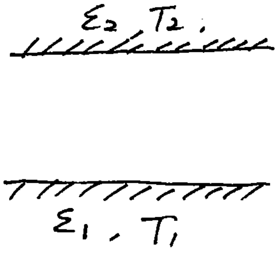
Review

So far: equation of radiative transfer.

(1) form of Boltzmann equation

(2) Solution: simple case



Integral equation

discussed physics

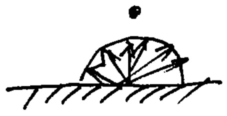
temperature discontinuity

(3) approximate solutions

optically thin

optically thick

(4) Deisler jump boundary conditi



(5) spherical & cylindrical coordinates

mention, read yourself.

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where we are going:

(1) last lecture on radiative equati of transfer

(2) next lecture Solar cell principles.

(3) laser principle

(4) ▨ direct thermal emission calculation

$$\int_a^b f(x)dx = \sum_{i=1}^{N} \underset{\substack{\uparrow}}{f}(x_i) \underset{\substack{\llcorner \text{ weight}}}{w_i}$$

~~value~~
root of
quadrature

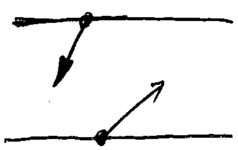
Guass quadrature.

$$\int_0^1 \cancel{\beta}\, I\, \mu\, d\mu = \sum_{i=1}^{N} I_{\cancel{0}}(x, \mu_i)\, \mu_i w_i.$$

$$\mu_i \frac{dI_\eta}{d\xi} = -I_\eta + I_{b\eta}(T)$$

$\uparrow$

discretize ξ using finite difference



Remember $\Rightarrow$

$$\int I_\eta\, d\Omega = I_{b\eta}(T) \cdot 4\pi$$

$$\mu_i \frac{dI_\eta}{d\xi} = -I_\eta^{(\xi,\mu_i)} + 2\pi \sum_{i=1}^{N} I_\eta(\mu_i) w_i.$$

If ~~both μ_i, φ_i vary~~ $I(\mu; \varphi, \xi)$

$\uparrow$

choose propriate quadrature.

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Spherical Harmonics Method.

What is spherical harmonics?

If we solve a Laplace equation using separation of variables

$$\nabla^2 T = 0$$

In Cartesian coordinate.

$\Rightarrow$ Fourier series. $\quad \sin\frac{2\pi m x}{L}$

orthogonality $\frac{2}{L}\int_0^L \sin\frac{2\pi m x}{L}\sin\frac{2\pi n x}{L}\,dx = \delta_{mn} = \begin{cases} 1 & m=n \\ 0 & m\neq n. \end{cases}$

$\Downarrow$

$\cos^2$

In spherical coordinate

$$T = R(r)\,\Theta(\theta)\,\Phi(\varphi)$$

$\uparrow$ $\cos\theta$.

$$\Theta(\theta) = P_\ell^m(\overset{\mu}{\cancel{\cos\theta}}) = \frac{1}{2^n \ell!}(1-\mu^2)^{m/2}\frac{d^{m+\ell}}{d\mu^{m+\ell}}(\mu^2-1)^\ell$$

$-m \le \ell \le m$

$$\Phi(\varphi) = e^{im\varphi}$$

spherical harmonics

$$Y_\ell^m = (-1)^{(m+|m|)/2}\sqrt{\frac{(\ell-|m|)!}{(\ell+|m|)!}}\,e^{im\varphi}P_\ell^{|m|}(\cos\theta)$$

$$\int_{-1}^{1} P_\ell P_m\,d\mu = \frac{2\delta_{\ell m}}{2m+1} = \begin{cases} 0 & \ell\neq m \\ \frac{2}{2m+1} & \ell = m \end{cases}$$

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Expand $I(\vec{r}, \hat{e}_\Omega) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} I_l^m(\vec{r})\, Y_l^m(\hat{e}_\Omega)$

One-dime When I is φ independent

function or $\vec{r}$

$$I(\tau, \mu) = \sum_{l=0}^{N} I_l(\tau) P_l(\mu)$$

↳ function of angle.

phase functi $\Phi(\mu, \mu') = \sum_{m=0}^{M} A_m P_m(\mu') P_m(\mu)$

$$\int_{-1}^{1} \Phi(\mu, \mu') I(\tau, \mu') d\mu' = \sum_{l=0}^{N} \frac{2A_l}{2l+1} I_l(\tau) P_l(\mu)$$

P_1 : $l=1$, $\underline{I(\vec{r}, \hat{e}_\Omega) = a(\vec{r}) + \vec{b}(\vec{r}) \cdot \hat{e}_\Omega}$

$$= \frac{1}{4\pi}\left[G(\vec{r}) + 3\vec{q} \cdot \hat{e}_\Omega\right]$$

$$= I_0^0 Y_0^0 + I_1^{-1} Y_1^{-1} + I_1^0 Y_1^0 + I_1^1 Y_1^1$$

$Y_0^0 = 1$, Y_1^{-1}: ϕ, Y_1^0: $\cos\theta$, Y_1^1: $\sin\theta$

$$G(\vec{r}) = \int_{4\pi} I \, d\Omega$$

$$\vec{q} = \int \vec{I} \cdot \vec{e}_\Omega \, d\Omega = \frac{4\pi}{3} \vec{b}(\vec{r})$$

