Lecture 3: The Concept of Stress, Generalized Stresses and Equilibrium

3.1 Stress Tensor

We start with the presentation of simple concepts in one and two dimensions before introducing a general concept of the stress tensor. Consider a prismatic bar of a square cross-section subjected to a tensile force F,

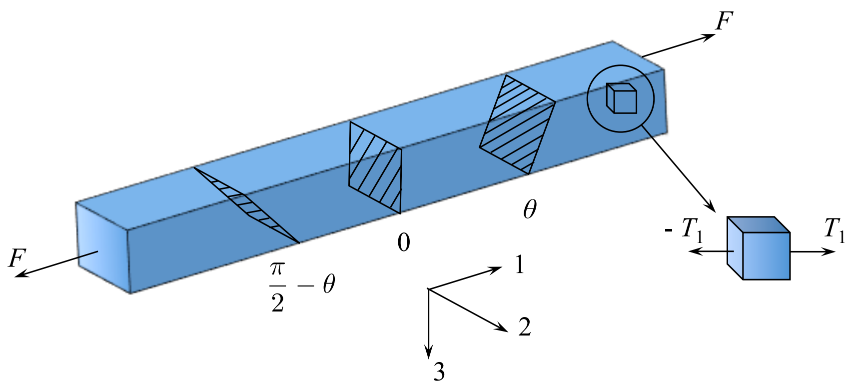

Figure 3.1: A long bar with three different cuts at $\theta, \theta = 0$ and $\pi/2 - \theta$.

The force per unit area is called the surface traction T:

$$T = \sigma = \frac{\text{force}}{\text{area}} = \frac{F}{A_o}\left[\frac{\text{N}}{\text{mm}^2}\right] \tag{3.1}$$

In the uniaxial case, the surface fraction is the only component of the stress tensor in the global coordinate system, commonly referred to as σ.

How can one apply a force to the end section of a bar? This can be done in a number of different ways (see Fig. (3.2)). A pin connection can be glued (or welded) to the end section, or a hole can be drilled through the bar to attach a pin.

Or an internal or external thread can be machined. Finally, axial force could be applied through frictional or mechanical grips. Except the welded or glued connector, a complex state of stress is created near the bar ends where the stress state is multi-axial. Such stress states is confined to a relatively short segment of the bar comparable with the height or diameter of the bar. Along this section a gradual transition takes place from the multi-axial state of stress to the uniaxial state, for which Eq.(3.1) holds.

The above example an serve as a practical application of the Saint-Venant's principle (1856). This principle named after the French elasticity theorist, Jean Claude Barre' de Saint-Venant can be stated as: "the difference between the effects of two different but statically equivalent loads become very small at sufficiently large distances from load."

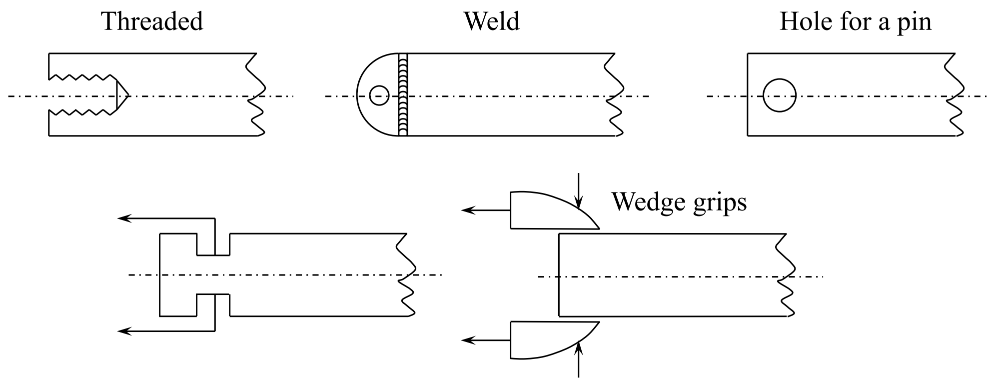

Figure 3.2: How to apply tension to the end of a bar.

Think what are the "two" equivalent loads that are applied to the bar ends? We usually think of a cross-section being cut perpendicular to the axis of the bar. Consider now two cuts at the angles θ and $\left(\frac{\pi}{2} - \theta\right)$ to the normal direction. The planes are defined by the unit normal vector $\boldsymbol{n}$.

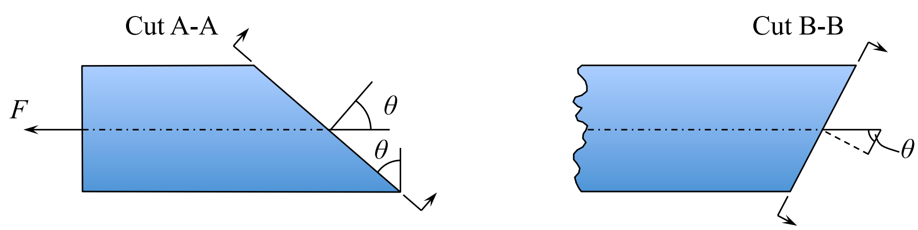

Figure 3.3: Normal and tangential forces acting on the slant section of the bar.

