17.4 Torque, Angular Acceleration, and Moment of Inertia

17.4.1 Torque Equation for Fixed Axis Rotation

For fixed-axis rotation, there is a direct relation between the component of the torque along the axis of rotation and angular acceleration. Consider the forces that act on the rotating body. Generally, the forces on different volume elements will be different, and so we will denote the force on the volume element of mass Δm_i by $\vec{\mathbf{F}}_i$. Choose the z-axis to lie along the axis of rotation. Divide the body into volume elements of mass Δm_i. Let the point S denote a specific point along the axis of rotation (Figure 17.19). Each volume element undergoes a tangential acceleration as the volume element moves in a circular orbit of radius $r_i = |\vec{\mathbf{r}}_i|$ about the fixed axis.

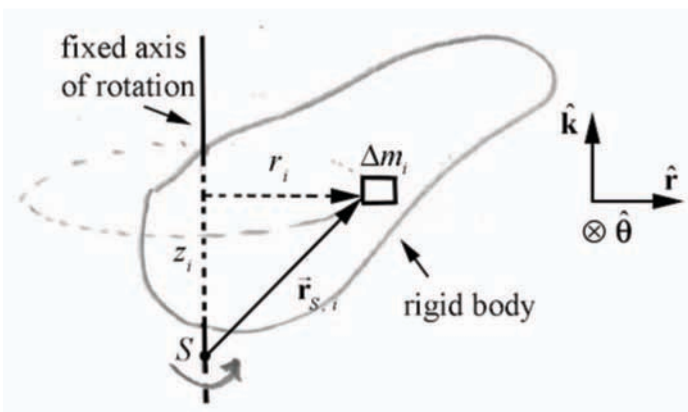


Figure 17.19: Volume element undergoing fixed-axis rotation about the z-axis.

The vector from the point S to the volume element is given by

$$\vec{\mathbf{r}}_{S,\,i} = z_i\,\hat{\mathbf{k}} + \vec{\mathbf{r}}_i = z_i\,\hat{\mathbf{k}} + r_i\,\hat{\mathbf{r}} \tag{17.3.1}$$

where z_i is the distance along the axis of rotation between the point S and the volume element. The torque about S due to the force $\vec{\mathbf{F}}_i$ acting on the volume element is given by

$$\vec{\boldsymbol{\tau}}_{S,i} = \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_i . \quad (17.3.2)$$

Substituting Eq. (17.3.1) into Eq. (17.3.2) gives

$$\vec{\boldsymbol{\tau}}_{S,i} = (z_i\,\hat{\mathbf{k}} + r_i\,\hat{\mathbf{r}}) \times \vec{\mathbf{F}}_i . \quad (17.3.3)$$

For fixed-axis rotation, we are interested in the z-component of the torque, which must be the term

$$(\vec{\boldsymbol{\tau}}_{S,i})_z = (r_i\,\hat{\mathbf{r}} \times \vec{\mathbf{F}}_i)_z \quad (17.3.4)$$

because the vector product $z_i\,\hat{\mathbf{k}} \times \vec{\mathbf{F}}_i$ must be directed perpendicular to the plane formed by the vectors $\hat{\mathbf{k}}$ and $\vec{\mathbf{F}}_i$, hence perpendicular to the z-axis. The force acting on the volume element has components

$$\vec{\mathbf{F}}_i = F_{r,i}\,\hat{\mathbf{r}} + F_{\theta,i}\,\hat{\boldsymbol{\theta}} + F_{z,i}\,\hat{\mathbf{k}} . \quad (17.3.5)$$

The z-component $F_{z,i}$ of the force cannot contribute a torque in the z-direction, and so substituting Eq. (17.3.5) into Eq. (17.3.4) yields

$$(\vec{\boldsymbol{\tau}}_{S,i})_z = (r_i\,\hat{\mathbf{r}} \times (F_{r,i}\,\hat{\mathbf{r}} + F_{\theta,i}\,\hat{\boldsymbol{\theta}}))_z . \quad (17.3.6)$$

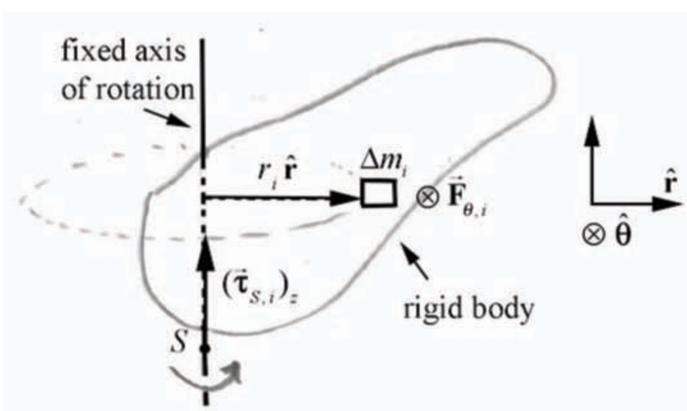


Figure 17.20 Tangential force acting on a volume element.

