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Lecture 3

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Contents

Reading in the Textbook

- Chapter 3, pp.49 - pp.72

Lecture 3

Differential geometry of surfaces

3.1 Definition of surfaces

- *Implicit surfaces* $F(x, y, z) = 0$
  Example: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ Ellipsoid, see Figure 3.1.

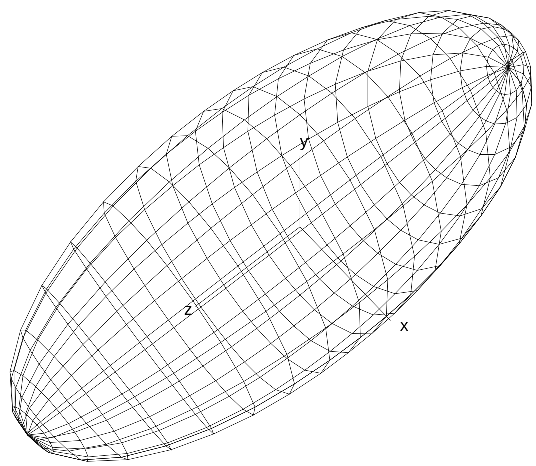

Figure 3.1: Ellipsoid.

- *Explicit surfaces*
  If the implicit equation $F(x, y, z) = 0$ can be solved for one of the variables as a function of the other two, we obtain an explicit surface, as shown in Figure 3.2. *Example:* $z = \frac{1}{2}(\alpha x^2 + \beta y^2)$

- *Parametric surfaces* $x = x(u, v),\ y = y(u, v),\ z = z(u, v)$
  Here functions $x(u, v)$, $y(u, v)$, $z(u, v)$ have continuous partial derivatives of the r^{th} order, and the parameters u and v are restricted to some intervals (i.e., $u_1 \leq u \leq u_2$, $v_1 \leq v \leq v_2$) leading to parametric surface patches. This rectangular domain D of u, v is called *parametric space* and it is frequently the unit square, see Figure 3.3. If derivatives of the surface are continuous up to the r^{th} order, the surface is said to be of class r, denoted C^r.

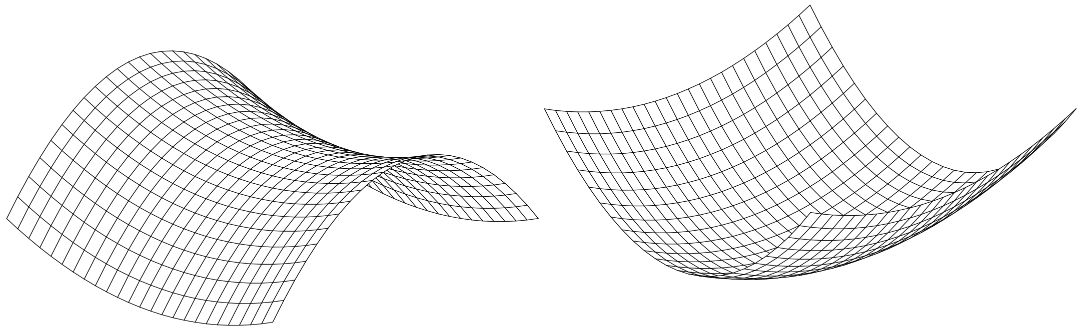

Figure 3.2: Explicit quadratic surfaces $z = \frac{1}{2}(\alpha x^2 + \beta y^2)$. (a) Left: Hyperbolic paraboloid ($\alpha = -3$, $\beta = 1$). (b) Right: Elliptic paraboloid ($\alpha = 1$, $\beta = 3$).

In vector notation:

$$\begin{aligned} &\mathbf{r} = \mathbf{r}(u, v) \\ &\textit{where} \quad \mathbf{r} = (x, y, z), \quad \mathbf{r}(u, v) = (x(u, v), y(u, v), z(u, v)) \end{aligned}$$

Example:

$$\mathbf{r} = (u + v, u - v, u^2 + v^2)$$

$$\left.\begin{array}{l} x = u + v \\ y = u - v \\ z = u^2 + v^2 \end{array}\right\} \Rightarrow \text{eliminate } u, v \Rightarrow z = \frac{1}{2}(x^2 + y^2) \ \text{ paraboloid}$$

