19.8 Principle of Conservation of Angular Momentum

Consider a system of particles. We begin with the result that we derived in Section 19.7 that the torque about a point S is equal to the time derivative of the angular momentum about that point S,

$$\vec{\boldsymbol{\tau}}_S^{\text{ext}} = \frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt}. \qquad (19.5.33)$$

With this assumption, the torque due to the external forces is equal to the rate of change of the angular momentum

$$\vec{\boldsymbol{\tau}}_S^{\text{ext}} = \frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt}. \qquad (19.5.34)$$

Principle of Conservation of Angular Momentum

If the external torque acting on a system is zero, then the angular momentum of the system is constant. So for any change of state of the system the change in angular momentum is zero

$$\Delta\vec{\mathbf{L}}_S^{\text{sys}} \equiv (\vec{\mathbf{L}}_S^{\text{sys}})_f - (\vec{\mathbf{L}}_S^{\text{sys}})_i = \vec{\mathbf{0}}. \qquad (19.5.35)$$

Equivalently the angular momentum is constant

$$(\vec{\mathbf{L}}_S^{\text{sys}})_f = (\vec{\mathbf{L}}_S^{\text{sys}})_i. \qquad (19.5.36)$$

So far no isolated system has been encountered such that the angular momentum is not constant so our assumption that internal torques cancel is pairs can be taken as an experimental observation.

Example 19.7 Collision Between Pivoted Rod and Object

A point-like object of mass m_1 moving with constant speed v_i strikes a rigid uniform rod of length l and mass m_2 that is hanging by a frictionless pivot from the ceiling. Immediately after striking the rod, the object continues forward but its speed decreases to $v_i / 2$ (Figure 19.19). The moment of inertia of the rod about its center of mass is $I_{cm} = (1/12)m_2 l^2$. Gravity acts with acceleration g downward. (a) For what value of v_i will the rod just touch the ceiling on its first swing? (b) For what ratio m_2 / m_1 will the collision be elastic?

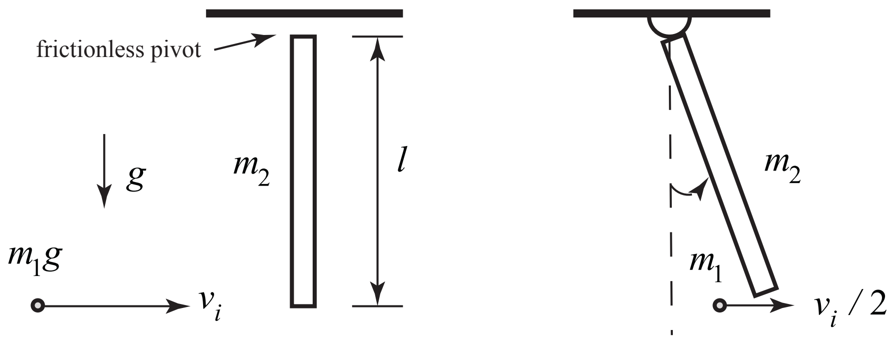


Figure 19.19 Example 19.7

Solution: We begin by identifying our system, which consists of the object and the uniform rod. We identify three states; an initial state i: immediately before the collision, state a: immediately after the collision, and state f: the instant the rod touches the ceiling when the final angular speed is zero. We would like to know if any of our fundamental quantities: momentum, energy, and angular momentum, are constant during these state changes, state i to state a, state a to state f.

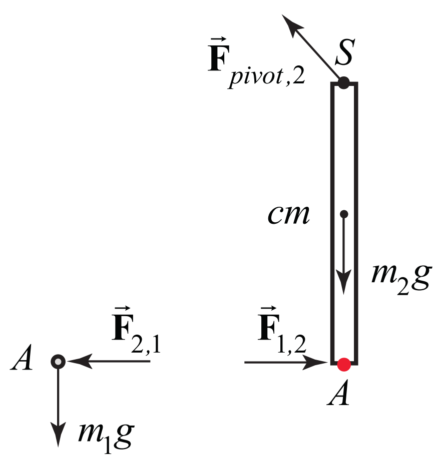


Figure 19.20 Free-body force diagrams on particle and rod

We start with the transition from state i to state a. The pivot force holding the rod to the ceiling is an external force acting at the pivot point S. There is also the gravitational force acting at the center of mass of the rod and on the object. There are also internal forces due to the collision of the rod and the object at point A (Figure 19.20).

