7.1 Force and Quantity of Matter

In our daily experience, we can cause a body to move by either pushing or pulling that body. Ordinary language use describes this action as the effect of a person's strength or *force.* However, bodies placed on inclined planes, or when released at rest and undergo free fall, will move without any push or pull. Galileo referred to a force acting on these bodies, a description of which he published in *Mechanics* in 1623. In 1687, Isaac Newton published his three laws of motion in the *Philosophiae Naturalis Principia Mathematica* ("Mathematical Principles of Natural Philosophy"), which extended Galileo's observations. The First Law expresses the idea that when no force acts on a body, it will remain at rest or maintain uniform motion; when a force is applied to a body, it will change its state of motion.

Many scientists, especially Galileo, recognized the idea that force produces motion before Newton but Newton extended the concept of force to any circumstance that produces acceleration. When a body is initially at rest, the direction of our push or pull corresponds to the direction of motion of the body. If the body is moving, the direction of the applied force may change both the direction of motion of the body and how fast it is moving. Newton defined the force acting on an object as proportional to the acceleration of the object.

> *An impressed force is an action exerted upon a body, in order to change its state, either of rest, or of uniform motion in a right line.*[2]

In order to define the magnitude of the force, he introduced a constant of proportionality, the *inertial mass*, which Newton called "quantity of matter".

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[2] Isaac Newton. *Mathematical Principles of Natural Philosophy*. Translated by Andrew Motte (1729). Revised by Florian Cajori. Berkeley: University of California Press, 1934. p. 2.

The quantity of matter is the measure of the same, arising from its density and bulk conjointly.

Thus air of double density, in a double space, is quadruple in quantity; in a triple space, sextuple in quantity. The same thing is to be understood of snow, and fine dust or powders, that are condensed by compression or liquefaction, and of all bodies that are by any causes whatever differently condensed. I have no regard in this place to a medium, if any such there is, that freely pervades the interstices between the parts of bodies. It is this quantity that I mean hereafter everywhere under the name of body or mass. And the same is known by the weight of each body, for it is proportional to the weight, as I have found by experiment on pendulums, very accurately made, which shall be shown hereafter.[3]

Suppose we apply a force to a body that is an identical copy of the standard mass, (we shall refer to this body as a standard body). The force will induce the standard body to accelerate with magnitude $|\vec{\mathbf{a}}|$ that can be measured by an accelerometer (any device that measures acceleration). The magnitude of the force $|\vec{\mathbf{F}}|$ acting on the standard body is **defined** to be the product of the standard mass m_s with the magnitude of the acceleration $|\vec{\mathbf{a}}|$. Force is a vector quantity. The direction of the force on the standard body is **defined** to be the direction of the acceleration of the body. Thus

$$\vec{\mathbf{F}} \equiv m_s \vec{\mathbf{a}} \tag{7.1.1}$$

In order to justify the statement that force is a vector quantity, we need to apply two forces $\vec{\mathbf{F}}_1$ and $\vec{\mathbf{F}}_2$ simultaneously to our standard body and show that the resultant force $\vec{\mathbf{F}}^T$ is the vector sum of the two forces when the forces are applied one at a time.

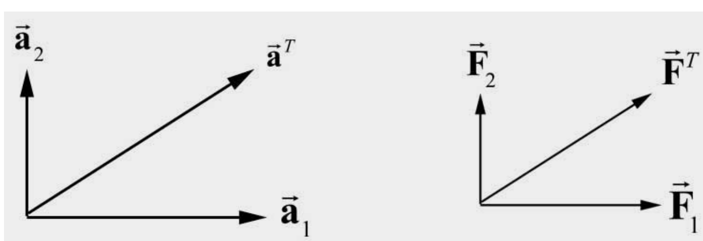

Figure 7.1 Acceleration add as vectors **Figure 7.2** Force adds as vectors.

We apply each force separately and measure the accelerations $\vec{\mathbf{a}}_1$ and $\vec{\mathbf{a}}_2$, noting that

$$\vec{\mathbf{F}}_1 = m_s \vec{\mathbf{a}}_1 \tag{7.1.2}$$

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[3] Ibid. p. 1.

$$\vec{\mathbf{F}}_2 = m_s \vec{\mathbf{a}}_2 . \quad (7.1.3)$$

When we apply the two forces simultaneously, we measure the acceleration $\vec{\mathbf{a}}$. The force by definition is now

$$\vec{\mathbf{F}}^T \equiv m_s \vec{\mathbf{a}} . \quad (7.1.4)$$

We then compare the accelerations. The results of these three measurements, and for that matter any similar experiment, confirms that the accelerations add as vectors (Figure 7.1)

$$\vec{\mathbf{a}} = \vec{\mathbf{a}}_1 + \vec{\mathbf{a}}_2 . \quad (7.1.5)$$

Therefore the forces add as vectors as well (Figure 7.2),

$$\vec{\mathbf{F}}^T = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 . \quad (7.1.6)$$

This last statement is not a definition but a consequence of the experimental result described by Equation (7.1.5) and our definition of force.