3.2 Curves on a surface

Let $\mathbf{r} = \mathbf{r}(u, v)$ be the equation of a surface, defined on a domain D (i.e., $u_1 \leq u \leq u_2$, $v_1 \leq v \leq v_2$). Let $\beta(t) = (u(t), v(t))$ be a curve in the parameter plane. Then $\mathbf{r} = \mathbf{r}(u(t), v(t))$ is a curve lying on the surface, see Figure 3.3. A tangent vector of curve $\beta(t)$ is given by $\dot{\beta}(t) = (\dot{u}(t), \dot{v}(t))$ A tangent vector of a curve on a surface is given by:

$$\frac{d\mathbf{r}(u(t), v(t))}{dt} \tag{3.1}$$

By using the chain rule:

$$\frac{d\mathbf{r}(u(t), v(t))}{dt} = \frac{\partial \mathbf{r}}{\partial u}\frac{du}{dt} + \frac{\partial \mathbf{r}}{\partial v}\frac{dv}{dt} = \mathbf{r}_u \dot{u}(t) + \mathbf{r}_v \dot{v}(t) \tag{3.2}$$

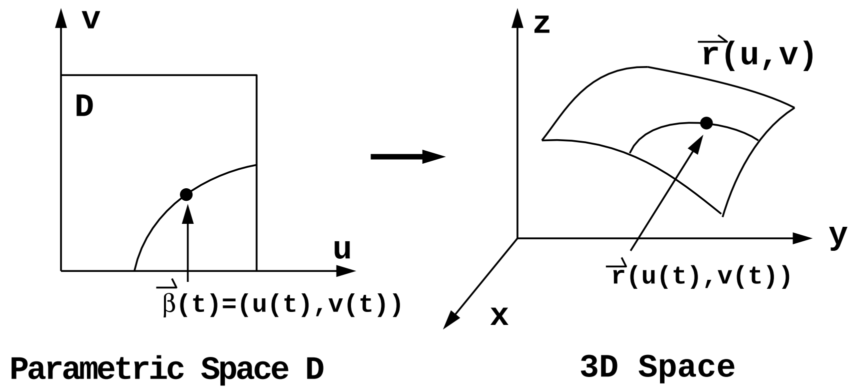

Figure 3.3: The mapping of a curve in 2D parametric space onto a 3D biparametric surface
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3.3 First fundamental form (arc length)

Consider a curve on a surface $\mathbf{r} = \mathbf{r}(u(t), v(t))$. The arc length of the curve on a surface is given by

$$\begin{aligned}
ds &= |\frac{d\mathbf{r}}{dt}|dt = |\mathbf{r}_u \frac{du}{dt} + \mathbf{r}_v \frac{dv}{dt}|dt \\
&= \sqrt{(\mathbf{r}_u \dot{u} + \mathbf{r}_v \dot{v}) \cdot (\mathbf{r}_u \dot{u} + \mathbf{r}_v \dot{v})} dt \\
&= \sqrt{(\mathbf{r}_u \cdot \mathbf{r}_u) du^2 + 2\mathbf{r}_u \mathbf{r}_v du dv + (\mathbf{r}_v \cdot \mathbf{r}_v) dv^2} \\
&= \sqrt{E du^2 + 2F du dv + G dv^2} \qquad (3.3)
\end{aligned}$$

where

$$E = \mathbf{r}_u \cdot \mathbf{r}_u, \quad F = \mathbf{r}_u \cdot \mathbf{r}_v, \quad G = \mathbf{r}_v \cdot \mathbf{r}_v \qquad (3.4)$$

The first fundamental form is defined as

$$\begin{aligned}
I &= d\mathbf{r} \cdot d\mathbf{r} = (\mathbf{r}_u du + \mathbf{r}_v dv) \cdot (\mathbf{r}_u du + \mathbf{r}_v dv) \\
&= E du^2 + 2F du dv + G dv^2 \qquad (3.5)
\end{aligned}$$

E, F, G are called first fundamental form coefficients Note that $E = \mathbf{r}_u \cdot \mathbf{r}_u > 0$ and $G = \mathbf{r}_v \cdot \mathbf{r}_v > 0$ if $\mathbf{r}_u \neq 0$ and $\mathbf{r}_v \neq 0$. The first fundamental form I is positive definite. That is $I \geq 0$ and $I = 0$ if and only if $du = 0$ and $dv = 0$ since

$$I = \frac{1}{E}(E\ du + F\ dv)^2 + \frac{EG - F^2}{E} dv^2 \text{ and } EG - F^2 = |r_u \times r_v|^2 > 0.$$