The external force means that momentum is not constant. The point of action of the external pivot force is fixed and so does no work. However, we do not know whether or not the collision is elastic and so we cannot assume that mechanical energy is constant. Choose the pivot point S as the point about which to calculate torque, then the torque diagrams are shown in Figure 19.21.

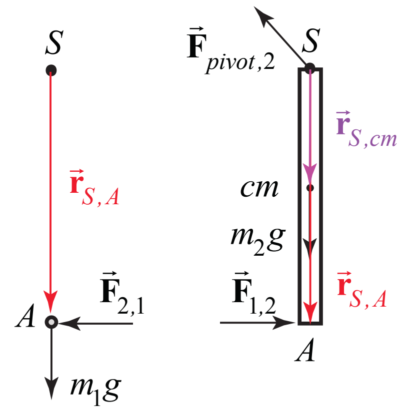


Figure 19.21 Torque diagrams on particle and rod with torque calculated about pivot point S

The torque on the system about the pivot S is then the sum of terms

$$\vec{\boldsymbol{\tau}}_S^{\text{sys}} = \vec{\mathbf{r}}_{S,S} \times \vec{\mathbf{F}}_{pivot,2} + \vec{\mathbf{r}}_{S,A} \times \vec{\mathbf{F}}_{1,2} + \vec{\mathbf{r}}_{S,A} \times \vec{\mathbf{F}}_{2,1} + \vec{\mathbf{r}}_{S,cm} \times m_2 \vec{\mathbf{g}} + \vec{\mathbf{r}}_{S,A} \times m_1 \vec{\mathbf{g}} \, . \quad (19.5.37)$$

The external pivot force does not contribute any torque because $\vec{\mathbf{r}}_{S,S} = \vec{\mathbf{0}}$. The internal forces between the rod and the object are equal in magnitude and opposite in direction, $\vec{\mathbf{F}}_{1,2} = -\vec{\mathbf{F}}_{2,1}$ (Newton's Third Law), and so their contributions to the torque add to zero. If the collision is instantaneous then the gravitational force is parallel to $\vec{\mathbf{r}}_{S,cm}$ and $\vec{\mathbf{r}}_{S,A}$ so the two gravitational torques are zero. Therefore the torque on the system about the pivot point is zero, $\vec{\boldsymbol{\tau}}_S^{\text{sys}} = \vec{\mathbf{0}}$. Thus the angular momentum about the pivot point is constant,

$$\vec{\mathbf{L}}_{S,i}^{\text{sys}} = \vec{\mathbf{L}}_{S,a}^{\text{sys}} \, . \quad (19.5.38)$$

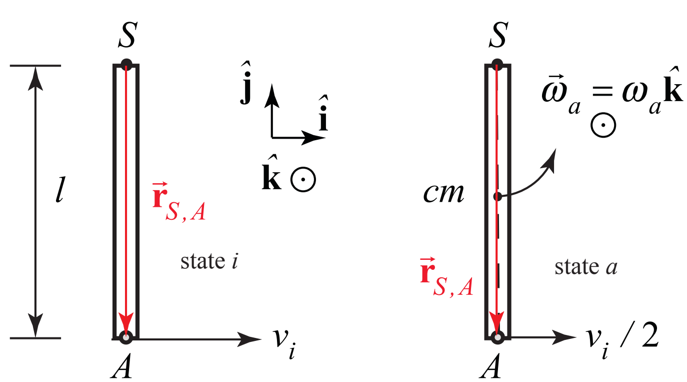


Figure 19.22 Angular momentum diagram

In order to calculate the angular momentum we draw a diagram showing the momentum of the object and the angular speed of the rod in (Figure 19.22). The angular momentum about S immediately before the collision is

$$\vec{\mathbf{L}}^{\text{sys}}_{S,i} = \vec{\mathbf{r}}_{S,1} \times m_1 \vec{\mathbf{v}}_i = l(-\hat{\mathbf{j}}) \times m_1 v_i \hat{\mathbf{i}} = l m_1 v_i \hat{\mathbf{k}} .$$