From the free body diagram, the components of the normal and tangential forces:

$$F_{\mathrm{N}} = F\cos\theta \tag{3.2a}$$

$$F_{\mathrm{n}} = F\cos(\frac{\pi}{2} - \theta) \tag{3.2b}$$

$$F_{\mathrm{T}} = F\sin\theta \tag{3.2c}$$

$$F_{\mathrm{t}} = F\sin(\frac{\pi}{2} - \theta) \tag{3.2d}$$

The slant cross-section A is larger and is related to the reference cross-section by

$$A_o = A_{\mathrm{A}}\cos\theta, \;\; A_o = A_{\mathrm{B}}\cos\left(\frac{\pi}{2} - \theta\right) \tag{3.3}$$

Consider now a unit volume cubic element located at the intersections of cuts A-A and B-B, Fig.(3.4).

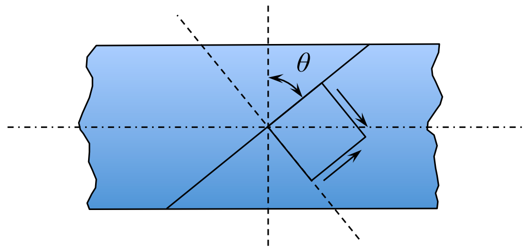

Figure 3.4: The volume element with surface traction acting on two adjacent facets.

The surface traction (force per unit area) on the two perpendicular facets are

$$\text{Facet parallel to A-A:} \quad T_\mathrm{n} = T\cos^2\theta \tag{3.4a}$$

$$T_\mathrm{t} = T\sin\theta\cos\theta \tag{3.4b}$$

$$\text{Facet parallel to B-B:} \quad T_\mathrm{n} = T\cos^2\left(\frac{\pi}{2}-\theta\right) \tag{3.4c}$$

$$T_\mathrm{t} = T\sin\left(\frac{\pi}{2}-\theta\right)\cos\left(\frac{\pi}{2}-\theta\right) \tag{3.4d}$$

It can be observed that the tangential components of the surface traction vector on A-A and B-B cuts are identical. The normalized plots of the above quantities versus the orientation angle of the cross-section are shown in Fig.(3.5).

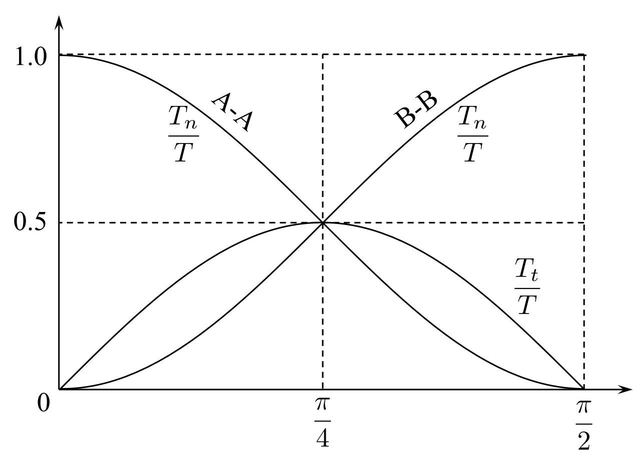

Figure 3.5: Relative values of normal and shear components of the surface traction as a function of the orientation of the cut.

It can be noted that the tangential component attains maximum at 45°. This means that if the material fails due to shear loading, the fracture surface will always be oriented at 45°. The above example teaches us that there are infinite combinations of normal and tangential components of surface tractions which are in equilibrium with the applied load. For each orientation of the cross-section there is a different pair of $\{T_\mathrm{n}, T_\mathrm{t}\}$. The orientation of the surface element is uniquely defined by the unit normal vector $\boldsymbol{n}\{n_1, n_2, n_3\}$. At the same time the components of the surface traction vector acting on the same element are $\boldsymbol{T}\{T_1, T_2, T_3\}$.

The components of the surface traction vector acting on this surface element are $\boldsymbol{T}\{T_1, T_2, T_3\}$. For example, the orientation of facets of the unit material cube is shown in Fig.(3.6).

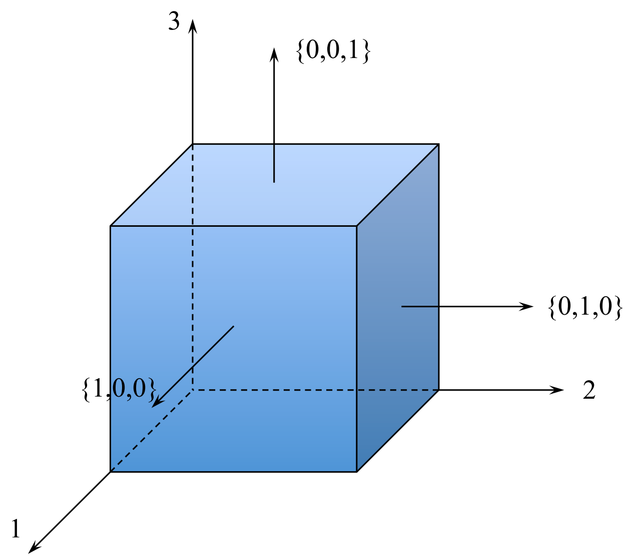

Figure 3.6: Components of the unit normal vector on facets of a unit cube.

The relation between the vectors of surface tractions, unit normal vector defining the surface element and the stress tensor are given by the famous *Cauchy formula*

$$T_i = T_{ij} n_j \tag{3.5}$$

or in the expanded notation,

$$T_1 = \sigma_{1j} n_j = \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 \tag{3.6a}$$

$$T_2 = \sigma_{2j} n_j = \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3 \tag{3.6b}$$

$$T_3 = \sigma_{3j} n_j = \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3 \tag{3.6c}$$

To a large extent the Cauchy relation is analogous to the strain-displacement relation put in the form of Eq.(3.2).

$$\mathrm{d}u_i = F_{ij} \mathrm{d}x_j \tag{3.7}$$

The displacement gradient F_{ij} transforms the increment of the length element $\mathrm{d}x_j$ into the increment of displacement $\mathrm{d}u_i$. In the same way the stress tensor transforms the orientation of the surface element $\boldsymbol{n}$ into the surface traction acting on this element.