I depends only on the surface and not on the parametrization.
The area of the surface can be derived as follows:

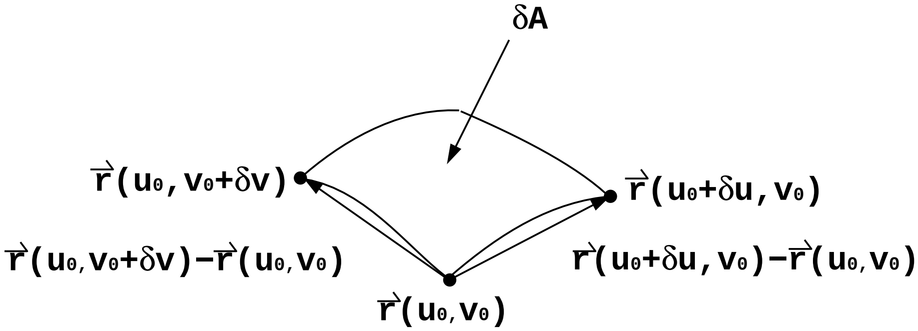

Figure 3.4: Area of an infinitessimal surface patch.

$$\mathbf{r}(u_0, v_0 + \delta v) - \mathbf{r}(u_0, v_0) \simeq \frac{\partial \mathbf{r}}{\partial v}\delta v$$
$$\mathbf{r}(u_0 + \delta u, v_0) - \mathbf{r}(u_0, v_0) \simeq \frac{\partial \mathbf{r}}{\partial u}\delta u$$

$$\delta A = |\mathbf{r}_u \delta u \times \mathbf{r}_v \delta v| = |\mathbf{r}_u \times \mathbf{r}_v|\delta u \delta v$$

$$|\mathbf{r}_u \times \mathbf{r}_v|^2 = (\mathbf{r}_u \times \mathbf{r}_v) \cdot (\mathbf{r}_u \times \mathbf{r}_v)$$

Using the vector identity $(a \times b) \cdot (c \times d) = (a \cdot c)(b \cdot d) - (a \cdot d)(b \cdot c)$, we get

$$|\mathbf{r}_u \times \mathbf{r}_v|^2 = (\mathbf{r}_u \cdot \mathbf{r}_u)(\mathbf{r}_v \cdot \mathbf{r}_v) - (\mathbf{r}_u \cdot \mathbf{r}_v)^2 \quad (3.6)$$
$$= EG - F^2 \quad (3.7)$$

$$\delta A = \sqrt{EG - F^2}\, \delta u \delta v, \qquad A = \int\int \sqrt{EG - F^2}\, du dv \quad (3.8)$$

Example: For the hyperbolic paraboloid $\mathbf{r}(u, v) = (u, v, u^2 - v^2)$, let us derive an expression for the area of a region of its surface corresponding to a the circle $u^2 + v^2 \leq 1$ in the parametric domain D.

We begin by forming expressions for the derivatives of the position vector $\mathbf{r}$ and the first fundamental form coeffients.

$$\begin{aligned}
\mathbf{r}_u &= (1, 0, 2u) \\
\mathbf{r}_v &= (0, 1, -2v) \\
E &= \mathbf{r}_u \cdot \mathbf{r}_u = 1 + 4u^2 \\
F &= \mathbf{r}_u \cdot \mathbf{r}_v = -4uv \\
G &= \mathbf{r}_v \cdot \mathbf{r}_v = 1 + 4v^2
\end{aligned}$$

Using Equation (3.8), we find

$$\begin{aligned}
EG - F^2 &= (1 + 4u^2)(1 + 4v^2) - 16u^2v^2 = 1 + 4u^2 + 4v^2 > 0 \\
A &= \int\int_D \sqrt{1 + 4u^2 + 4v^2} du dv
\end{aligned}$$

To compute the area, we need to evaluate the double integral over the unit disk $u^2 + v^2 \leq 1$ in the parametric domain D;

$$A = \int\int_{u^2+v^2 \leq 1} \sqrt{1 + 4u^2 + 4v^2}\, du\, dv.$$

To perform the integration, let us change variables.

$$\begin{aligned}
u &= r\cos(\theta),\ v = r\sin(\theta), \text{and } du\, dv = r\, dr\, d\theta \\
A &= \int\int_{r \leq 1} \sqrt{1 + 4r^2}\, r\, dr\, d\theta \\
&= \int_0^{2\pi}\int_0^1 \sqrt{1 + 4r^2}\, r\, dr\, d\theta \\
&= \frac{\pi}{6}(5\sqrt{5} - 1)
\end{aligned}$$

