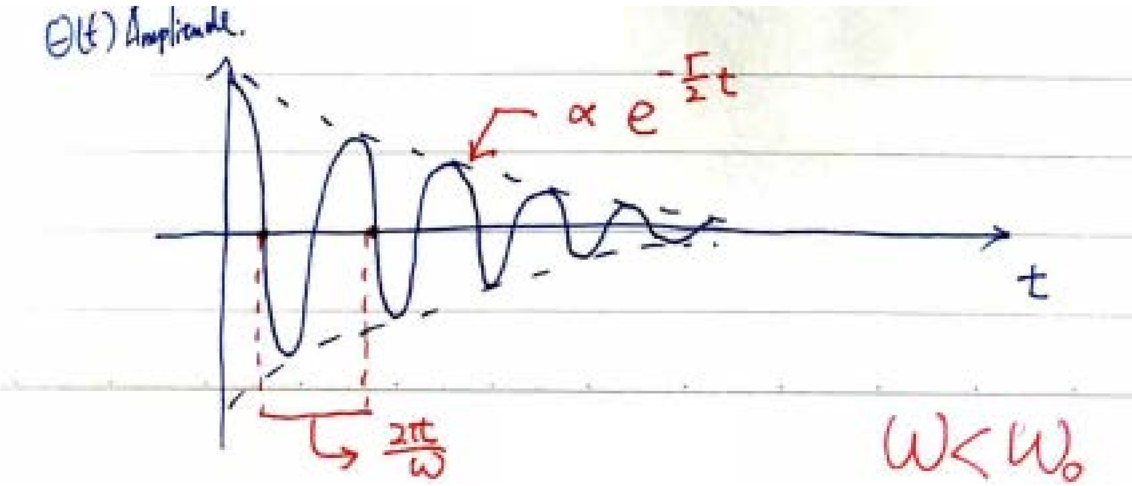
$$\Rightarrow \omega^2 \equiv \omega_0^2 - \frac{\Gamma^2}{4}$$
$$z_+(t) = e^{-\Gamma t/2}e^{i\omega t}$$
$$z_-(t) = e^{-\Gamma t/2}e^{-i\omega t}$$
$$\theta_1(t) = \frac{1}{2}(z_+(t) + z_-(t))$$
$$= e^{-\Gamma t/2}\cos\omega t$$
$$\theta_2(t) = \frac{-i}{2}(z_+(t) - z_-(t))$$
$$= e^{-\Gamma t/2}\sin\omega t$$
$$\theta(t) = e^{-\Gamma t/2}(a\cos\omega t + b\sin\omega t) \quad \text{or}$$
$$\theta(t) = Ae^{-\Gamma t/2}(\cos(\omega t + \phi))$$

Use the initial condition:

$$\theta(0) = \theta_{INITIAL} = A\cos\phi$$
$$\dot{\theta}(0) = \frac{A\Gamma}{2}\cos\phi - A\omega\sin\phi = 0$$

We can solve for A and ϕ

$$\tan\phi = -\frac{\Gamma}{2\omega} \quad \phi = \arctan -\frac{\Gamma}{2\omega}$$
$$A = \frac{\theta_{INITIAL}}{\cos\phi}$$



A graphic representation of underdamped oscillation.

2. "Critically damped oscillation"
This means that $\omega = 0$
Starting from 1.:

$$\theta_1 = e^{-\Gamma t/2}\cos\omega t \xrightarrow{\omega\to 0} e^{-\Gamma t/2}$$
$$\theta_1 = e^{-\Gamma t/2}\sin\omega t \xrightarrow{\omega\to 0} 0$$

Knowing $\theta_2(t) = 0$ is not helpful. Instead we do:

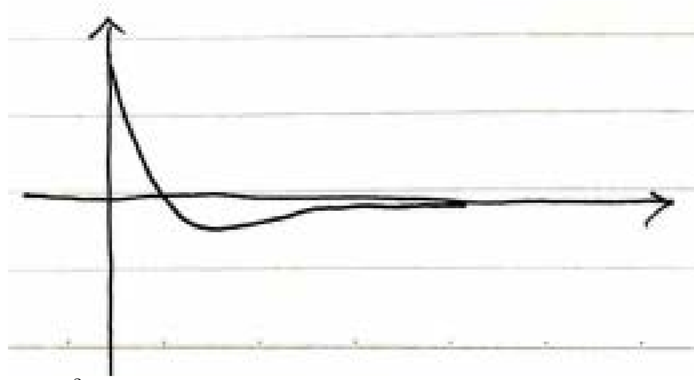
$$\frac{\theta_2(t)}{\omega} = \frac{1}{\omega}e^{-\Gamma t/2}\sin\omega t \xrightarrow{\omega\to 0} te^{-\Gamma t/2}$$

$\Rightarrow$ Now we have a linear combination of $\theta_1(t)$ and $\frac{\theta_2(t)}{\omega}$:

$$\theta(t) = (A + Bt)e^{-\Gamma t/2}$$

Prediction: No oscillation!

Applications of critical damping:
Door closing, suspension systems.



3. "Overdamped oscillation" which means $\omega_0^2 < \frac{\Gamma^2}{4}$, there is a huge drag force!

$$\begin{aligned}\alpha &= \frac{i\Gamma}{2} \pm \sqrt{\omega_0^2 - \frac{\Gamma^2}{4}} \\ &= i\left(\frac{\Gamma}{2} \pm \sqrt{\frac{\Gamma^2}{4} - \omega_0^2}\right) \\ \text{Define } \Gamma_\pm &= \frac{\Gamma}{2} \pm \sqrt{\frac{\Gamma^2}{4} - \omega_0^2} \\ \Rightarrow \text{Solution: } \theta(t) &= A_+ e^{-\Gamma_+ t} + A_- e^{-\Gamma_- t}\end{aligned}$$

No oscillation! Two exponentially decaying terms: