Example 8.9 Pulleys and Ropes Constraint Conditions

Consider the arrangement of pulleys and blocks shown in Figure 8.39. The pulleys are assumed massless and frictionless and the connecting strings are massless and inextensible. Denote the respective masses of the blocks as m_1, m_2 and m_3. The upper pulley in the figure is free to rotate but its center of mass does not move. Both pulleys have the same radius R. (a) How are the accelerations of the objects related? (b) Draw force diagrams on each moving object. (c) Solve for the accelerations of the objects and the tensions in the ropes.

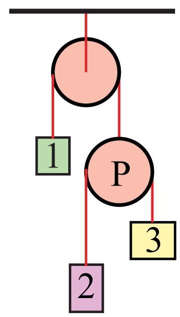

Figure 8.39 Constrained pulley system

Solution: (a) Choose an origin at the center of the upper pulley. Introduce coordinate functions for the three moving blocks, y_1, y_2 and y_3. Introduce a coordinate function y_P for the moving pulley (the pulley on the lower right in Figure 8.40). Choose downward for positive direction; the coordinate system is shown in the figure below then.

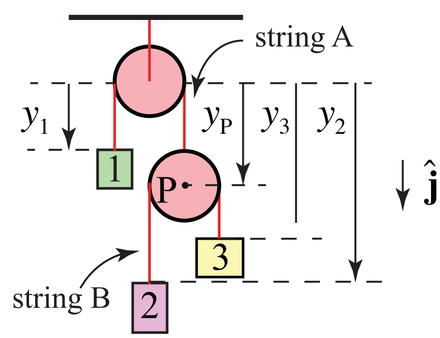

Figure 8.40 Coordinated system for pulley system

The length of string A is given by

$$l_A = y_1 + y_P + \pi R \tag{8.6.46}$$

where πR is the arc length of the rope that is in contact with the pulley. This length is constant, and so the second derivative with respect to time is zero,

$$0 = \frac{d^2 l_A}{dt^2} = \frac{d^2 y_1}{dt^2} + \frac{d^2 y_P}{dt^2} = a_{y,1} + a_{y,P}\,. \tag{8.6.47}$$

Thus block 1 and the moving pulley's components of acceleration are equal in magnitude but opposite in sign,

$$a_{y,P} = -a_{y,1}\,. \tag{8.6.48}$$

The length of string B is given by

$$l_B = (y_3 - y_P) + (y_2 - y_P) + \pi R = y_3 + y_2 - 2y_P + \pi R \tag{8.6.49}$$

where πR is the arc length of the rope that is in contact with the pulley. This length is also constant so the second derivative with respect to time is zero,

$$0 = \frac{d^2 l_B}{dt^2} = \frac{d^2 y_2}{dt^2} + \frac{d^2 y_3}{dt^2} - 2\frac{d^2 y_P}{dt^2} = a_{y,2} + a_{y,3} - 2a_{y,P}\,. \tag{8.6.50}$$

We can substitute Equation (8.6.48) for the pulley acceleration into Equation (8.6.50) yielding the *constraint relation* between the components of the acceleration of the three blocks,

$$0 = a_{y,2} + a_{y,3} + 2a_{y,1}\,. \tag{8.6.51}$$

b) Free-body Force diagrams: the forces acting on block 1 are: the gravitational force $m_1\vec{\mathbf{g}}$ and the pulling force $\vec{\mathbf{T}}_{A,1}$ of string A acting on the block 1. Denote the magnitude of this force by T_A. Because the string is assumed to be massless and the pulley is assumed to be massless and frictionless, the tension T_A in the string is uniform and equal in magnitude to the pulling force of the string on the block. The free-body diagram on block 1 is shown in Figure 8.41(a).