3.4 Tangent plane

Tangent plane at a point $\mathbf{r}(u_o, v_o)$ is the union of tangent vectors of all curves on the surface pass through $\mathbf{r}(u_o, v_o)$, as shown in Figure 3.5. Since the tangent vector of a curve on a parametric surface is given by $\frac{d\mathbf{r}}{dt} = \mathbf{r}_u \frac{du}{dt} + \mathbf{r}_v \frac{dv}{dt}$, the tangent plane lies on the plane of the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$. The equation of the tangent plane is

$$\mathbf{T}_p(u, v) = \mathbf{r}(u, v) + \lambda \mathbf{r}_u(u, v) + \mu \mathbf{r}_v(u, v) \tag{3.9}$$

where λ and μ are real variables parameterizing the plane.

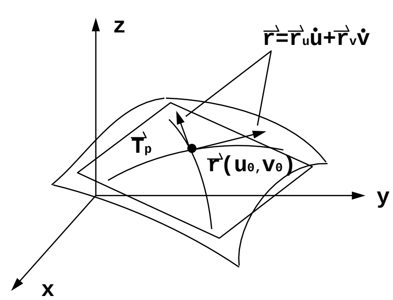

Figure 3.5: The tangent plane at a point on a surface.

3.5 Normal vector

The surface normal is the vector at point $\mathbf{r}(u_o, v_o)$ perpendicular to the tangent plane, see Figure 3.6. And therefore

$$\mathbf{N} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \tag{3.10}$$

Note that $\mathbf{r}_u$ and $\mathbf{r}_v$ are not necessarily perpendicular.

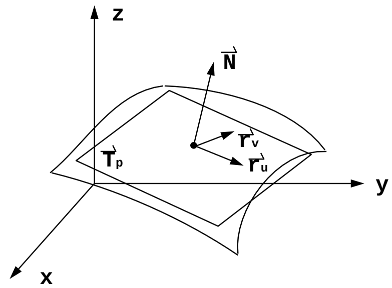

Figure 3.6: The normal to the point on a surface.

A *regular* (ordinary) point $\mathbf{P}$ on the surface is defined as one for which $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$. A point where $\mathbf{r}_u \times \mathbf{r}_v = \mathbf{0}$ is called a *singular* point. The condition $\mathbf{r}_u \times \mathbf{r}_v \neq \mathbf{0}$ requires that at that point $\mathbf{P}$ the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$ do not vanish and have different directions.

Example: Elliptic Paraboloid $\mathbf{r}(u,v) = (u+v, u-v, u^2+v^2)$

$$\begin{aligned}
\mathbf{r}_u &= (1,1,2u) \\
\mathbf{r}_v &= (1,-1,2v)
\end{aligned}$$

$$\begin{aligned}
\mathbf{r}_u \times \mathbf{r}_v &= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ 1 & 1 & 2u \\ 1 & -1 & 2v \end{vmatrix} \\
&= 2(u+v)\mathbf{e}_x + 2(u-v)\mathbf{e}_y - 2\mathbf{e}_z \neq \mathbf{0}
\end{aligned}$$

$$\begin{aligned}
|\mathbf{r}_u \times \mathbf{r}_v| &= 2\sqrt{(u+v)^2 + (u-v)^2 + 1} \\
&= 2\sqrt{2u^2 + 2v^2 + 1} > 0 \Rightarrow \text{Regular !}
\end{aligned}$$

$$\begin{aligned}
\mathbf{N} &= \frac{(2(u+v), 2(u-v), -2)}{2\sqrt{2u^2+2v^2+1}} \\
&= \frac{(u+v, u-v, -1)}{\sqrt{2u^2+2v^2+1}}
\end{aligned}$$

$$\text{at } (u,v) = (0,0), \mathbf{N} = (0,0,-1)$$

Example: Circular Cone $\mathbf{r}(u,v) = (u \sin\alpha \cos v, u \sin\alpha \sin v, u \cos\alpha)$, see Figure 3.7

$$\begin{aligned}
\mathbf{r}_u &= (\sin\alpha\cos v, \sin\alpha\sin v, \cos\alpha) \\
\mathbf{r}_v &= (-u\sin\alpha sinv, u\sin\alpha sinv, 0)
\end{aligned}$$