The radial force does not contribute to the torque about the z-axis, since

$$r_i\,\hat{\mathbf{r}} \times F_{r,i}\,\hat{\mathbf{r}} = \vec{\mathbf{0}} . \quad (17.3.7)$$

So, we are interested in the contribution due to torque about the z-axis due to the tangential component of the force on the volume element (Figure 17.20). The component of the torque about the z-axis is given by

$$(\vec{\boldsymbol{\tau}}_{S,i})_z = (r_i\,\hat{\mathbf{r}} \times F_{\theta,i}\,\hat{\boldsymbol{\theta}})_z = r_i\,F_{\theta,i}\,. \tag{17.3.8}$$

The z-component of the torque is directed upwards in Figure 17.20, where $F_{\theta,i}$ is positive (the tangential force is directed counterclockwise, as in the figure). Applying Newton's Second Law in the tangential direction,

$$F_{\theta,\,i} = \Delta m_i\,a_{\theta,\,i}\,. \tag{17.3.9}$$

Using our kinematics result that the tangential acceleration is $a_{\theta,\,i} = r_i\,\alpha_z$, where α_z is the z-component of angular acceleration, we have that

$$F_{\theta,\,i} = \Delta m_i\,r_i\,\alpha_z\,. \tag{17.3.10}$$

From Eq. (17.3.8), the component of the torque about the z-axis is then given by

$$(\vec{\boldsymbol{\tau}}_{S,i})_z = r_i\,F_{\theta,\,i} = \Delta m_i\,r_i^{\,2}\alpha_z\,. \tag{17.3.11}$$

The component of the torque about the z-axis is the summation of the torques on all the volume elements,

$$\begin{aligned}(\vec{\boldsymbol{\tau}}_S)_z &= \sum_{i=1}^{i=N}(\vec{\boldsymbol{\tau}}_{S,i})_z = \sum_{i=1}^{i=N} r_{\perp,\,i}\,F_{\theta,\,i}\\ &= \sum_{i=1}^{i=N}\Delta m_i r_i^{\,2}\alpha_z.\end{aligned} \tag{17.3.12}$$

Because each element has the same z-component of angular acceleration, α_z, the summation becomes

$$(\vec{\boldsymbol{\tau}}_S)_z = \left(\sum_{i=1}^{i=N}\Delta m_i\,r_i^{\,2}\right)\alpha_z\,. \tag{17.3.13}$$

Recalling our definition of the moment of inertia, (Chapter 16.3) the z-component of the torque is proportional to the z-component of angular acceleration,

$$\tau_{S,z} = I_S\,\alpha_z\,, \tag{17.3.14}$$

and the moment of inertia, I_S, is the constant of proportionality. The torque about the point S is the sum of the external torques and the internal torques

$$\vec{\boldsymbol{\tau}}_S = \vec{\boldsymbol{\tau}}_S^{\ \text{ext}} + \vec{\boldsymbol{\tau}}_S^{\ \text{int}} . \tag{17.3.15}$$

The external torque about the point S is the sum of the torques due to the net external force acting on each element

$$\vec{\boldsymbol{\tau}}_S^{\text{ext}} = \sum_{i=1}^{i=N} \vec{\boldsymbol{\tau}}_{S,i}^{\text{ext}} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_i^{\text{ext}} . \tag{17.3.16}$$

The internal torque arise from the torques due to the internal forces acting between pairs of elements

$$\vec{\boldsymbol{\tau}}_S^{\text{int}} = \sum_{i=1}^{N} \vec{\boldsymbol{\tau}}_{S,j}^{\text{int}} = \sum_{i=1}^{i=N} \sum_{\substack{j=1 \\ j\neq i}}^{j=N} \vec{\boldsymbol{\tau}}_{S,j,i}^{\text{int}} = \sum_{i=1}^{i=N} \sum_{\substack{j=1 \\ j\neq i}}^{j=N} \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_{j,i} . \tag{17.3.17}$$

We know by Newton's Third Law that the internal forces cancel in pairs, $\vec{\mathbf{F}}_{j,i} = -\vec{\mathbf{F}}_{i,j}$, and hence the sum of the internal forces is zero

$$\vec{\mathbf{0}} = \sum_{i=1}^{i=N} \sum_{\substack{j=1 \\ j\neq i}}^{j=N} \vec{\mathbf{F}}_{j,i} . \tag{17.3.18}$$

Does the same statement hold about pairs of internal torques? Consider the sum of internal torques arising from the interaction between the i^{th} and j^{th} particles

$$\vec{\boldsymbol{\tau}}_{S,j,i}^{\text{int}} + \vec{\boldsymbol{\tau}}_{S,i,j}^{\text{int}} = \vec{\mathbf{r}}_{S,i} \times \vec{\mathbf{F}}_{j,i} + \vec{\mathbf{r}}_{S,j} \times \vec{\mathbf{F}}_{i,j} . \tag{17.3.19}$$