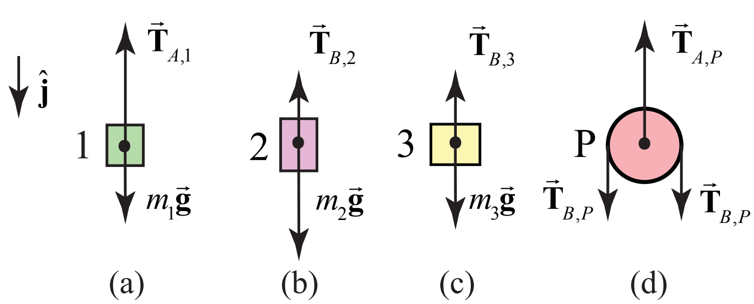

Figure 8.41 Free-body force diagram on (a) block 1; (b) block 2; (c) block 3; (d) pulley

Newton's Second Law applied to block 1 is then

$$\hat{\mathbf{j}}:\ m_1 g - T_A = m_1\, a_{y,1}\,. \tag{8.6.52}$$

The forces on the block 2 are the gravitational force $m_2\vec{\mathbf{g}}$ and string B holding the block, $\vec{\mathbf{T}}_{B,2}$, with magnitude T_B. The free-body diagram for the forces acting on block 2 is shown in Figure 8.41(b). Newton's second Law applied to block 2 is

$$\hat{\mathbf{j}}:\ m_2 g - T_B = m_2\, a_{y,2}\,. \tag{8.6.53}$$

The forces on the block 3 are the gravitational force $m_3\vec{\mathbf{g}}$ and string holding the block, $\vec{\mathbf{T}}_{B,3}$, with magnitude equal to T_B because pulley P has been assumed to be both frictionless and massless. The free-body diagram for the forces acting on block 3 is shown in Figure 8.41(c). Newton's second Law applied to block 3 is

$$\hat{\mathbf{j}}:\ m_3 g - T_B = m_3\, a_{y,3}\,. \tag{8.6.54}$$

The forces on the moving pulley P are the gravitational force $m_P\,\vec{\mathbf{g}} = \vec{\mathbf{0}}$ (the pulley is assumed massless); string B pulls down on the pulley on each side with a force, $\vec{\mathbf{T}}_{B,P}$, which has magnitude T_B. String A holds the pulley up with a force $\vec{\mathbf{T}}_{A,P}$ with the magnitude T_A equal to the tension in string A. The free-body diagram for the forces acting on the moving pulley is shown in Figure 8.41(d). Newton's second Law applied to the pulley is

$$\hat{\mathbf{j}}:\ 2T_B - T_A = m_P\, a_{y,P} = 0\,. \tag{8.6.55}$$

Because the pulley is assumed to be massless, we can use this last equation to determine the condition that the tension in the two strings must satisfy,

$$2T_B = T_A \tag{8.6.56}$$

We are now in position to determine the accelerations of the blocks and the tension in the two strings. We record the relevant equations as a summary.

$$0 = a_{y,2} + a_{y,3} + 2a_{y,1} \tag{8.6.57}$$

$$m_1 g - T_A = m_1\, a_{y,1} \tag{8.6.58}$$

$$m_2 g - T_B = m_2\, a_{y,2} \tag{8.6.59}$$

$$m_3 g - T_B = m_3\, a_{y,3} \tag{8.6.60}$$

$$2T_B = T_A\,. \tag{8.6.61}$$

There are five equations with five unknowns, so we can solve this system. We shall first use Equation (8.6.61) to eliminate the tension T_A in Equation (8.6.58), yielding

$$m_1 g - 2T_B = m_1 \, a_{y,1} \,. \tag{8.6.62}$$

We now solve Equations (8.6.59), (8.6.60) and (8.6.62) for the accelerations,

$$a_{y,2} = g - \frac{T_B}{m_2} \tag{8.6.63}$$

$$a_{y,3} = g - \frac{T_B}{m_3} \tag{8.6.64}$$

$$a_{y,1} = g - \frac{2T_B}{m_1} \,. \tag{8.6.65}$$

We now substitute these results for the accelerations into the constraint equation, Equation (8.6.57),

$$0 = g - \frac{T_B}{m_2} + g - \frac{T_B}{m_3} + 2g - \frac{4T_B}{m_1} = 4g - T_B\left(\frac{1}{m_2} + \frac{1}{m_3} + \frac{4}{m_1}\right). \tag{8.6.66}$$

We can now solve this last equation for the tension in string B,

$$T_B = \frac{4g}{\left(\dfrac{1}{m_2} + \dfrac{1}{m_3} + \dfrac{4}{m_1}\right)} = \frac{4g \, m_1 \, m_2 \, m_3}{m_1 \, m_3 + m_1 m_2 + 4 m_2 \, m_3} \,. \tag{8.6.67}$$