Example 7.1 Vector Decomposition Solution

Two horizontal ropes are attached to a post that is stuck in the ground. The ropes pull the post producing the vector forces $\vec{\mathbf{F}}_1 = 70\text{ N}\,\hat{\mathbf{i}} + 20\text{ N}\,\hat{\mathbf{j}}$ and $\vec{\mathbf{F}}_2 = -30\text{ N}\,\hat{\mathbf{i}} + 40\text{ N}\,\hat{\mathbf{j}}$ as shown in Figure 7.3. Find the direction and magnitude of the horizontal component of a contact force of the ground on the post.

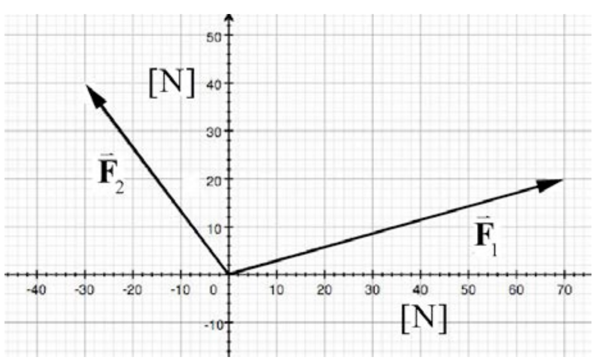

Figure 7.3 Example 7.1

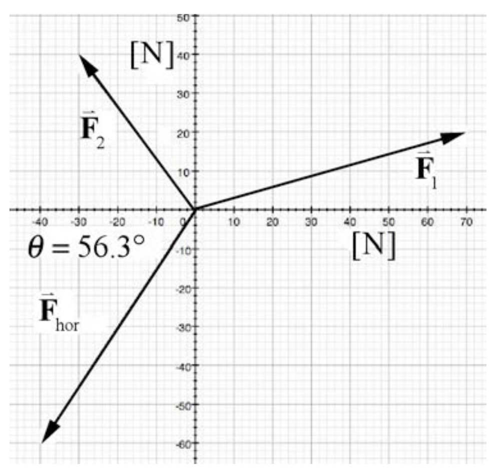

Figure 7.4 Vector sum of the horizontal forces

Solution: Because the ropes are pulling the post horizontally, the contact force must have a horizontal component that is equal to the negative of the sum of the two horizontal forces exerted by the rope on the post (Figure 7.4). There is an additional vertical component of the contact force that balances the gravitational force exerted on the post by the earth. We restrict our attention to the horizontal component of the contact force. Let $\vec{\mathbf{F}}_3$ denote the sum of the forces due to the ropes. Then we can write the vector $\vec{\mathbf{F}}_3$ as

$$\begin{aligned}\vec{\mathbf{F}}_3 &= (F_{1x} + F_{2x})\,\hat{\mathbf{i}} + (F_{1y} + F_{2y})\,\hat{\mathbf{j}} = (70\text{ N} + -30\text{ N})\,\hat{\mathbf{i}} + (20\text{ N} + 40\text{ N})\,\hat{\mathbf{j}} \\ &= (40\text{ N})\,\hat{\mathbf{i}} + (60\text{ N})\,\hat{\mathbf{j}}\end{aligned}$$

Therefore the horizontal component of the contact force satisfies the condition that

$$\vec{\mathbf{F}}_{hor} = -\vec{\mathbf{F}}_3 = -(\vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2) = (-40\text{ N})\,\hat{\mathbf{i}} + (-60\text{ N})\,\hat{\mathbf{j}}.$$

The magnitude is $\left|\vec{\mathbf{F}}_{hor}\right| = \sqrt{(-40\text{ N})^2 + (-60\text{ N})^2} = 72\text{ N}$. The horizontal component of the contact force makes an angle

$$\theta = \tan^{-1}\left[\frac{60\text{ N}}{40\text{ N}}\right] = 56.3^\circ$$

as shown in the figure above.

7.1.1 Mass Calibration

So far, we have only used the standard body to measure force. Instead of performing experiments on the standard body, we can calibrate the masses of all other bodies in terms of the standard mass by the following experimental procedure. We shall refer to the mass measured in this way as the **inertial mass** and denote it by m_{in}.

We apply a force of magnitude F to the standard body and measure the magnitude of the acceleration a_s. Then we apply the same force to a second body of unknown mass m_{in} and measure the magnitude of the acceleration a_{in}. Because the same force is applied to both bodies,

$$F = m_{in}\, a_{in} = m_s\, a_s, \tag{1.7}$$

the ratio of the inertial mass to the standard mass is equal to the inverse ratio of the magnitudes of the accelerations,

$$\frac{m_{in}}{m_s} = \frac{a_s}{a_{in}}. \tag{1.8}$$

Therefore the second body has inertial mass equal to

$$m_{in} = m_s \frac{a_s}{a_{in}}. \tag{1.9}$$

This method is justified by the fact that we can repeat the experiment using a different force and still find that the ratios of the acceleration are the same. For simplicity we shall denote the inertial mass by m.