By the Newton's Third Law this sum becomes

$$\vec{\boldsymbol{\tau}}_{S,j,i}^{\text{int}} + \vec{\boldsymbol{\tau}}_{S,i,j}^{\text{int}} = (\vec{\mathbf{r}}_{S,i} - \vec{\mathbf{r}}_{S,j}) \times \vec{\mathbf{F}}_{j,i} . \tag{17.3.20}$$

In the Figure 17.21, the vector $\vec{\mathbf{r}}_{S,i} - \vec{\mathbf{r}}_{S,j}$ points from the j^{th} element to the i^{th} element. If the internal forces between a pair of particles are directed along the line joining the two particles then the torque due to the internal forces cancel in pairs.

$$\vec{\boldsymbol{\tau}}_{S,j,i}^{\text{int}} + \vec{\boldsymbol{\tau}}_{S,i,j}^{\text{int}} = (\vec{\mathbf{r}}_{S,i} - \vec{\mathbf{r}}_{S,j}) \times \vec{\mathbf{F}}_{j,i} = \vec{\mathbf{0}} . \tag{17.3.21}$$

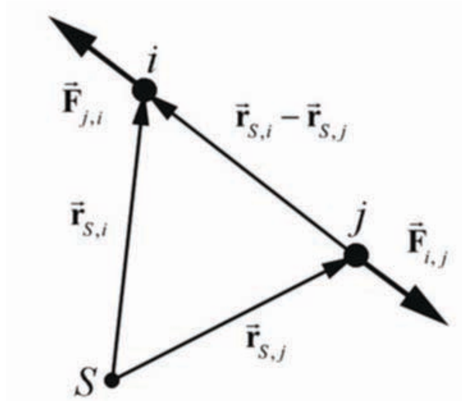


Figure 17.21 The internal force is directed along the line connecting the i^{th} and j^{th} particles

This is a stronger version of Newton's Third Law than we have so far since we have added the additional requirement regarding the direction of all the internal forces between pairs of particles. With this assumption, the torque is just due to the external forces

$$\vec{\boldsymbol{\tau}}_S = \vec{\boldsymbol{\tau}}_S^{\text{ext}} . \qquad (17.3.22)$$

Thus Eq. (17.3.14) becomes

$$(\tau_S^{\text{ext}})_z = I_S \alpha_z , \qquad (17.3.23)$$

This is very similar to Newton's Second Law: the total force is proportional to the acceleration,

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}} . \qquad (17.3.24)$$

where the mass, m, is the constant of proportionality.

17.4.2 Torque Acts at the Center of Gravity

Suppose a rigid body in static equilibrium consists of N particles labeled by the index $i = 1, 2, 3, ..., N$. Choose a coordinate system with a choice of origin O such that mass m_i has position $\vec{\mathbf{r}}_i$. Each point particle experiences a gravitational force $\vec{\mathbf{F}}_{\text{gravity},i} = m_i \vec{\mathbf{g}}$. The total torque about the origin is then zero (static equilibrium condition),

$$\vec{\boldsymbol{\tau}}_O = \sum_{i=1}^{i=N} \vec{\boldsymbol{\tau}}_{O,i} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_i \times \vec{\mathbf{F}}_{\text{gravity},i} = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_i \times m_i \vec{\mathbf{g}} = \vec{\mathbf{0}} . \qquad (17.3.25)$$

If the gravitational acceleration $\vec{\mathbf{g}}$ is assumed constant, we can rearrange the summation in Eq. (17.3.25) by pulling the constant vector $\vec{\mathbf{g}}$ out of the summation ($\vec{\mathbf{g}}$ appears in each term in the summation),

$$\vec{\boldsymbol{\tau}}_O = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_i \times m_i \vec{\mathbf{g}} = \left(\sum_{i=1}^{i=N} m_i \vec{\mathbf{r}}_i \right) \times \vec{\mathbf{g}} = \vec{\mathbf{0}}. \tag{17.3.26}$$

We now use our definition of the center of the center of mass, Eq. (10.5.3), to rewrite Eq. (17.3.26) as

$$\vec{\boldsymbol{\tau}}_O = \sum_{i=1}^{i=N} \vec{\mathbf{r}}_i \times m_i \vec{\mathbf{g}} = M_{\mathrm{T}} \vec{\mathbf{R}}_{\mathrm{cm}} \times \vec{\mathbf{g}} = \vec{\mathbf{R}}_{\mathrm{cm}} \times M_{\mathrm{T}} \vec{\mathbf{g}} = \vec{\mathbf{0}}. \tag{17.3.27}$$

Thus the torque due to the gravitational force acting on each point-like particle is equivalent to the torque due to the gravitational force acting on a point-like particle of mass M_{T} located at a point in the body called the ***center of gravity***, which is equal to the center of mass of the body in the typical case in which the gravitational acceleration $\vec{\mathbf{g}}$ is constant throughout the body.