In order to get a physical interpretation of the concept of the stress tensor, let us see how the Cauchy formula works in the case of one and two-dimensional problems of the axially loaded bar. Consider first the normal cut of the bar with the longitudinal axis as

1-axis. The components of the surface tractions are given in Fig.(3.7). The corresponding components of the unit normal vector were defined in Fig. (3.6), where $T = \dfrac{F}{A_o}$.

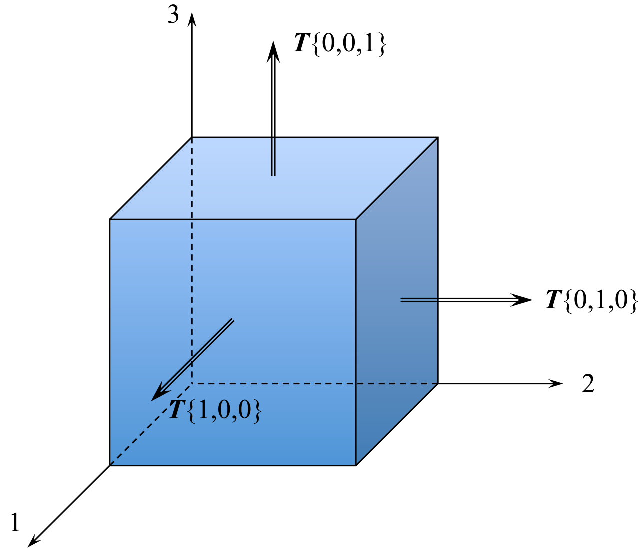

Figure 3.7: The unit volume element aligned with the axis of the bar.

Substituting the values of the components of the two vectors into Eq.(3.6b) one gets the following expressions:

$$\begin{array}{c|c|c} \text{Facet}(1,0,0) & \text{Facet}(0,1,0) & \text{Facet}(0,0,1) \\ T_1 = \sigma_{11} & 0 = \sigma_{12} & 0 = \sigma_{13} \\ 0 = \sigma_{21} & 0 = \sigma_{22} & 0 = \sigma_{23} \\ 0 = \sigma_{31} & 0 = \sigma_{32} & 0 = \sigma_{33} \end{array} \tag{3.8}$$

Therefore the components of the stress 3×3 matrix in the global coordinate system are

$$\boldsymbol{\sigma} = \begin{vmatrix} T & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} \tag{3.9}$$

This is the uniaxial state of stress. The two-dimensional example of the slant cut is much more interesting. This time a local coordinate system, rotated with respect to the 3-axis will be used. In this system the components $\boldsymbol{n}$ are the same as in the global system. The components of the surface traction vector on three facets, calculated in Eq.(3.4b) are defined in Fig.(3.8).

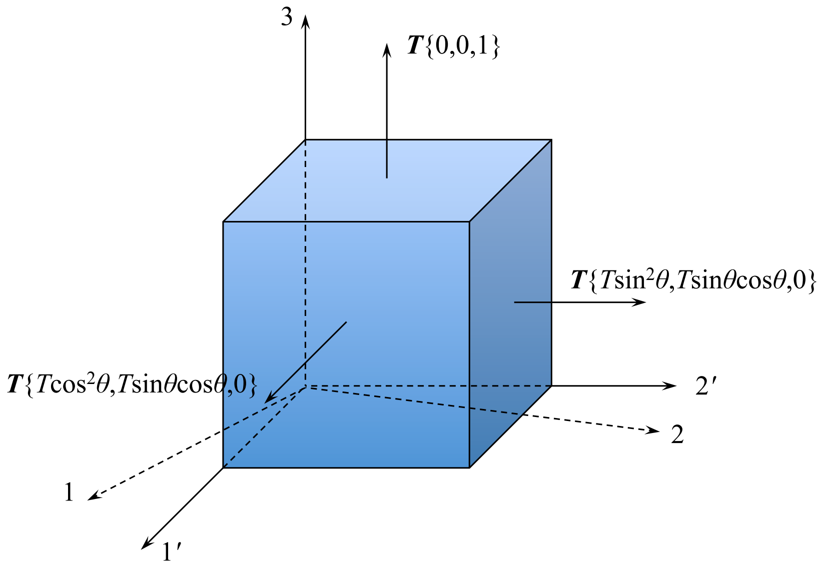

Figure 3.8: Components of the surface tractions on the rotated volume element.

Substituting the above values into the Cauchy formula we obtain

$$\begin{array}{c|c|c} \text{Facet}(1,0,0) & \text{Facet}(0,1,0) & \text{Facet}(0,0,1) \\ T\cos^2\theta = \sigma_{11} & T\sin\theta\cos\theta = \sigma_{12} & 0 = \sigma_{13} \\ T\sin\theta\cos\theta = \sigma_{21} & T\sin^2\theta = \sigma_{22} & 0 = \sigma_{23} \\ 0 = \sigma_{31} & 0 = \sigma_{32} & 0 = \sigma_{33} \end{array} \tag{3.10}$$

The plane stress components of the stress tensor are

$$\boldsymbol{\sigma} = \begin{vmatrix} T\cos^{\theta} & T\sin\theta\cos\theta & 0 \\ T\sin\theta\cos\theta & T\sin^2\theta & 0 \\ 0 & 0 & 0 \end{vmatrix} \tag{3.11}$$

It is interesting that the matrices Eq.(3.9) and Eq.(3.11) represent the same state of stress seen in two coordinate systems rotated with respect to one another. The transformation of the stress tensor from one coordinate system to the other is the subject Recitation 1 where the relation between Eq.(3.9) and Eq.(3.11) will be derived in a different way.