7.2 Newton's First Law

The First Law of Motion, commonly called the "Principle of Inertia," was first realized by Galileo. (Newton did not acknowledge Galileo's contribution.) Newton was particularly concerned with how to phrase the First Law in Latin, but after many rewrites Newton choose the following expression for the First Law (in English translation):

> *Law 1: Every body continues in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed upon it.*
>
> *Projectiles continue in their motions, so far as they are not retarded by the resistance of air, or impelled downwards by the force of gravity. A top, whose parts by their cohesion are continually drawn aside from rectilinear motions, does not cease its rotation, otherwise than as it is retarded by air. The greater bodies of planets and comets, meeting with less resistance in freer spaces, preserve their motions both progressive and circular for a much longer time.*[4]

The first law is an experimental statement about the motions of bodies. When a body moves with constant velocity, there are either no forces present or the sum of all the forces acting on the body is zero. If the body changes its velocity, it has non-zero acceleration, and hence the sum of all the forces acting on the body must be non-zero as well. If the velocity of a body changes in time, then either the direction or magnitude changes, or both can change.

After a bus or train starts, the acceleration is often so small we can barely perceive it. We are often startled because it seems as if the station is moving in the opposite direction while we seem to be at rest. Newton's First Law states that there is no physical way to distinguish between whether we are moving or the station is moving, because there is nearly zero total force acting on the body. Once we reach a constant velocity, our minds dismiss the idea that the ground is moving backwards because we think it is impossible, but there is no actual way for us to distinguish whether the train is moving or the ground is moving.

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[4] Ibid. p. 13.

7.3 Momentum, Newton's Second Law and Third Law

Newton began his analysis of the cause of motion by introducing the quantity of motion:

> ***Definition: Quantity of Motion***
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> *The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjointly.*
>
> *The motion of the whole is the sum of the motion of all its parts; and therefore in a body double in quantity, with equal velocity, the motion is double, with twice the velocity, it is quadruple.*[5]

Our modern term for quantity of motion is *momentum* and it is a vector quantity

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}}\,, \tag{7.3.1}$$

where m is the inertial mass and $\vec{\mathbf{v}}$ is the velocity of the body. Newton's Second Law states that

> *Law II: The change of motion is proportional to the motive force impressed, and is made in the direction of the right line in which that force is impressed.*
>
> *If any force generates a motion, a double force will generate double the motion, a triple force triple the motion, whether that force is impressed altogether and at once or gradually and successively. And this motion (being always directed the same way with the generating force), if the body moved before, is added or subtracted from the former motion, according as they directly conspire with or are directly contrary to each other; or obliquely joined, when they are oblique, so as to produce a new motion compounded from the determination of both.*[6]

Suppose that a force is applied to a body for a time interval Δt. The impressed force or *impulse* (a vector quantity $\vec{\mathbf{I}}$) produces a change in the momentum of the body,

$$\vec{\mathbf{I}} = \vec{\mathbf{F}}\Delta t = \Delta\vec{\mathbf{p}}\,. \tag{7.3.2}$$

From the commentary to the second law, Newton also considered forces that were applied continually to a body instead of impulsively. The instantaneous action of the total

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[5] Ibid. p. 1.
[6] Ibid. p. 13.

force acting on a body at a time t is defined by taking the mathematical limit as the time interval Δt becomes smaller and smaller,

$$\vec{\mathbf{F}} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\mathbf{p}}}{\Delta t} \equiv \frac{d\vec{\mathbf{p}}}{dt}. \tag{7.3.3}$$

When the mass remains constant in time, the Second Law can be recast in its more familiar form,

$$\vec{\mathbf{F}} = m\frac{d\vec{\mathbf{v}}}{dt}. \tag{7.3.4}$$

Because the derivative of velocity is the acceleration, the force is the product of mass and acceleration,

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}}. \tag{7.3.5}$$

Because we defined force in terms of change in motion, the Second Law appears to be a restatement of this definition, and devoid of predictive power since force is only determined by measuring acceleration. What transforms the Second Law from just a definition is the additional input that comes from *force laws* that are based on experimental observations on the interactions between bodies. Throughout this book, we shall investigate these force laws and learn to use them in order to determine the forces and accelerations acting on a body (left-hand-side of Newton’s Second Law). When a physical body is constrained to move along a surface, or inside a container (for example gas molecules in a container), there are *constraint forces* that are not determined beforehand by any force law but are only determined by their effect on the motion of the body. For any given constrained motion, these constraint forces are unknown and must be determined by the particular motion of the body that we are studying, for example the contact force of the surface on the body, or the force of the wall on the gas particles.

The right-hand-side of Newton’s Second Law is the product of mass with acceleration. Acceleration is a mathematical description of how the velocity of a body changes. Knowledge of all the forces acting on the body enables us to predict the acceleration. Eq. (7.3.5) is known as the *equation of motion.* Once we know this equation we may be able to determine the velocity and position of that body at all future times by integration techniques, or computational techniques. For constrained motion, if we know the acceleration of the body, we can also determine the constraint forces acting on the body.