Example 17.9 Turntable

The turntable in Example 16.1, of mass $1.2\ \mathrm{kg}$ and radius $1.3\times10^{1}\ \mathrm{cm}$, has a moment of inertia $I_S = 1.01\times10^{-2}\ \mathrm{kg}\cdot\mathrm{m}^2$ about an axis through the center of the turntable and perpendicular to the turntable. The turntable is spinning at an initial constant frequency $f_i = 33\ \mathrm{cycles}\cdot\mathrm{min}^{-1}$. The motor is turned off and the turntable slows to a stop in $8.0\ \mathrm{s}$ due to frictional torque. Assume that the angular acceleration is constant. What is the magnitude of the frictional torque acting on the turntable?

Solution: We have already calculated the angular acceleration of the turntable in Example 16.1, where we found that

$$\alpha_z = \frac{\Delta\omega_z}{\Delta t} = \frac{\omega_f - \omega_i}{t_f - t_i} = \frac{-3.5\ \mathrm{rad}\cdot\mathrm{s}^{-1}}{8.0\ \mathrm{s}} = -4.3\times10^{-1}\ \mathrm{rad}\cdot\mathrm{s}^{-2} \tag{17.3.28}$$

and so the magnitude of the frictional torque is

$$\begin{aligned} \left|\tau_z^{\mathrm{fric}}\right| &= I_S\left|\alpha_z\right| = (1.01\times10^{-2}\ \mathrm{kg}\cdot\mathrm{m}^2)(4.3\times10^{-1}\ \mathrm{rad}\cdot\mathrm{s}^{-2}) \\ &= 4.3\times10^{-3}\ \mathrm{N}\cdot\mathrm{m}. \end{aligned} \tag{17.3.29}$$

Example 17.10 Pulley and blocks

A pulley of mass m_{p}, radius R, and moment of inertia about its center of mass I_{cm}, is attached to the edge of a table. An inextensible string of negligible mass is wrapped around the pulley and attached on one end to block 1 that hangs over the edge of the table (Figure 17.22). The other end of the string is attached to block 2 that slides along a table.

The coefficient of sliding friction between the table and the block 2 is μ_k. Block 1 has mass m_1 and block 2 has mass m_2, with $m_1 > \mu_k m_2$. At time $t = 0$, the blocks are released from rest and the string does not slip around the pulley. At time $t = t_1$, block 1 hits the ground. Let g denote the gravitational constant. (a) Find the magnitude of the acceleration of each block. (b) How far did the block 1 fall before hitting the ground?

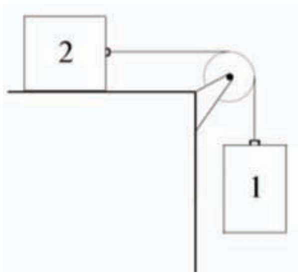


Figure 17.22 Example 17.10

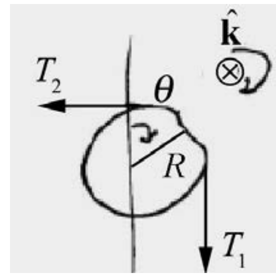


Figure 17.23 Torque diagram for pulley

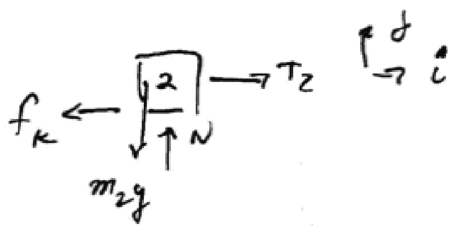
Solution: The torque diagram for the pulley is shown in the figure below where we choose $\hat{\mathbf{k}}$ pointing into the page. Note that the tensions in the string on either side of the pulley are not equal. The reason is that the pulley is massive. To understand why, remember that the difference in the magnitudes of the torques due to the tension on either side of the pulley is equal to the moment of inertia times the magnitude of the angular acceleration, which is non-zero for a massive pulley. So the tensions cannot be equal. From our torque diagram, the torque about the point O at the center of the pulley is given by

$$\vec{\boldsymbol{\tau}}_O = \vec{\mathbf{r}}_{O,1} \times \vec{\mathbf{T}}_1 + \vec{\mathbf{r}}_{O,2} \times \vec{\mathbf{T}}_2 = R(T_1 - T_2)\hat{\mathbf{k}} . \tag{17.3.30}$$

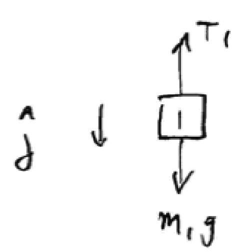
Therefore the torque equation (17.3.23) becomes

$$R(T_1 - T_2) = I_z \alpha_z . \tag{17.3.31}$$

The free body force diagrams on the two blocks are shown in Figure 17.23.