The angular momentum about S immediately after the collision is

$$\vec{\mathbf{L}}^{\text{sys}}_{S,a} = \vec{\mathbf{r}}_{S,1} \times m_1 \vec{\mathbf{v}}_i / 2 + I_S \vec{\boldsymbol{\omega}}_a = l(-\hat{\mathbf{j}}) \times m_1 (v_i / 2) \hat{\mathbf{i}} + I_S \omega_a \hat{\mathbf{k}} = (l m_1 v_i / 2) \hat{\mathbf{k}} + I_S \omega_a \hat{\mathbf{k}} .$$

Therefore the condition that the angular momentum about S is constant during the collision becomes

$$l m_1 v_i \hat{\mathbf{k}} = (l m_1 v_i / 2 + I_S \omega_a) \hat{\mathbf{k}} .$$

We can solve for the angular speed immediately after the collision

$$\omega_a = \frac{l m_1 v_i}{2 I_S} .$$

By the parallel axis theorem the moment of inertial of a uniform rod about the pivot point is

$$I_S = m_2 (l/2)^2 + I_{cm} = (1/4) m_2 l^2 + (1/12) m_2 l^2 = (1/3) m_2 l^2 . \quad (19.5.39)$$

Therefore the angular speed immediately after the collision is

$$\omega_2 = \frac{3 m_1 v_i}{2 m_2 l} . \quad (19.5.40)$$

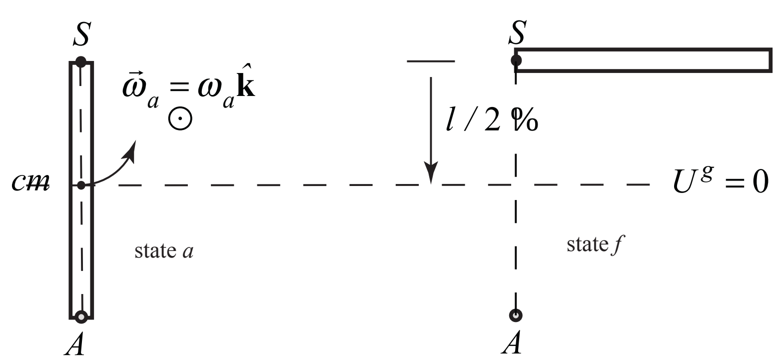


Figure 19.23 Energy diagram for transition from state a to state f .

For the transition from state a to state f, we know that the gravitational force is conservative and the pivot force does no work so mechanical energy is constant.

$$E_a^{mech} = E_f^{mech}$$

We draw an energy diagram only for the rod because the kinetic energy for the particle is not changing between states a and f, (Figure 19.23), with a choice of zero for the potential energy at the center of mass. The mechanical energy of the rod and particle immediately after the collision is

$$E_a^{mech} = \frac{1}{2} I_S \omega_a^2 + \frac{1}{2} m_1 (v_i / 2)^2 .$$

Using our results for the moment of inertia I_S (Eq. (19.5.39)) and ω_2 (Eq. (19.5.40)), we have that

$$E_a^{mech} = \frac{1}{2}(1/3) m_2 l^2 \left(\frac{3 m_1 v_i}{2 m_2 l} \right)^2 + \frac{1}{2} m_1 (v_i / 2)^2 = \frac{3 m_1^{\,2} v_i^{\,2}}{8 m_2} + \frac{1}{2} m_1 (v_i / 2)^2 . \quad (19.5.41)$$

The mechanical energy when the rod just reaches the ceiling when the final angular speed is zero is then

$$E_f^{mech} = m_2 g (l/2) + \frac{1}{2} m_1 (v_i / 2)^2 .$$

Then the condition that the mechanical energy is constant becomes

$$\frac{3 m_1^{\,2} v_i^{\,2}}{8 m_2} + \frac{1}{2} m_1 (v_i / 2)^2 = m_2 g (l/2) + \frac{1}{2} m_1 (v_i / 2)^2 . \quad (19.5.42)$$

