

19.3 Torque and the Time Derivative of Angular Momentum about a Point for a Particle

We will now show that the torque about a point S is equal to the time derivative of the angular momentum about S ,

$$\vec{\tau}_S = \frac{d\vec{L}_S}{dt}. \quad (19.3.1)$$

Take the time derivative of the angular momentum about S ,

$$\frac{d\vec{L}_S}{dt} = \frac{d}{dt}(\vec{r}_S \times \vec{p}). \quad (19.3.2)$$

In this equation we are taking the time derivative of a vector product of two vectors. There are two important facts that will help us simplify this expression. First, the time derivative of the vector product of two vectors satisfies the product rule,

$$\frac{d\vec{L}_S}{dt} = \frac{d}{dt}(\vec{r}_S \times \vec{p}) = \left(\left(\frac{d\vec{r}_S}{dt} \right) \times \vec{p} \right) + \left(\vec{r}_S \times \left(\frac{d\vec{p}}{dt} \right) \right). \quad (19.3.3)$$

Second, the first term on the right hand side vanishes,

$$\frac{d\vec{r}_S}{dt} \times \vec{p} = \vec{v} \times m\vec{v} = \vec{0}. \quad (19.3.4)$$

The rate of angular momentum change about the point S is then

$$\frac{d\vec{L}_S}{dt} = \vec{r}_S \times \frac{d\vec{p}}{dt}. \quad (19.3.5)$$

From Newton's Second Law, the force on the particle is equal to the derivative of the linear momentum,

$$\vec{F} = \frac{d\vec{p}}{dt}. \quad (19.3.6)$$

Therefore the rate of change in time of angular momentum about the point S is

$$\frac{d\vec{L}_S}{dt} = \vec{r}_S \times \vec{F}. \quad (19.3.7)$$

Recall that the torque about the point S due to the force \vec{F} acting on the particle is

$$\vec{\tau}_S = \vec{r}_S \times \vec{F}. \quad (19.3.8)$$

Combining the expressions in (19.3.7) and (19.3.8), it is readily seen that the torque about the point S is equal to the rate of change of angular momentum about the point S ,

$$\vec{\tau}_S = \frac{d\vec{L}_S}{dt}. \quad (19.3.9)$$

19.4 Conservation of Angular Momentum about a Point

So far we have introduced two conservation principles, showing that energy is constant for closed systems (no change in energy in the surroundings) and linear momentum is constant isolated system. The change in mechanical energy of a closed system is

$$W_{nc} = \Delta E_m = \Delta K + \Delta U, \quad (\text{closed system}). \quad (19.3.10)$$

If the non-conservative work done in the system is zero, then the mechanical energy is constant,

$$0 = W_{nc} = \Delta E_{\text{mechanical}} = \Delta K + \Delta U, \quad (\text{closed system}). \quad (19.3.11)$$

The conservation of linear momentum arises from Newton's Second Law applied to systems,

$$\vec{F}^{\text{ext}} = \sum_{i=1}^N \frac{d}{dt} \vec{p}_i = \frac{d}{dt} \vec{p}_{\text{sys}} \quad (19.3.12)$$

Thus if the external force in any direction is zero, then the component of the momentum of the system in that direction is a constant. For example, if there are no external forces in the x - and y -directions then

$$\begin{aligned} \vec{0} &= (\vec{F}^{\text{ext}})_x = \frac{d}{dt} (\vec{p}_{\text{sys}})_x \\ \vec{0} &= (\vec{F}^{\text{ext}})_y = \frac{d}{dt} (\vec{p}_{\text{sys}})_y. \end{aligned} \quad (19.3.13)$$

We can now use our relation between torque about a point S and the change of the angular momentum about S , Eq. (19.3.9), to introduce a new conservation law. Suppose we can find a point S such that torque about the point S is zero,

$$\vec{0} = \vec{\tau}_S = \frac{d\vec{L}_S}{dt}, \quad (19.3.14)$$

then the angular momentum about the point S is a constant vector, and so the change in angular momentum is zero,

$$\Delta \vec{L}_S \equiv \vec{L}_{S,f} - \vec{L}_{S,i} = \vec{0}. \quad (19.3.15)$$

Thus when the torque about a point S is zero, the final angular momentum about S is equal to the initial angular momentum,

$$\vec{L}_{S,f} = \vec{L}_{S,i}. \quad (19.3.16)$$

Example 19.4 Meteor Flyby of Earth

A meteor of mass m is approaching Earth as shown in the figure. The radius of Earth is R_E . The mass of Earth is M_E . Assume that the meteor started very far away from Earth with

speed v_i and at a perpendicular distance h from the axis of symmetry of the orbit. At some later time the meteor just grazes Earth (Figure 19.9). You may ignore all other gravitational forces except due to Earth. Find the distance h . Hint: What quantities are constant for this orbit?