(a)



(b)

Figure 17.23 Free-body force diagrams on (a) block 2, (b) block 1

Newton's Second Law on block 1 yields

$$m_1 g - T_1 = m_1 a_{y1} \,. \tag{17.3.32}$$

Newton's Second Law on block 2 in the $\hat{\mathbf{j}}$ direction yields

$$N - m_2 g = 0 \,. \tag{17.3.33}$$

Newton's Second Law on block 2 in the $\hat{\mathbf{i}}$ direction yields

$$T_2 - f_k = m_2 a_{x2} \,. \tag{17.3.34}$$

The kinetic friction force is given by

$$f_k = \mu_k N = \mu_k m_2 g \tag{17.3.35}$$

Therefore Eq. (17.3.34) becomes

$$T_2 - \mu_k m_2 g = m_2 a_{x2} \,. \tag{17.3.36}$$

Block 1 and block 2 are constrained to have the same acceleration so

$$a \equiv a_{x1} = a_{x2} \,. \tag{17.3.37}$$

We can solve Eqs. (17.3.32) and (17.3.36) for the two tensions yielding

$$T_1 = m_1 g - m_1 a \,, \tag{17.3.38}$$

$$T_2 = \mu_k m_2 g + m_2 a \,. \tag{17.3.39}$$

At point on the rim of the pulley has a tangential acceleration that is equal to the acceleration of the blocks so

$$a = a_\theta = R\alpha_z \,. \tag{17.3.40}$$

The torque equation (Eq. (17.3.31)) then becomes

$$T_1 - T_2 = \frac{I_z}{R^2} a \,. \tag{17.3.41}$$

Substituting Eqs. (17.3.38) and (17.3.39) into Eq. (17.3.41) yields

$$m_1 g - m_1 a - (\mu_k m_2 g + m_2 a) = \frac{I_z}{R^2} a \,, \tag{17.3.42}$$

which we can now solve for the accelerations of the blocks

$$a = \frac{m_1 g - \mu_k m_2 g}{m_1 + m_2 + I_z / R^2}. \qquad (17.3.43)$$

Block 1 hits the ground at time t_1, therefore it traveled a distance

$$y_1 = \frac{1}{2}\left(\frac{m_1 g - \mu_k m_2 g}{m_1 + m_2 + I_z / R^2}\right) t_1^{\,2}. \qquad (17.3.44)$$

Example 17.11 Experimental Method for Determining Moment of Inertia

A steel washer is mounted on a cylindrical rotor of radius $r = 12.7\ \text{mm}$. A massless string, with an object of mass $m = 0.055\ \text{kg}$ attached to the other end, is wrapped around the side of the rotor and passes over a massless pulley (Figure 17.24). Assume that there is a constant frictional torque about the axis of the rotor. The object is released and falls. As the object falls, the rotor undergoes an angular acceleration of magnitude α_1. After the string detaches from the rotor, the rotor coasts to a stop with an angular acceleration of magnitude α_2. Let $g = 9.8\ \text{m}\cdot\text{s}^{-2}$ denote the gravitational constant. Based on the data in the Figure 17.25, what is the moment of inertia I_R of the rotor assembly (including the washer) about the rotation axis?

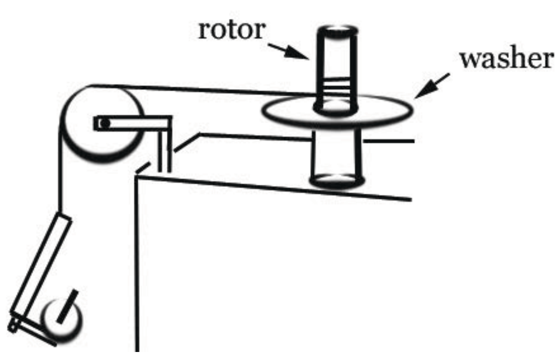


Figure 17.24 Steel washer, rotor, pulley, and hanging object

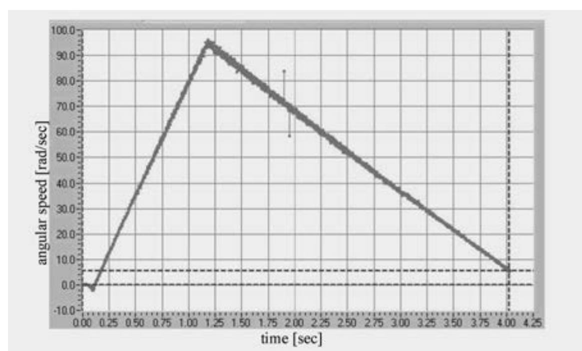
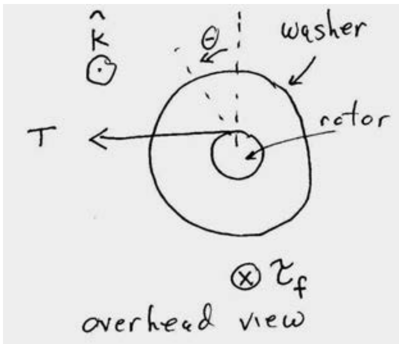
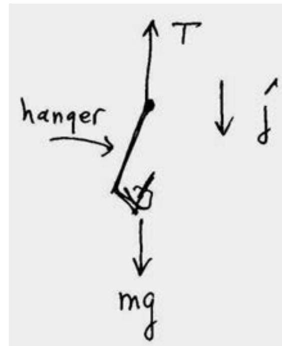


Figure 17.26 Graph of angular speed vs. time for falling object

Solution: We begin by drawing a force-torque diagram (Figure 17.26a) for the rotor and a free-body diagram for hanger (Figure 17.26b). (The choice of positive directions are indicated on the figures.) The frictional torque on the rotor is then given by $\vec{\boldsymbol{\tau}}_f = -\tau_f \hat{\mathbf{k}}$ where we use τ_f as the magnitude of the frictional torque. The torque about the center of the rotor due to the tension in the string is given by $\vec{\boldsymbol{\tau}}_T = rT\,\hat{\mathbf{k}}$ where r is the radius of the rotor. The angular acceleration of the rotor is given by $\vec{\boldsymbol{\alpha}}_1 = \alpha_1\,\hat{\mathbf{k}}$ and we expect that $\alpha_1 > 0$ because the rotor is speeding up.