Symmetry of the stress tensor

It should also be noted from Eq.(3.11) that stress tensor is symmetric meaning that $\sigma_{12} = \sigma_{21}$. The symmetry of the stress tensor comes from the moment equilibrium equation of are infinitesimal volume element. In general

$$\sigma_{ij} = \sigma_{ji} \tag{3.12}$$

The symmetry of the stress tensor reduce the nine components of the 3×3 metric to only six independent components. The meaning of the two subscripts of the stress tensor is explained below

$$\sigma_{??}$$

The first subscript defines the plane on which the surface tractions are acting. For example "1" denotes the surface element perpendicular to the axis x_1. The second subscript indicates direction of a particular component of the surface tractions. This convention is explained in Fig.(3.9).

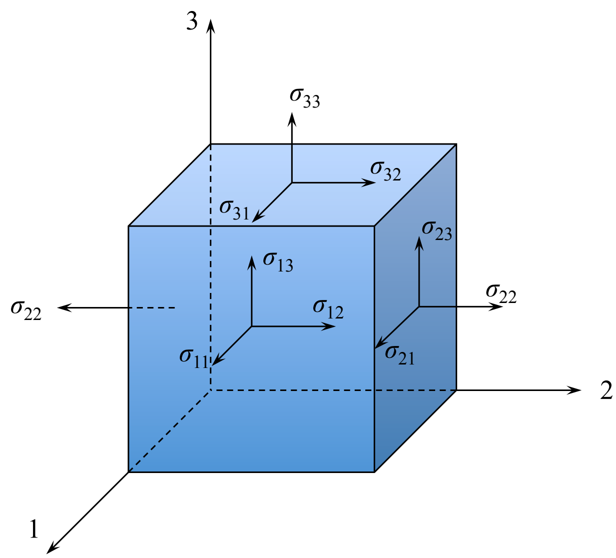

Figure 3.9: Components of the stress tensor on three facets of the infinitesimal surface element.

Sign convention

The Cauchy formula can also be consistently used to determine the sign of the components of the stress tensor. The point is that the sign of the components of the vectors is known from the chosen coordinate system. For illustration, let us orient the volume element along the x_1 axis. With positive direction to the right.

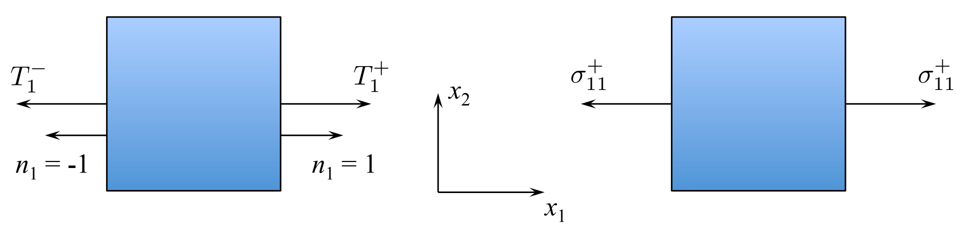

Figure 3.10: Explanation of the sign convention of the stress tensor.

From the Cauchy formula

$$T_1 = \sigma_{11} n_1 \tag{3.13}$$

On the right facet both the surface traction and the unit normal vector is positive and so must be the normal component of the stress tensor σ_{11}. On the left facet both T_1 and to the x_1 axis. In order for Eq.(3.13) to hold the *component* σ_{11} *must be positive*, even if its visualization points out in the negative direction. in the above example the stress state is uniform along the x_1 axis. This is the case of a bar under tension. In general there is a gradient of the components of the stress tensor so that stresses on both sides of the infinitesimal element differ by a small amount of $d\sigma_{11}$. The sign convention is opening the way for deriving the equations of equilibrium for the 3-D continuum. This topic is the subject of the next section.

3-D Equilibrium

The equilibrium equation for an infinitesimal volume element are derived first using two methods. Referring to Fig.(3.9), indicated on Fig.(3.11) are only those components of the stress tensor that are directed along x_2 axis. These are σ_{12}, σ_{22} and σ_{32}.

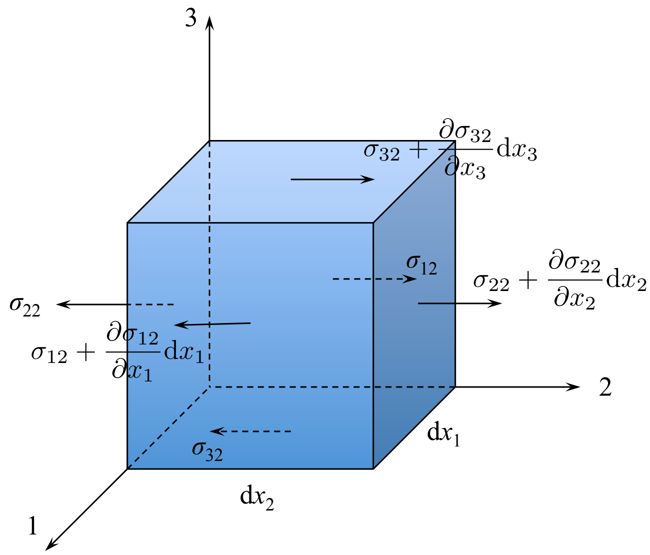

Figure 3.11: All components of the stress tensor contributing to the force equilibrium in x_2 direction must be in equilibrium.