From Equation (8.6.61), the tension in string A is

$$T_A = 2T_B = \frac{8g \, m_1 \, m_2 \, m_3}{m_1 \, m_3 + m_1 \, m_2 + 4 m_2 \, m_3} \,. \tag{8.6.68}$$

We find the acceleration of block 1 from Equation (8.6.65), using Equation (8.6.67) for the tension in string B,

$$a_{y,1} = g - \frac{2T_B}{m_1} = g - \frac{8g \, m_2 \, m_3}{m_1 \, m_3 + m_1 \, m_2 + 4 m_2 \, m_3} = g \frac{m_1 \, m_3 + m_1 \, m_2 - 4 m_2 \, m_3}{m_1 \, m_3 + m_1 m_2 + 4 m_2 \, m_3} \,. \tag{8.6.69}$$

We find the acceleration of block 2 from Equation (8.6.63), using Equation (8.6.67) for the tension in string B,

$$a_{y,2} = g - \frac{T_B}{m_2} = g - \frac{4g\,m_1\,m_3}{m_1\,m_3 + m_1\,m_2 + 4m_2\,m_3} = g\frac{-3m_1\,m_3 + m_1\,m_2 + 4m_2\,m_3}{m_1\,m_3 + m_1\,m_2 + 4m_2\,m_3}. \quad (8.6.70)$$

Similarly, we find the acceleration of block 3 from Equation (8.6.64), using Equation (8.6.67) for the tension in string *B*,

$$a_{y,3} = g - \frac{T_B}{m_3} = g - \frac{4g\,m_1\,m_2}{m_1\,m_3 + m_1\,m_2 + 4m_2\,m_3} = g\frac{m_1\,m_3 - 3m_1 m_2 + 4m_2\,m_3}{m_1\,m_3 + m_1\,m_2 + 4m_2\,m_3}. \quad (8.6.71)$$

As a check on our algebra we note that

$$2a_{1,y} + a_{2,y} + a_{3,y} =$$

$$2g\frac{m_1\,m_3 + m_1\,m_2 - 4m_2\,m_3}{m_1\,m_3 + m_1\,m_2 + 4m_2\,m_3} + g\frac{-3m_1\,m_3 + m_1\,m_2 + 4m_2\,m_3}{m_1\,m_3 + m_1 m_2 + 4m_2\,m_3} + g\frac{m_1\,m_3 - 3m_1\,m_2 + 4m_2\,m_3}{m_1\,m_3 + m_1 m_2 + 4m_2 m_3}$$

$$= 0.$$

Example 8.10 Accelerating Wedge

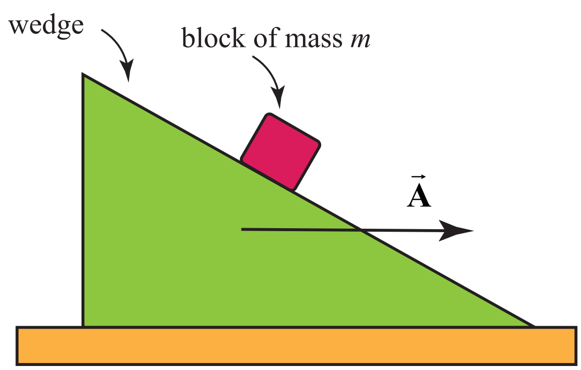

Figure 8.42 Block on accelerating wedge

A 45^o wedge is pushed along a table with constant acceleration $\vec{\mathbf{A}}$ according to an observer at rest with respect to the table. A block of mass m slides without friction down the wedge (Figure 8.42). Find its acceleration with respect to an observer at rest with respect to the table. Write down a plan for finding the magnitude of the acceleration of the block. Make sure you clearly state which concepts you plan to use to calculate any relevant physical quantities. Also clearly state any assumptions you make. Be sure you include any free-body force diagrams or sketches that you plan to use.

Solution: Choose a coordinate system for the block and wedge as shown in Figure 8.43. Then $\vec{\mathbf{A}} = A_{x,w}\,\hat{\mathbf{i}}$ where $A_{x,w}$ is the x-component of the acceleration of the wedge.