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \sin\alpha\cos v & \sin\alpha\sin v & \cos\alpha \\ -u\sin\alpha\sin v & u\sin\alpha\cos v & 0 \end{vmatrix}$$

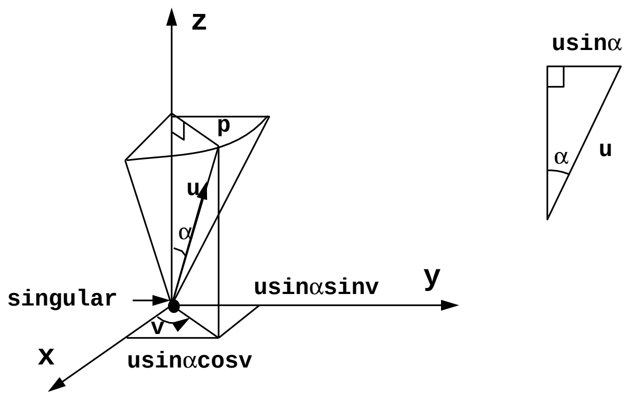

Figure 3.7: Circular cone.

$$= \quad -u\sin\alpha\cos\alpha\cos v\mathbf{e}_x - u\sin\alpha\cos\alpha\sin v\mathbf{e}_y + u\sin^2\alpha\mathbf{e}_z$$

At the origin $\mathbf{n} = 0$,

$$\mathbf{r}_u \times \mathbf{r}_v \quad = \quad \mathbf{0}$$

Therefore, the apex of the cone is a singular point.

3.6 Second fundamental form II (curvature)

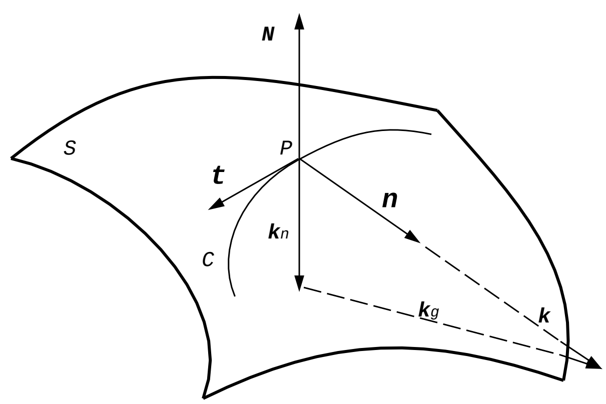

Figure 3.8: Definition of normal curvature

In order to quantify the curvatures of a surface S, we consider a curve C on S which passes through point P as shown in Figure 3.8. $\mathbf{t}$ is the unit tangent vector and $\mathbf{n}$ is the unit normal vector of the curve C at point P.

$$\frac{d\mathbf{t}}{ds} \quad = \quad \kappa\mathbf{n} = \mathbf{k}_n + \mathbf{k}_g \tag{3.11}$$

$$\mathbf{k}_n \quad = \quad \kappa_n\mathbf{N} \tag{3.12}$$

where $\mathbf{k}_n$ is the normal curvature vector normal to the surface, $\mathbf{k}_g$ is the geodesic curvature vector tangent to the surface, and $\mathbf{k} = \kappa\mathbf{n}$ is the curvature vector of the curve C at point $\mathbf{P}$. κ_n is called the normal curvature of the surface at $\mathbf{P}$ in the direction $\mathbf{t}$.

Meusnier's Theorem : All curves lying on a surface S passing through a given point $p \in S$ with the same tangent line have the same normal curvature at this point.

Since $\mathbf{N} \cdot \mathbf{t} = 0$, differentiate w.r.t. s

$$\begin{aligned} \frac{d}{ds}(\mathbf{N} \cdot \mathbf{t}) &= \mathbf{N}' \cdot \mathbf{t} + \mathbf{N} \cdot \mathbf{t}' \\ \frac{d\mathbf{t}}{ds} \cdot \mathbf{N} &= -\mathbf{t} \cdot \frac{d\mathbf{N}}{ds} = -\frac{d\mathbf{r}}{ds} \cdot \frac{d\mathbf{N}}{ds} \end{aligned} \tag{3.13}$$