According to Newton's law, the sum of all forces (stress times the surface area) acting along x_2 must be zero

$$\left(\sigma_{22} + \frac{\partial \sigma_{22}}{\partial x_2}\mathrm{d}x_2\right)\mathrm{d}x_1\mathrm{d}x_3 - \sigma_2\mathrm{d}x_1\mathrm{d}x_3 + \left(\sigma_{12} + \frac{\partial \sigma_{12}}{\partial x_1}\mathrm{d}x_1\right)\mathrm{d}x_2\mathrm{d}x_3 - \sigma_{12}\mathrm{d}x_2\mathrm{d}x_3 + \left(\sigma_{32} + \frac{\partial \sigma_{32}}{\partial x_3}\mathrm{d}x_3\right)\mathrm{d}x_1\mathrm{d}x_2 - \sigma_{32}\mathrm{d}x_1\mathrm{d}x_2 + B_2\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3 = 0 \tag{3.14}$$

For generality, the body force (force per unit volume) was included as well. The body force represent for example gravity force $B = \rho g$ or d'Alambert inertia force $B = \rho \ddot{u}$ so that the derivation is valid both for static and dynamic problems. Summing up the forces one gets the first equilibrium equation

$$\frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_3} + B_2 = 0 \tag{3.15}$$

Invoking the index notation

$$\frac{\partial \sigma_{j2}}{\partial x_j} + B_2 = 0 \to \sigma_{j2,j} + B_2 = 0 \tag{3.16}$$

with the summation and coma convention. A similar procedure of summing-up forces can be repeated in the x_1 and x_3 direction, yielding two additional equations of equilibrium. One can immediately notice that by replacing the life subscripts "2" in Eq.(3.16) respectively by "1" and "3", the final compact form of the equation of equilibrium reads

$$\sigma_{ij,j} + B_i = 0 \ \text{ or } \ \frac{\partial \sigma_{ij}}{\partial x_j} + B_i = 0 \tag{3.17}$$

In the expanded notation and replacing x_i by (x, y, z), the familiar form of the equilibrium equation is

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + B_x = 0 \tag{3.18a}$$

$$\frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + B_y = 0 \tag{3.18b}$$

$$\frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + B_z = 0 \tag{3.18c}$$

The *plane stress* case, prevailing in thin plate and shells is defined by

$$\sigma_{3j} = 0 \ \text{ or } \ \sigma_{31} = \sigma_{32} = \sigma_{33} = 0 \tag{3.19}$$

In other words all components of the stress tensor pointing out in the z-directions are zero, $\sigma_{zz} = \sigma_{zx} = \sigma_{zy} = 0$. The components of the plane stress tensor are highlighted by the framed area, thus $\boldsymbol{\sigma}$ is equal to

$$\begin{pmatrix} \boxed{\begin{matrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{matrix}} & \begin{matrix} \sigma_{xz} \\ \sigma_{yz} \end{matrix} \\ \begin{matrix} \sigma_{zx} & \sigma_{zy} \end{matrix} & \sigma_{zz} \end{pmatrix}$$

For plane stress, the subscripts run only over two dimensions and the Greek letters are commonly used, $\alpha, \beta = 1, 2$. In the compact notation, the plane stress equilibrium equation reads

$$\sigma_{\alpha\beta,\beta} + B_\alpha = 0 \tag{3.20}$$

In the uniaxial case only one component of the equilibrium survives, giving

$$\frac{\mathrm{d}\sigma_{xx}}{\mathrm{d}x} + B = 0 \tag{3.21}$$

With no body force, $B = 0$, Eq.(3.21) predicts a constant stress along the length of the bar. The addition of the d'Alambert inertia force will lead to the one-dimensional wave equation.

ADVANCED TOPIC

3.2 Local Equilibrium from the Principle of Virtual Work

The derivation of the *local* equation of equilibrium from the *global* principle of virtual work is an elegant method in continuum and structural mechanics. This procedure also formulates static and kinematic boundary condition. There are two mathematical tools involved. One is the divergence theorem (Gauss-Green identity) and the other one is the concept of the calculus of variation.

The Gauss theorem transforms the volume integral into a surface integral

$$\int_V A_{i,i} \mathrm{d}V = \int_S A_i n_i \mathrm{d}S, \; A_{i,i} = \frac{\partial A_i}{\partial x_i} \tag{3.22}$$

where A_i is a vector and n_i is the unit normal vector of the surface element $\mathrm{d}\boldsymbol{S}$. In the simplest one-dimensional case, Eq.(3.22) is reduced to

$$\int_{x_1}^{x_2} \frac{\mathrm{d}A}{\mathrm{d}x} \mathrm{d}x = A|_{x_1}^{x_2} = A(x_2) - A(x_1) \tag{3.23}$$

Starting from the definition of the infinitesimal strain given by Eq.(2.14), the increments of the strain tensor and displacement vector are also linearly related

$$\delta\epsilon_{ij} = \frac{1}{2}(\delta u_{i,j} + \delta u_{j,i}) \tag{3.24}$$

There is a fine difference between the symbol δu and $\mathrm{d}u$, which is explained in Fig.(3.12).