We can now solve Eq. (19.5.42) for the initial speed of the object

$$v_i = \frac{m_2}{m_1} \sqrt{\frac{4gl}{3}} . \quad (19.5.43)$$

We now return to the transition from state i to state a. and determine the constraint on the mass ratio in order for the collision to be elastic. The mechanical energy before the collision is

$$E_i^{mech} = \frac{1}{2} m_1 v_i^{\,2} . \quad (19.5.44)$$

If we impose the condition that the collision is elastic then

$$E_i^{mech} = E_a^{mech} . \quad (19.5.45)$$

Substituting Eqs. (19.5.41) and (19.5.44) into Eq. (19.5.45) yields

$$\frac{1}{2} m_1 v_i^2 = \frac{3 m_1^2 v_i^2}{8 m_2} + \frac{1}{2} m_1 (v_i / 2)^2 .$$

This simplifies to

$$\frac{3}{8} m_1 v_i^2 = \frac{3 m_1^2 v_i^2}{8 m_2}$$

Hence we can solve for the mass ratio necessary to ensure that the collision is elastic if the final speed of the object is half it's initial speed

$$\frac{m_2}{m_1} = 1 . \qquad (19.5.46)$$

Notice that this mass ratio is independent of the initial speed of the object.

19.9 External Angular Impulse and Change in Angular Momentum

Define the ***external angular impulse*** *about a point* S *applied as the integral of the external torque about* S

$$\vec{\mathbf{J}}_S^{\text{ext}} \equiv \int_{t_i}^{t_f} \vec{\boldsymbol{\tau}}_S^{\text{ext}} \, dt . \qquad (19.5.47)$$

Then the external angular impulse about S is equal to the change in angular momentum

$$\vec{\mathbf{J}}_S^{\text{ext}} \equiv \int_{t_i}^{t_f} \vec{\boldsymbol{\tau}}_S^{\text{ext}} \, dt = \int_{t_i}^{t_f} \frac{d\vec{\mathbf{L}}_S^{\text{sys}}}{dt} dt = \vec{\mathbf{L}}_{S,f}^{\text{sys}} - \vec{\mathbf{L}}_{S,i}^{\text{sys}} . \qquad (19.5.48)$$

Notice that this is the rotational analog to our statement about impulse and momentum,

$$\vec{\mathbf{I}}_S^{\text{ext}} \equiv \int_{t_i}^{t_f} \vec{\mathbf{F}}^{\text{ext}} \, dt = \int_{t_i}^{t_f} \frac{d\vec{\mathbf{p}}_{\text{sys}}}{dt} dt = \vec{\mathbf{p}}_{\text{sys},f} - \vec{\mathbf{p}}_{\text{sys},i} . \qquad (19.5.49)$$

Example 19.8 Angular Impulse on Steel Washer

A steel washer is mounted on the shaft of a small motor. The moment of inertia of the motor and washer is I_0. The washer is set into motion. When it reaches an initial angular speed ω_0, at $t = 0$, the power to the motor is shut off, and the washer slows down until it reaches an angular speed of ω_a at time t_a. At that instant, a second steel washer with a moment of inertia I_w is dropped on top of the first washer. Assume that the second washer is only in contact with the first washer. The collision takes place over a time $\Delta t_{\text{int}} = t_b - t_a$. Assume the frictional torque on the axle is independent of speed, and remains the same when the second washer is dropped. The two washers continue to slow down during the time interval $\Delta t_2 = t_f - t_b$ until they stop at time $t = t_f$. (a) What is the angular acceleration while the washer and motor are slowing down during the interval $\Delta t_1 = t_a$? (b) Suppose the collision is nearly instantaneous, $\Delta t_{\text{int}} = (t_b - t_a) \simeq 0$. What is the angular speed ω_b of the two washers immediately after the collision is finished (when the washers rotate together)?

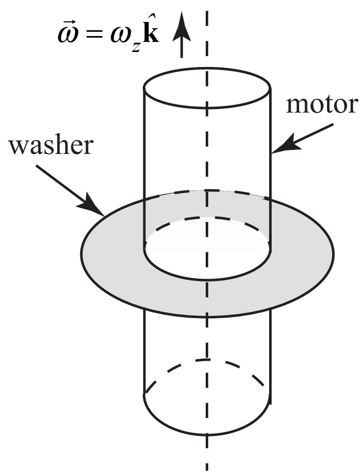


Figure 19.24 Example 19.8

Now suppose the collision is not instantaneous but that the frictional torque is independent of the speed of the rotor. (c) What is the angular impulse during the collision? (d) What is the angular velocity ω_b of the two washers immediately after the collision is finished (when the washers rotate together)? (e) What is the angular deceleration α_2 after the collision?