Recoginizing that $ds \cdot ds = dx^2 + dy^2 + dz^2 = d\mathbf{r} \cdot d\mathbf{r}$, we can rewrite Equation 3.13 as:

$$\begin{aligned} \frac{d\mathbf{t}}{ds} \cdot \mathbf{N} &= = -\frac{d\mathbf{r} \cdot d\mathbf{N}}{d\mathbf{r} \cdot d\mathbf{r}} \\ \text{while } \frac{d\mathbf{t}}{ds} \cdot \mathbf{N} &= \kappa \mathbf{n} \cdot \mathbf{N} \equiv \kappa_n \end{aligned}$$

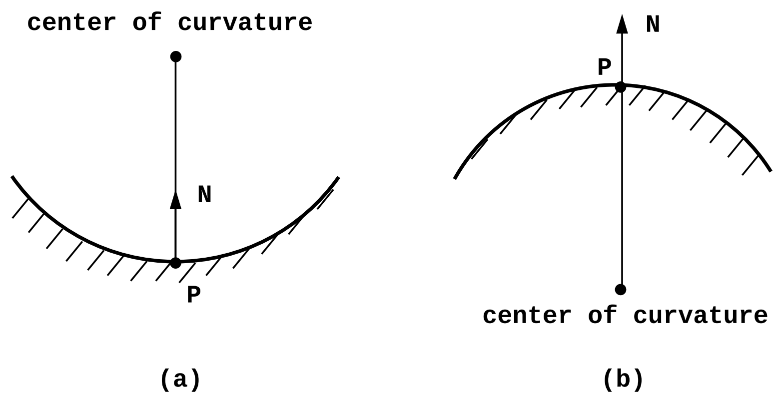

Figure 3.9: Definition of positive normal: (a) $\kappa \mathbf{n} \cdot \mathbf{N} = \kappa_n$; (b) $\kappa \mathbf{n} \cdot \mathbf{N} = -\kappa_n$.

$$\begin{aligned} II &= -d\mathbf{r} \cdot d\mathbf{N} = -(\mathbf{r}_u du + \mathbf{r}_v dv) \cdot (\mathbf{N}_u du + \mathbf{N}_v dv) \\ &= L du^2 + 2M du dv + N dv^2 \end{aligned} \tag{3.14}$$

where

$$L = \mathbf{N} \cdot \mathbf{r}_{uu}, \quad M = \mathbf{N} \cdot \mathbf{r}_{uv}, \quad N = \mathbf{N} \cdot \mathbf{r}_{vv} \tag{3.15}$$

Therefore the normal curvature is given by

$$\kappa_n = \frac{II}{I} = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2} \tag{3.16}$$

where $\lambda = \frac{dv}{du}$.

Suppose P is a point on a surface and Q is a point in the neighborhood of P, as in Figure 3.10. Taylor's expansion gives

$$\mathbf{r}(u + du, v + dv) = \mathbf{r}(u, v) + \mathbf{r}_u du + \mathbf{r}_v dv + \frac{1}{2}(\mathbf{r}_{uu} du^2 + 2\mathbf{r}_{uv} du dv + \mathbf{r}_{vv} dv^2) + H.O.T. \tag{3.17}$$

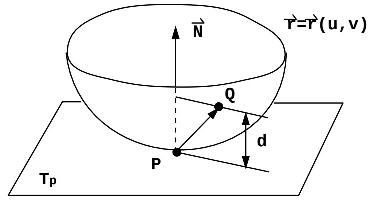

Figure 3.10: Geometrical illustration of the second fundamental form.

Therefore

$$\mathbf{PQ} = \mathbf{r}(u + du, v + dv) - \mathbf{r}(u, v) = \mathbf{r}_u du + \mathbf{r}_v dv + \frac{1}{2}(\mathbf{r}_{uu} du^2 + 2\mathbf{r}_{uv} du dv + \mathbf{r}_{vv} dv^2) + H.O.T.$$

Thus, the projection of $\mathbf{PQ}$ onto $\mathbf{N}$

$$d = \mathbf{PQ} \cdot \mathbf{N} = (\mathbf{r}_u du + \mathbf{r}_v dv) \cdot \mathbf{N} + \frac{1}{2} II$$

and since $\mathbf{r}_u \cdot \mathbf{N} = \mathbf{r}_v \cdot \mathbf{N} = 0$, we get

$$d = \frac{1}{2} II = \frac{1}{2}(L du^2 + 2M du dv + N dv^2)$$

We want to observe in which situation d is positive and negative. When $d = 0$

$$L du^2 + 2M du dv + N dv^2 = 0$$

Solve for du

$$du = \frac{-M \pm \sqrt{(M dv)^2 - LN dv^2}}{L} = \frac{-M \pm \sqrt{M^2 - LN}}{L} dv \tag{3.18}$$