(a)



(b)

Figure 17.26 (a) Force-torque diagram on rotor and (b) free-body force diagram on hanging object

While the hanger is falling, the rotor-washer combination has a net torque due to the tension in the string and the frictional torque, and using the rotational equation of motion,

$$Tr - \tau_f = I_R \alpha_1 . \tag{17.4.1}$$

We apply Newton's Second Law to the hanger and find that

$$mg - T = ma_1 = m\alpha_1 r , \tag{17.4.2}$$

where $a_1 = r\alpha_1$ has been used to express the linear acceleration of the falling hanger to the angular acceleration of the rotor; that is, the string does not stretch. Before proceeding, it might be illustrative to multiply Eq. (17.4.2) by r and add to Eq. (17.4.1) to obtain

$$mgr - \tau_f = (I_R + mr^2)\alpha_1 . \tag{17.4.3}$$

Eq. (17.4.3) contains the unknown frictional torque, and this torque is determined by considering the slowing of the rotor/washer after the string has detached.

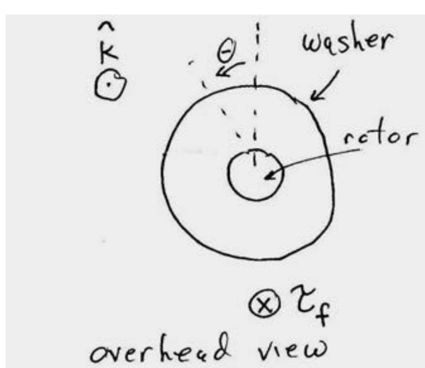


Figure 17.27 Torque diagram on rotor when string has detached

The torque on the system is just this frictional torque (Figure 17.27), and so

$$-\tau_f = I_R \alpha_2 \tag{17.4.4}$$

Note that in Eq. (17.4.4), $\tau_f > 0$ and $\alpha_2 < 0$. Subtracting Eq. (17.4.4) from Eq. (17.4.3) eliminates τ_f,

$$mgr = mr^2\alpha_1 + I_R(\alpha_1 - \alpha_2)\,. \tag{17.4.5}$$

We can now solve for I_R yielding

$$I_R = \frac{mr(g - r\alpha_1)}{\alpha_1 - \alpha_2}\,. \tag{17.4.6}$$

For a numerical result, we use the data collected during a trial run resulting in the graph of angular speed vs. time for the falling object shown in Figure 17.25. The values for α_1 and α_2 can be determined by calculating the slope of the two straight lines in Figure 17.28 yielding

$$\alpha_1 = (96\,\text{rad}\cdot\text{s}^{-1})/(1.15\,\text{s}) = 83\ \text{rad}\cdot\text{s}^{-2},$$
$$\alpha_2 = -(89\,\text{rad}\cdot\text{s}^{-1})/(2.85\,\text{s}) = -31\,\text{rad}\cdot\text{s}^{-2}.$$

Inserting these values into Eq. (17.4.6) yields

$$I_R = 5.3\times10^{-5}\,\text{kg}\cdot\text{m}^2\,. \tag{17.4.7}$$

17.5 Torque and Rotational Work

When a constant torque $\tau_{s,z}$ is applied to an object, and the object rotates through an angle $\Delta\theta$ about a fixed z-axis through the center of mass, then the torque does an amount of work $\Delta W = \tau_{S,z}\Delta\theta$ on the object. By extension of the linear work-energy theorem, the amount of work done is equal to the change in the rotational kinetic energy of the object,

$$W_{\text{rot}} = \frac{1}{2}I_{\text{cm}}\omega_f^2 - \frac{1}{2}I_{\text{cm}}\omega_i^2 = K_{\text{rot},f} - K_{\text{rot},i}\,. \tag{17.4.8}$$

The rate of doing this work is the rotational power exerted by the torque,

$$P_{\text{rot}} \equiv \frac{dW_{\text{rot}}}{dt} = \lim_{\Delta t\to 0}\frac{\Delta W_{\text{rot}}}{\Delta t} = \tau_{S,z}\frac{d\theta}{dt} = \tau_{S,z}\omega_z\,. \tag{17.4.9}$$