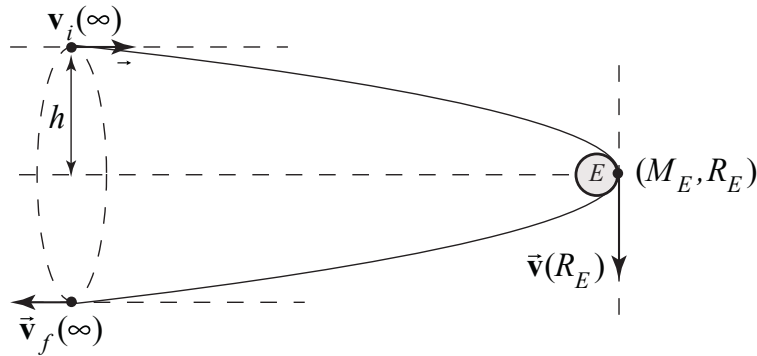


Figure 19.9 Meteor flyby of earth

Solution: In this problem both energy and angular momentum about the center of Earth are constant (see below for justification).

The meteor's mass is so much small than the mass of Earth that we will assume that the earth's motion is not affected by the meteor. We'll also need to neglect any air resistance when the meteor approaches Earth. Choose the center of Earth, (point S) to calculate the torque and angular momentum. The force on the meteor is

$$\vec{F}_{E,m}^G = -\frac{GmM_E}{r^2} \hat{r}$$

The vector from the center of Earth to the meteor is $\vec{r}_S = r\hat{r}$. The torque about S is zero because they two vectors are anti-parallel

$$\vec{\tau}_S = \vec{r}_S \times \vec{F}_{E,m}^G = r\hat{r} \times -\frac{GmM_E}{r^2} \hat{r} = \vec{0}$$

Therefore the angular momentum about the center of Earth is a constant.

The initial angular momentum is

$$\vec{L}_{S,i} = \vec{r}_{S,i} \times m\vec{v}_i = (x_i\hat{i} + h\hat{j}) \times mv_i\hat{i} = -hmv_i\hat{k}$$

When the meteor just grazes Earth, the angular momentum is

$$\vec{L}_{S,E} = \vec{r}_{S,E} \times m\vec{v}_p = R_E\hat{i} \times mv_p(-\hat{j}) = -R_Emv_p\hat{k}$$

where we have used v_p for the speed of the meteor at its nearest approach to Earth. The constancy of angular momentum requires that

$$mv_i h = mv_p R_E$$

In order to solve for h , we need to find v_p . Because we are neglecting all forces on the meteor other than Earth's gravity, mechanical energy is constant, and

$$\frac{1}{2}mv_i^2 = \frac{1}{2}mv_p^2 - \frac{GmM_E}{R_E},$$

where we have taken the meteor to have speed v_i at a distance “very far away from Earth” to mean that we neglect any gravitational potential energy in the meteor-Earth system, when $r \rightarrow \infty$, $U(r) = -GmM_E/r \rightarrow 0$. From the angular momentum condition, $v_p = v_i h / R_E$ and therefore the energy condition can be rewritten as

$$v_i^2 = v_i^2 \left(\frac{h}{R_E} \right)^2 - \frac{2GM_E}{R_E}$$

which we solve for the impact parameter h

$$h = \sqrt{R_E^2 + \frac{2GM_E R_E}{v_i^2}}.$$

19.5 Angular Impulse and Change in Angular Momentum

If there is a total applied torque $\vec{\tau}_S$ about a point S over an interval of time $\Delta t = t_f - t_i$, then the torque applies an *angular impulse* about a point S , given by

$$\vec{\mathbf{J}}_S = \int_{t_i}^{t_f} \vec{\tau}_S dt. \quad (19.4.1)$$

Because $\vec{\tau}_S = d\vec{\mathbf{L}}_S^{\text{total}}/dt$, the angular impulse about S is equal to the change in angular momentum about S ,

$$\vec{\mathbf{J}}_S = \int_{t_i}^{t_f} \vec{\tau}_S dt = \int_{t_i}^{t_f} \frac{d\vec{\mathbf{L}}_S}{dt} dt = \Delta\vec{\mathbf{L}}_S = \vec{\mathbf{L}}_{S,f} - \vec{\mathbf{L}}_{S,i}. \quad (19.4.2)$$

This result is the rotational analog to linear impulse, which is equal to the change in momentum,

$$\vec{\mathbf{I}} = \int_{t_i}^{t_f} \vec{\mathbf{F}} dt = \int_{t_i}^{t_f} \frac{d\vec{\mathbf{p}}}{dt} dt = \Delta\vec{\mathbf{p}} = \vec{\mathbf{p}}_f - \vec{\mathbf{p}}_i. \quad (19.4.3)$$

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