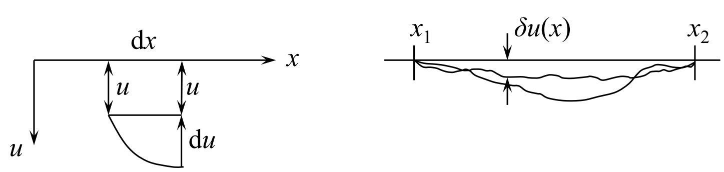

Figure 3.12: The local increment δu over the infinitesimal length $\mathrm{d}x$ and the global small (virtual) displacement from the equilibrium configuration satisfying kinematic boundary conditions.

Both are linear operators and the rule for differentiations are the same.

The principle of virtual work states that the incremental work of strains on the stresses over the volume of the body must be equal to the work of surface tractions in the incremental displacements over the surface of the body. Fig.(3.13) helps to visualize the notation.

A part of the surface on which the displacement are zero $\delta u_i = 0$ is denoted by S_{U}. The remainder of the surface $S - S_{\mathrm{U}}$ is denoted by S_{T}. Mathematically the principle of virtual work states

$$\int_V \sigma_{ij}\delta\epsilon_{ij} \mathrm{d}V = \int_S T_i \delta u_i \mathrm{d}S \tag{3.25}$$

where $\delta\epsilon_{ij}$ are calculated from δu_i using Eq.(3.24). The one-dimensional graphical interpretation of the principle is shown in Fig.(3.14).

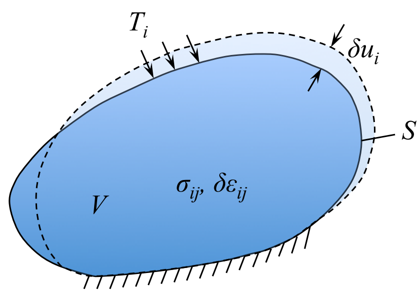

Figure 3.13: The 3-D potato (body) subjected to stress and displacement boundary condition develops internal stresses and incremental displacements.

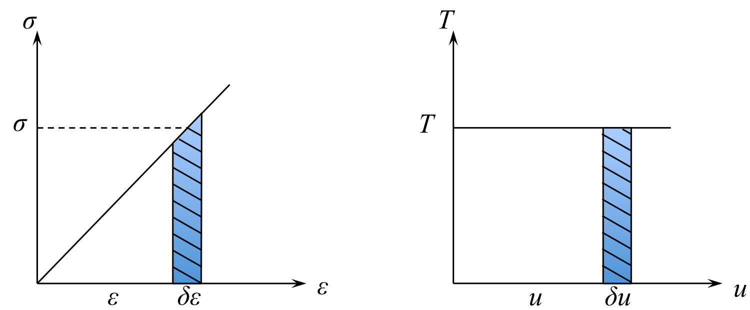

Figure 3.14: Incremental internal and external energies.

The integrand of the left hand side of Eq.(3.25) can be transformed to a simpler form using the symmetry property of the stress tensor $\sigma_{ij} = \sigma_{ji}$

$$\frac{1}{2}\sigma_{ij}\delta u_{i,j} + \frac{1}{2}\sigma_{ji}\delta u_{j,i} = \sigma_{ij}\delta u_{i,j} \tag{3.26}$$

Recall an elementary rule of differentiation of the product of two functions

$$(ab)' = a'b + ab' \tag{3.27}$$

which in application to our problem reads

$$\sigma_{ij}(\delta u_i)_{,j} = (\sigma_{ij}\delta u_i)_{,j} - (\sigma_{ij})_{,j}\delta u_i \tag{3.28}$$

Now, the left hand side of Eq.(3.25) is transformed to

$$\int_V \sigma_{ij}\delta\epsilon_{ij}\,\mathrm{d}V = \int_V \underbrace{(\sigma_{ij}\delta u_i)}_{A_i}{}_{,j}\,\mathrm{d}V - \int_V (\sigma_{ij})_{,j}\delta u_i\,\mathrm{d}V \tag{3.29}$$

The first volume integral is now transformed to the surface integral according to Eq.(3.22). Substituting this result into the statement of virtual work one gets

$$\int_S \sigma_{ij}\delta u_i n_j\,\mathrm{d}S - \int_V \sigma_{ij,j}\delta u_i\,\mathrm{d}V = \int_S T_i\delta u_i\,\mathrm{d}S \tag{3.30}$$

Combining the two surface integrals into one integral we finally arrive at

$$\int_S (\sigma_{ij} n_i - T_i)\delta u_i \, \mathrm{d}S - \int_{\mathrm{V}} \sigma_{\mathrm{ij,j}} \delta u_{\mathrm{i}} \, dV = 0 \tag{3.31}$$

The first integral vanishes when either

$$\sigma_{ij} n_j - T_i = 0 \;\text{ on } S_{\mathrm{T}} \tag{3.32a}$$

$$\text{or } \; \delta u_i = 0 \;\text{ on } S_{\mathrm{U}} \tag{3.32b}$$

The above equations represent respectively the stress and displacement boundary condition. The meaning of second integral should be interpreted in the spirit of the first lemma of the calculus of variation. The increment of the displacement vector δu_i can not vanish over the whole volume of the body because this would mean rigid body motion. The point is that the second integral in Eq.(3.28) must be zero not for one particular shape of δu_i but for all possible variation of the displacement field, as shown in Fig.(3.12). Thus, the calculus of variation tell us that this is possible only when

$$\sigma_{ij,j} = 0 \;\text{ in } V \tag{3.33}$$

The principle of virtual work is often called the weak (global) statement of equilibrium while Eq.(3.33) is the local equation of equilibrium but is not called strong. The weak statement of equilibrium is a starting point of developing most approximate methods in continuum and structural mechanics such as eigenvalue expansion, finite difference or finite element method. The critical assumption of the first lemma of the calculus of variation is that an infinity of different virtual velocities are considered. This is achieved by considering a large but finite degrees of freedom through many terms in the eigenvalue expansion or many discrete elements. Through this assumption the equivalence of the global and local formulation is achieved.