17.5.1 Rotational Work

Consider a rigid body rotating about an axis. Each small element of mass Δm_i in the rigid body is moving in a circle of radius $(r_{S,i})_\perp$ about the axis of rotation passing through the point S. Each mass element undergoes a small angular displacement $\Delta\theta$ under the action of a tangential force, $\vec{\mathbf{F}}_{\theta,i} = F_{\theta,i}\,\hat{\boldsymbol{\theta}}$, where $\hat{\boldsymbol{\theta}}$ is the unit vector pointing in the tangential direction (Figure 17.20). The element will then have an associated displacement vector for this motion, $\Delta\vec{\mathbf{r}}_{S,i} = r_i\Delta\theta\,\hat{\boldsymbol{\theta}}$ and the work done by the tangential force is

$$\Delta W_{\text{rot},i} = \vec{\mathbf{F}}_{\theta,i}\cdot\Delta\vec{\mathbf{r}}_{S,i} = (F_{\theta,i}\,\hat{\boldsymbol{\theta}})\cdot(r_i\Delta\theta\,\hat{\boldsymbol{\theta}}) = r_i F_{\theta,i}\Delta\theta\,. \tag{17.4.10}$$

Recall the result of Eq. (17.3.8) that the component of the torque (in the direction along the axis of rotation) about S due to the tangential force, $\vec{\mathbf{F}}_{\theta,i}$, acting on the mass element Δm_i is

$$(\tau_{S,i})_z = r_i F_{\theta,i}\,, \tag{17.4.11}$$

and so Eq. (17.4.10) becomes

$$\Delta W_{\text{rot},\,i} = (\tau_{S,i})_z\Delta\theta\,. \tag{17.4.12}$$

Summing over all the elements yields

$$W_{\text{rot}} = \sum_i \Delta W_{\text{rot},\,i} = \sum_i \Big((\tau_{S,i})_z\Big)\Delta\theta = \tau_{S,z}\Delta\theta\,, \tag{17.4.13}$$

the rotational work is the product of the torque and the angular displacement. In the limit of small angles, $\Delta\theta \to d\theta$, $\Delta W_{\text{rot}} \to dW_{\text{rot}}$ and the differential rotational work is

$$dW_{\text{rot}} = \tau_{S,z}\,d\theta\,. \tag{17.4.14}$$

We can integrate this amount of rotational work as the angle coordinate of the rigid body changes from some initial value $\theta = \theta_i$ to some final value $\theta = \theta_f$,

$$W_{\text{rot}} = \int dW_{\text{rot}} = \int_{\theta_i}^{\theta_f} \tau_{S,z}\,d\theta\,. \tag{17.4.15}$$

17.5.2 Rotational Work-Kinetic Energy Theorem

We will now show that the rotational work is equal to the change in rotational kinetic energy. We begin by substituting our result from Eq. (17.3.14) into Eq. (17.4.14) for the infinitesimal rotational work,

$$dW_{\text{rot}} = I_S\alpha_z\,d\theta\,. \tag{17.4.16}$$

Recall that the rate of change of angular velocity is equal to the angular acceleration, $\alpha_z \equiv d\omega_z / dt$ and that the angular velocity is $\omega_z \equiv d\theta / dt$. Note that in the limit of small displacements,

$$\frac{d\omega_z}{dt} d\theta = d\omega_z \frac{d\theta}{dt} = d\omega_z \, \omega_z \,. \tag{17.4.17}$$

Therefore the infinitesimal rotational work is

$$dW_{\text{rot}} = I_S \alpha_z \, d\theta = I_S \frac{d\omega_z}{dt} d\theta = I_S \, d\omega_z \frac{d\theta}{dt} = I_S \, d\omega_z \, \omega_z \,. \tag{17.4.18}$$

We can integrate this amount of rotational work as the angular velocity of the rigid body changes from some initial value $\omega_z = \omega_{z,i}$ to some final value $\omega_z = \omega_{z,f}$,

$$W_{\text{rot}} = \int dW_{\text{rot}} = \int_{\omega_{z,i}}^{\omega_{z,f}} I_S \, d\omega_z \, \omega_z = \frac{1}{2} I_S \, \omega_{z,f}^2 - \frac{1}{2} I_S \, \omega_{z,i}^2 \,. \tag{17.4.19}$$

When a rigid body is rotating about a fixed axis passing through a point S in the body, there is both rotation and translation about the center of mass unless S is the center of mass. If we choose the point S in the above equation for the rotational work to be the center of mass, then

$$W_{\text{rot}} = \frac{1}{2} I_{\text{cm}} \omega_{\text{cm},f}^2 - \frac{1}{2} I_{\text{cm}} \omega_{\text{cm},i}^2 = K_{\text{rot},f} - K_{\text{rot},i} \equiv \Delta K_{\text{rot}} \,. \tag{17.4.20}$$

Note that because the z-component of the angular velocity of the center of mass appears as a square, we can just use its magnitude in Eq. (17.4.20).