Solution: a) The angular acceleration of the motor and washer from the instant when the power is shut off until the second washer was dropped is given by

$$\alpha_1 = \frac{\omega_a - \omega_0}{\Delta t_1} < 0. \tag{19.5.50}$$

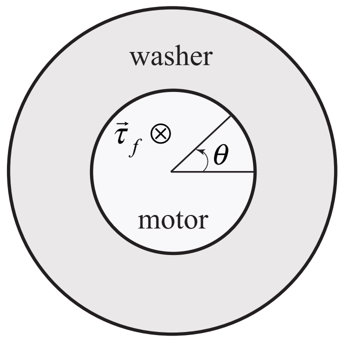
(b) If the collision is nearly instantaneous, then there is no angular impulse and therefore the z-component of the angular momentum about the rotation axis of the motor remains constant

$$0 = \Delta L_z = L_{f,z} - L_{0,z} = (I_0 + I_{\text{w}})\omega_b - I_0 \omega_a . \tag{19.5.51}$$

We can solve Eq. (19.5.51) for the angular speed ω_b of the two washers immediately after the collision is finished

$$\omega_b = \frac{I_0}{I_0 + I_{\mathrm{w}}}\omega_a . \tag{19.5.52}$$

(c) The angular acceleration found in part a) is due to the frictional torque in the motor.



overhead view

Figure 19.25 Frictional torque in the motor

Let $\vec{\boldsymbol{\tau}}_f = -\tau_f \hat{\mathbf{k}}$ where τ_f is the magnitude of the frictional torque (Figure 19.25) then

$$-\tau_f = I_0 \alpha_1 = \frac{I_0(\omega_a - \omega_0)}{\Delta t_1} . \tag{19.5.53}$$

During the collision with the second washer, the frictional torque exerts an angular impulse (pointing along the z-axis in the figure),

$$J_z = -\int_{t_a}^{t_b} \tau_f \, dt = -\tau_f \, \Delta t_{\mathrm{int}} = I_0(\omega_a - \omega_0)\frac{\Delta t_{\mathrm{int}}}{\Delta t_1} . \tag{19.5.54}$$

(d) The z-component of the angular momentum about the rotation axis of the motor changes during the collision,

$$\Delta L_z = L_{f,z} - L_{0,z} = (I_0 + I_{\mathrm{w}})\omega_b - I_0 \omega_a . \tag{19.5.55}$$

The change in the z-component of the angular momentum is equal to the z-component of the angular impulse

$$J_z = \Delta L_z . \tag{19.5.56}$$

Thus, equating the expressions in Equations (19.5.54) and (19.5.55), yields

$$I_0(\omega_a - \omega_0)\left(\frac{\Delta t_{\mathrm{int}}}{\Delta t_1}\right) = (I_0 + I_{\mathrm{w}})\omega_b - (I_0)\omega_a . \tag{19.5.57}$$

Solve Equation (19.5.57) for the angular velocity immediately after the collision,

$$\omega_b = \frac{I_0}{(I_0 + I_{\mathrm{w}})}\left((\omega_a - \omega_0)\left(\frac{\Delta t_{\mathrm{int}}}{\Delta t_1} \right) + \omega_a \right). \tag{19.5.58}$$

If there were no frictional torque, then the first term in the brackets would vanish, and the second term of Eq. (19.5.58) would be the only contribution to the final angular speed.

(e) The final angular acceleration α_2 is given by

$$\alpha_2 = \frac{0 - \omega_b}{\Delta t_2} = -\frac{I_0}{(I_0 + I_{\mathrm{w}})\Delta t_2}\left((\omega_a - \omega_0)\left(\frac{\Delta t_{\mathrm{int}}}{\Delta t_1} \right) + \omega_a \right). \tag{19.5.59}$$