An alternative form of the principle of virtual work, extensively in plasticity theory is the principle of *virtual velocity*. By observing that

$$\delta u = \frac{\mathrm{d}u}{\mathrm{d}t}\delta t = \dot{u}\delta t \tag{3.34}$$

Eq.(3.25) is transformed to

$$\int_V \sigma_{ij} \dot{\epsilon_{ij}} \, \mathrm{d}V = \int_S T_i \dot{u_i} \, \mathrm{d}S \tag{3.35}$$

where $\dot{\epsilon_{ij}}$ is the instantaneous velocity field obtained from the incremental velocities $\dot{u_i}$ through the linear geometric relation, Eq.(3.25).

Generalized Stresses

This concept is introduced in order to reduce the two-dimensional problem in x and z in beams to one-dimensional problem in x, governed by an ordinary differential equation.

At an arbitrary cross-section in a beam one can distinguish a vector of bending moments $\{M_x, M_y, T\}$ where M_x is bending the beam in the (x, z) plane, M_y and T is the torque. (Do not confuse torque with surface traction vector). The meaning of the moment vector is explained in Fig.(3.15).

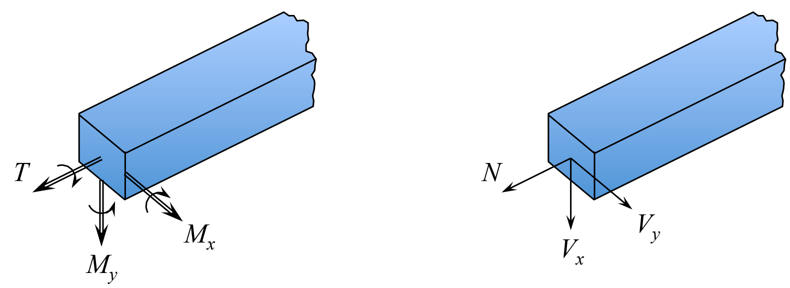

Figure 3.15: Imagine short shafts rotating a rigid slice of the beam. These are components of the bending moment vector.

The components of the force vector acting at an arbitrary cross-section are $\{V_x, V_y, N\}$ is the axial (membrane) force. In planar bending of a beam only three out of six components of the generalized stress resultants survival. They are defined by

$$M_x = M \stackrel{\text{def}}{=} \int_A \sigma_{xx} z \, \mathrm{d}A \ \ [\mathrm{Nm}] \tag{3.36a}$$

$$N \stackrel{\text{def}}{=} \int_A \sigma_{xx} \, \mathrm{d}A \ \ [\mathrm{N}] \tag{3.36b}$$

$$V_x = V \stackrel{\text{def}}{=} \int_A \sigma_{\mathrm{xz}} \, \mathrm{d}A \ \ [\mathrm{N}] \tag{3.36c}$$

The product $\sigma_{xx}\,\mathrm{d}A$ in Eq.(3.36a) is the incremental force. Multiplying this force by the "arm" z from the beam bending axis gives the incremental bounding moment $\mathrm{d}M = (\sigma_{xx}\mathrm{d}A)z$. The total bending moment is an integral of $\mathrm{d}M$ over the beam cross-section. The *sign* of the generalized quantities is decided by the sign of the stress.

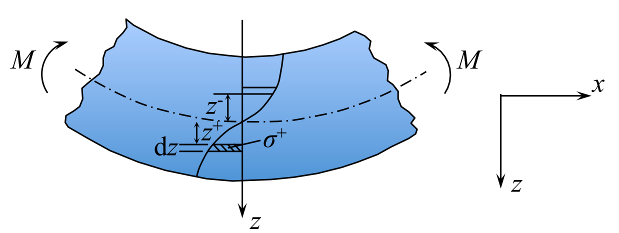

Figure 3.16: The bending moment in a "smiling" beam is positive.

Imagine that the beam is bent in the way shown is Fig.(3.16). On the tensile side of the beam the stress is positive, σ^+ and so is the distance from the beam axis. On the compressive side both the stress and force arms are negative $\sigma^- z^-$, but the product is positive. Therefore the tensile and compressive side of the beam contribute to the positive bending moment. The beam (or its portion) where the bending moment is negative is called the “smiling beam”. Therefore looking at the deformed shape of the beam one can determine immediately the sign of the bending moment. The sign of the axial and shear force can be easily determined from Fig.(3.17).

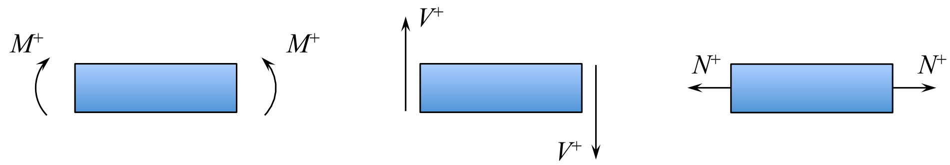

Figure 3.17: Positive shear and normal stresses in a beam.