17.5.3 Rotational Power

The rotational power is defined as the rate of doing rotational work,

$$P_{\text{rot}} \equiv \frac{dW_{\text{rot}}}{dt} \,. \tag{17.4.21}$$

We can use our result for the infinitesimal work to find that the rotational power is the product of the applied torque with the angular velocity of the rigid body,

$$P_{\text{rot}} \equiv \frac{dW_{\text{rot}}}{dt} = \tau_{S,z} \frac{d\theta}{dt} = \tau_{S,z} \omega_z \,. \tag{17.4.22}$$

Example 17.12 Work Done by Frictional Torque

A steel washer is mounted on the shaft of a small motor. The moment of inertia of the motor and washer is I_0. The washer is set into motion. When it reaches an initial angular velocity ω_0, at $t = 0$, the power to the motor is shut off, and the washer slows down during a time interval $\Delta t_1 = t_a$ until it reaches an angular velocity of ω_a at time t_a. At that instant, a second steel washer with a moment of inertia I_w is dropped on top of the first washer. Assume that the second washer is only in contact with the first washer. The collision takes place over a time $\Delta t_{\text{int}} = t_b - t_a$ after which the two washers and rotor rotate with angular speed ω_b. Assume the frictional torque on the axle (magnitude τ_f) is independent of speed, and remains the same when the second washer is dropped. (a) What angle does the rotor rotate through during the collision? (b) What is the work done by the friction torque from the bearings during the collision? (c) Write down an equation for conservation of energy. Can you solve this equation for ω_b? (d) What is the average rate that work is being done by the friction torque during the collision?

Solution: We begin by solving for the frictional torque during the first stage of motion when the rotor is slowing down. We choose a coordinate system shown in Figure 17.29.

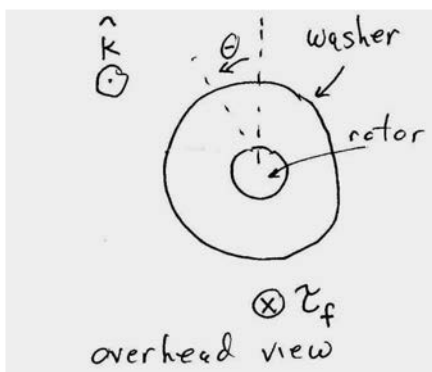


Figure 17.29 Coordinate system for Example 17.12

The component of average angular acceleration is given by

$$\alpha_1 = \frac{\omega_a - \omega_0}{t_a} < 0\,.$$

We can use the rotational equation of motion, and find that the frictional torque satisfies

$$-\tau_f = I_0\left(\frac{\omega_a - \omega_0}{\Delta t_1}\right).$$

During the collision, the component of the average angular acceleration of the rotor is given by

$$\alpha_2 = \frac{\omega_b - \omega_a}{(\Delta t_{\text{int}})} < 0 .$$

The angle the rotor rotates through during the collision is (analogous to linear motion with constant acceleration)

$$\Delta\theta_2 = \omega_a \Delta t_{\text{int}} + \frac{1}{2}\alpha_2 \Delta t_{\text{int}}{}^2 = \omega_a \Delta t_{\text{int}} + \frac{1}{2}\left(\frac{\omega_b - \omega_a}{\Delta t_{\text{int}}}\right)\Delta t_{\text{int}}{}^2 = \frac{1}{2}(\omega_b + \omega_a)\Delta t_{\text{int}} > 0 .$$

The non-conservative work done by the bearing friction during the collision is

$$W_{f,b} = -\tau_f \Delta\theta_{rotor} = -\tau_f \frac{1}{2}(\omega_a + \omega_b)\Delta t_{\text{int}} .$$

Using our result for the frictional torque, the work done by the bearing friction during the collision is

$$W_{f,b} = \frac{1}{2} I_0 \left(\frac{\omega_a - \omega_0}{\Delta t_1}\right)(\omega_a + \omega_b)\Delta t_{\text{int}} < 0 .$$

The negative work is consistent with the fact that the kinetic energy of the rotor is decreasing as the rotor is slowing down. Using the work energy theorem during the collision the kinetic energy of the rotor has deceased by

$$W_{f,b} = \frac{1}{2}(I_0 + I_w)\omega_b^2 - \frac{1}{2} I_0 \omega_a^2 .$$

Using our result for the work, we have that

$$\frac{1}{2} I_0 \left(\frac{\omega_a - \omega_0}{\Delta t_1}\right)(\omega_a + \omega_b)\Delta t_{\text{int}} = \frac{1}{2}(I_0 + I_w)\omega_b^2 - \frac{1}{2} I_0 \omega_a^2 .$$

This is a quadratic equation for the angular speed ω_b of the rotor and washer immediately after the collision that we can in principle solve. However remember that we assumed that the frictional torque is independent of the speed of the rotor. Hence the best practice would be to measure ω_0, ω_a, ω_b, Δt_1, Δt_{int}, I_0, and I_w and then determine how closely our model agrees with conservation of energy. The rate of work done by the frictional torque is given by

$$P_f = \frac{W_{f,b}}{\Delta t_{\text{int}}} = \frac{1}{2} I_0 \left(\frac{\omega_a - \omega_0}{\Delta t_1}\right)(\omega_a + \omega_b) < 0 .$$