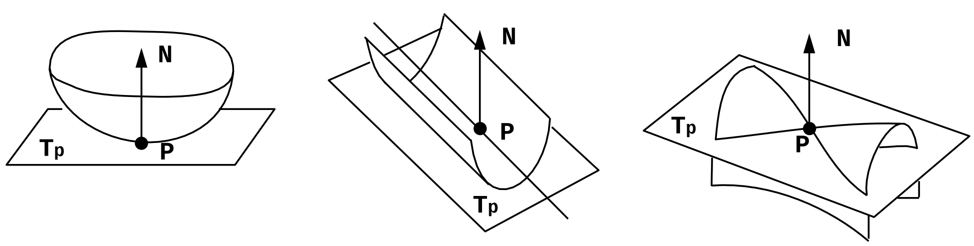

Figure 3.11: (a) Elliptic point; (b) Parabolic point; (c) Hyperbolic point.

- If $M^2 - LN < 0$, there is no real root. That means there is no intersection between the surface and its tangent plane except at point P. P is called *elliptic point* (Figure 3.11(a)).
- If $M^2 - LN = 0$, there is a double root. The surface intersects its tangent plane with one line $du = -\frac{M}{L} dv$, which passes through point P. P is called *parabolic point* (Figure 3.11(b)).
- If $M^2 - LN > 0$, there are two roots. The surface intersects its tangent plane with two lines $du = \frac{-M \pm \sqrt{M^2 - LN}}{L} dv$, which intersect at point P. P is called *hyperbolic point* (Figure 3.11(c)).

3.7 Principal curvatures

The extreme values of κ_n can be obtained by evaluating $\frac{d\kappa_n}{d\lambda} = 0$ of Equation 3.16, which gives:

$$(E + 2F\lambda + G\lambda^2)(N\lambda + M) - (L + 2M\lambda + N\lambda^2)(G\lambda + F) = 0 \tag{3.19}$$

Since

$$\begin{aligned} E + 2F\lambda + G\lambda^2 &= (E + F\lambda) + \lambda(F + G\lambda), \\ L + 2M\lambda + N\lambda^2 &= (L + M\lambda) + \lambda(M + N\lambda) \end{aligned}$$

equation (3.19) can be reduced to

$$(E + F\lambda)(M + N\lambda) = (L + M\lambda)(F + G\lambda) \tag{3.20}$$

Thus

$$\kappa_n = \frac{L + 2M\lambda + N\lambda^2}{E + 2F\lambda + G\lambda^2} = \frac{M + N\lambda}{F + G\lambda} = \frac{L + M\lambda}{E + F\lambda} \tag{3.21}$$

Therefore κ_n satisfies the two simultaneous equations

$$\begin{aligned} (L - \kappa_n E)du + (M - \kappa_n F)dv = 0 \\ (M - \kappa_n F)du + (N - \kappa_n G)dv = 0 \end{aligned} \tag{3.22}$$

These equations can be simultaneously satisfied if and only if

$$\begin{vmatrix} L - \kappa_n E & M - \kappa_n F \\ M - \kappa_n F & N - \kappa_n G \end{vmatrix} = 0 \tag{3.23}$$

where $|\ \ |$ denotes the determinant of a matrix. Expanding and defining K and H as

$$K = \frac{LN - M^2}{EG - F^2} \tag{3.24}$$

$$H = \frac{EN + GL - 2FM}{2(EG - F^2)} \tag{3.25}$$

we obtain a quadratic equation for κ_n as follows:

$$\kappa_n^2 - 2H\kappa_n + K = 0 \tag{3.26}$$

The values K and H are called Gauss (Gaussian) and mean curvature respectively. The discriminant D can be expressed as follows:

$$\begin{aligned} D &= H^2 - K \\ &= \frac{(EN + GL - 2FM)^2 - 4(EG - F^2)(LN - M^2)}{4(EG - F^2)^2} \end{aligned}$$

The denominator is always positive, so we only need to investigate the numerator. The numerator can be written as:

$$\begin{aligned} & (EN + GL - 2FM)^2 - 4(EG - F^2)(LN - M^2) \\ = 4\left(\frac{EG - F^2}{E^2}\right)&(EM - FL)^2 + [EN - GL - \frac{2F}{E}(EM - FL)]^2 \geq 0 \end{aligned}$$

Thus, $D \geq 0$.