Scattering

If linear anisotropic scattering $\Phi(\Omega' \to \Omega) = 1 + A_1 \hat{e}_{\Omega'} \cdot \hat{e}_\Omega$

$$\int_{4\pi} I(\Omega') \Phi(\Omega' \to \Omega) d\Omega'$$

$$\hookrightarrow = G + A_1 \vec{q} \cdot \hat{e}_\Omega$$

ERT becomes

$$\frac{1}{4\pi} \nabla_{\vec{r}} \cdot \left[\hat{e}_\Omega (G + 3\vec{q} \cdot \hat{e}_\Omega)\right] + \overset{I}{\frac{1}{4\pi}(G + 3\vec{q} \cdot \hat{e}_\Omega)}$$

$$= (1-\omega) I_b + \frac{\omega}{4\pi}(G + A_1 \vec{q} \cdot \hat{e}_\Omega)$$

Use Orthogonality

$\int$ Eq. × $Y_0^0(=1)\ d\Omega \Rightarrow$ $\hat{s}$ drops out, except $\overset{\text{inside}}{\nabla_r \cdot}$

$$\Rightarrow \qquad \nabla_r \cdot \vec{q} = (1-\omega)(4\pi I_b - G)$$

$\int$ Eq × $Y_1^m\ d\Omega$ $(m=-1,0,1) \Rightarrow$

$$\nabla_{\vec r} \overset{\cancel{\to}}{G} = -(3-A_1\omega)\,\vec{q}$$

$\Big\}$ G, $\vec{q}$ unknowns

└ diffusion approximation

If radiative equilibri $\nabla \cdot \vec{q} = 0$

$$\hookrightarrow \qquad \nabla^2 \overset{\cancel{\to}}{G} = 0 \quad \text{solve for } G.$$

b.C.

$$I = \frac{1}{4\pi}\left[G(\vec r) + 3\vec{q}\cdot\hat{e}_\Omega\right]$$

Not to match detail

continuity of heat flux

$$\int_{\hat n \cdot \hat e_\Omega > 0} I_w\, \hat{e}_\Omega \cdot \hat{n}\, d\Omega$$

$$= \frac{1}{4\pi}\int_{\hat n \cdot \hat e_\Omega > 0} (G + 3\vec{q}\cdot\hat{e}_\Omega)\,\hat{e}_\Omega\cdot\hat{n}\, d\Omega$$

$$\Rightarrow \quad -\cancel{B}\,\frac{2-\varepsilon}{\epsilon}\,\frac{2}{3-A_1\omega}\,\hat{n}\cdot\nabla_{\vec r} G + G = 4\pi \underset{\overset{\uparrow}{\text{wall}}}{I_{bw}}$$

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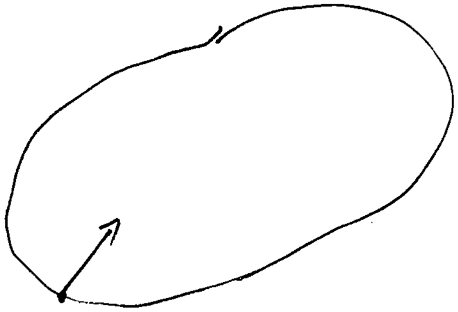
Modified Differential Approximation



Equation of radiative transfer.

$$\frac{dI}{d\tau} = \hat{e}_\Omega \cdot \nabla_r I = \underset{\substack{\uparrow \\ \text{source funct.}}}{S} - I$$

$$S = (1-\omega) I_b(\vec{r}) + \frac{\omega}{4\pi}\left[G + A_1 \vec{q} \cdot \hat{e}_\Omega \right]$$

$$I = I_w(\vec{r}, \hat{e}_\Omega) + I_m(\vec{r}, \hat{e}_\Omega) \qquad \Big| \qquad q = q_w + q_m$$

Wall component (ballistic)

$$\frac{dI_w}{d\tau} = -I_w$$

$$I_w = \frac{J_w}{\pi} e^{-\tau}$$

$$\frac{dI_m}{d\tau} = S - I_m$$

P₁ approximation

$$I_m(\vec{r}, \hat{e}_\Omega) \simeq \frac{1}{4\pi}\left[G_m + 3\vec{q}_m \cdot \hat{e}_\Omega \right]$$

$$G_m = \int_{4\pi} I_m \, d\Omega \qquad \vec{q}_m = \int_{4\pi} I_m \hat{e}_\Omega \, d\Omega$$

$$G = G_m + G_w$$

G_w ↖ can be calculated from I_w

Similar to previous treatment ⇒

$$\nabla_r G_m = A_1 \omega (\vec{q}_w + \vec{q}_W) - 3\vec{q}_m$$

$$\nabla_r \cdot \vec{q}_m = (1-\omega) 4\pi I_b + \omega (G_m + G_w) - 3\vec{q}_m$$

⟸ Isotropic scattering

$A_1 = 0$

$\vec{q}_m = -\frac{1}{3} \nabla_r G_m$.

diffusion approximation.

b.C.

$$\hat{q}_m \cdot \hat{n} = \int_{\hat{s}\cdot\hat{n}<0} I_m \hat{e}_\Omega \cdot \hat{n} \, d\Omega$$

$$2\left(\frac{2}{\varepsilon} - 1\right) \hat{q}_m \cdot \hat{n} + G_m = 0$$