Upon solving Equation (3.26) for the extreme values of curvature, we have:

$$\kappa_{max} = H + \sqrt{H^2 - K} \tag{3.27}$$
$$\kappa_{min} = H - \sqrt{H^2 - K} \tag{3.28}$$

From Equations (3.27), (3.28), it is readily seen that

$$K = \kappa_{max}\kappa_{min} \tag{3.29}$$
$$H = \frac{\kappa_{max} + \kappa_{min}}{2} \tag{3.30}$$

From Equation (3.24) (since $EG - F^2 > 0$, see Equation 3.6).

$$K > 0 \Rightarrow LN > M^2 \Rightarrow \text{Elliptic point}$$
$$K = 0 \Rightarrow LN = M^2 \Rightarrow \text{Parabolic point}$$
$$K < 0 \Rightarrow LN < M^2 \Rightarrow \text{Hyperbolic point}$$

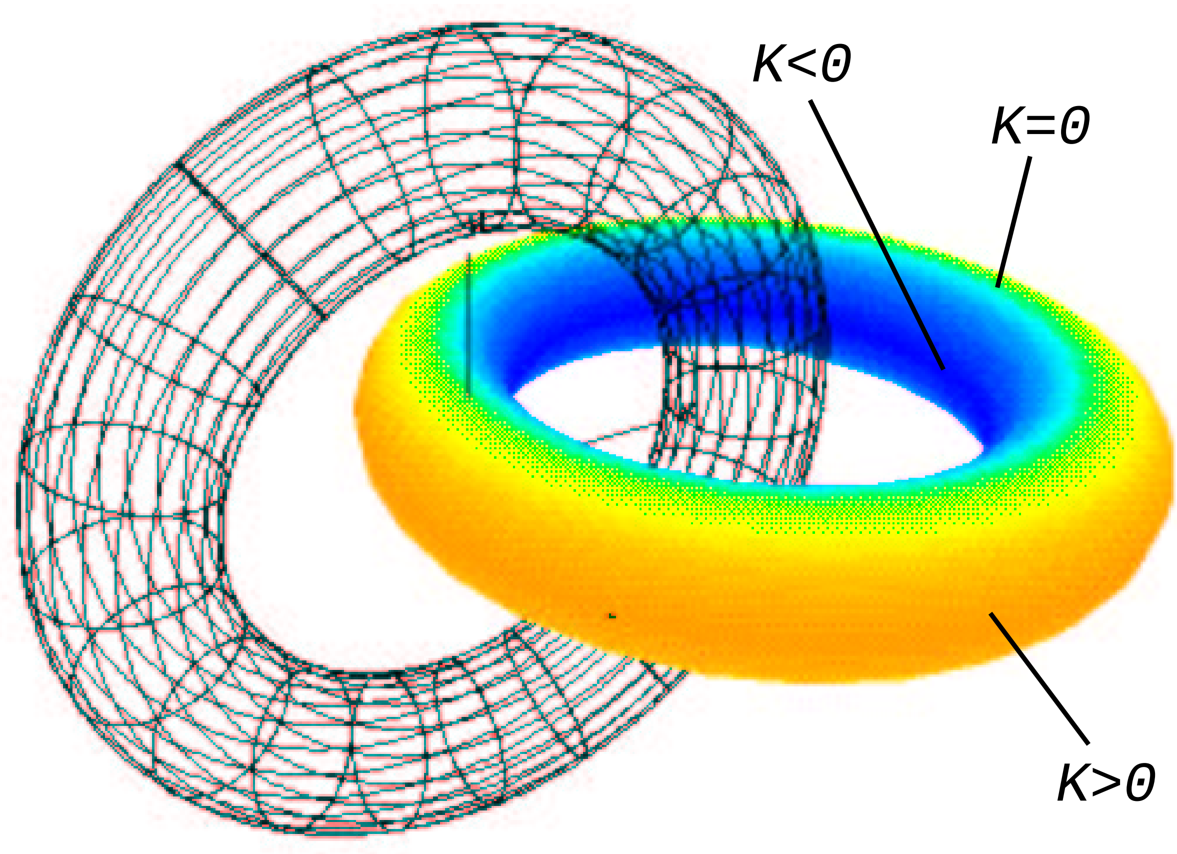

Figure 3.12: Curvature map of a torus showing elliptic, parabolic, and hyperbolic regions.

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