END OF ADVANCED TOPIC

3.3 Generalized Forces and Bending Moments in Plates

In plates there are three in-plane components of the stress tensor $\sigma_{\alpha\beta}\{\sigma_{xx}, \sigma_{yy}, \sigma_{xy}\}$. Replacing σ_{xx} by $\sigma_{\alpha\beta}$ or $\sigma_{z\alpha}$ in Eqs.(3.36a-3.36c) the generalized forces and couples are defined

$$M_{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta} z \, \mathrm{d}z \quad [\mathrm{Nm/m}] = [\mathrm{N}] \tag{3.37a}$$

$$N_{\alpha\beta} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{\alpha\beta} \, \mathrm{d}z \quad [\mathrm{N/m}] \tag{3.37b}$$

$$V_{\alpha} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{z\alpha} \, \mathrm{d}z \quad [\mathrm{N/m}] \tag{3.37c}$$

Note that in the plate theory the integration is performed over the thickness of the plate rather than the entire surface. Therefore the dimensions of the quantities defined by Eqs.(3.37a-3.37c) are "per unit length".

ADVANCED TOPIC

3.4 Principle of Virtual Work for Beams

This principle can be derived directly from the general 3-D principle, Eq.(3.24) assuming one-dimensional stress state and kinematic assumption of the elementary beam theory

$$\begin{aligned} &\sigma_{ij} \rightarrow \sigma_{xx} \\ &\delta\epsilon_{ij} \rightarrow \delta\epsilon_{xx} = \delta\epsilon^{\circ}(x) + z\delta\kappa \;\text{ from Eq.(2.44)} \\ &\mathrm{d}V = \mathrm{d}A\mathrm{d}x, \;\; 0 < x < l \end{aligned} \tag{3.38}$$

The left hand (LH) side of Eq.(3.25) becomes

$$\mathrm{LH} = \int_V \sigma_{ij}\delta\epsilon_{ij}\,\mathrm{d}V = \int_0^l \left\{ \int_A [\sigma_{xx}\delta\epsilon^{\circ}(x)\mathrm{d}A + \sigma_{xx}z\delta\kappa\mathrm{d}A] \right\} \mathrm{d}x \tag{3.39}$$

Both $\delta\epsilon^{\circ}(x)$ and $\delta\kappa(x)$ are extension and curvature of the beam axis and are constant with respect to integration over the area. The above equation can be further simplified

$$\mathrm{LH} = \int_0^l \left[\delta\epsilon^{\circ}(x) \int_A \sigma_{xx}\mathrm{d}A + \delta\kappa(x) \int_A \sigma_{xx}z\mathrm{d}A \right] \mathrm{d}x \tag{3.40}$$

Recalling the definition of the axial force, Eq.(3.36b) and the bending moment, Eq.(3.36a), the final expression for the virtual work inside the volume of the beam takes this simple form

$$\mathrm{LH} = \int_0^l (N\delta\epsilon^{\circ} + M\delta\kappa)\,\mathrm{d}x \tag{3.41}$$

where l is the length of the beam. Evaluation of the right hand side (RH) of Eq.(3.25) is more interesting.

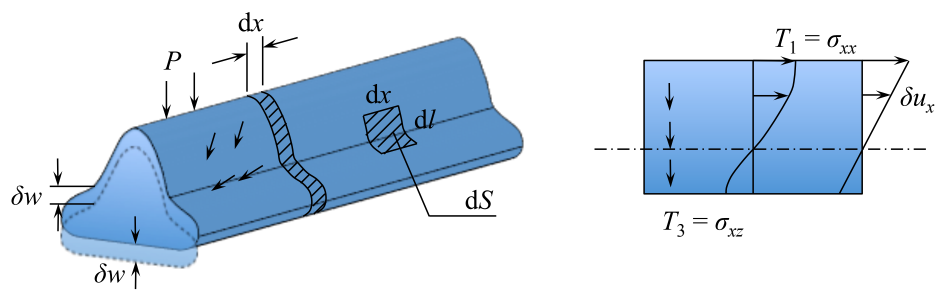

Figure 3.18: The outer surface of the beam consists of two parts: the lateral surface S_L on which the surface traction are acting and the end cuts A.

Note that all points on a slice of the beam move downward with the virtual displacement δw. The end cuts translate and rotate, according to Eq.(2.35). Then the right hand side of Eq.(3.25) becomes

$$\mathrm{RH} = \int_0^l q\delta w\,\mathrm{d}x + \int_A \sigma_{xx}[\delta u^{\circ} - \delta\theta z]\,\mathrm{d}A + \int_A \sigma_{xz}\delta w\,\mathrm{d}A \tag{3.42}$$

where q is the integrated pressure over the circumference of a slice

$$q = \oint T_i V_i \, \mathrm{d}s \tag{3.43}$$

and V_i are direction cosine of the surface traction vector with respect to the z-axis. In the case of the rectangular section ($h \times b$), Eq.(3.43) reduces to

$$q = pb \tag{3.44}$$

where p is the distributed pressure on the upper side of the beam and q is called the *line load*. The second term in Eq.(3.42) can be simplified using the definitions Eqs.(3.36a-3.36c)

$$\overline{M} = \int_{A_{\text{end}}} \sigma_{xx} z \, \mathrm{d}A \tag{3.45a}$$

$$\overline{N} = \int_{A_{\text{end}}} \sigma_{xx} \, \mathrm{d}A \tag{3.45b}$$

$$\overline{V} = \int_{A_{\text{end}}} \sigma_{xz} \, \mathrm{d}z \tag{3.45c}$$

where the bar over the symbol indicates that this is the value at the beam end.

The final expression for the principle of virtual work for a beam takes the form

$$\int_0^l (N\delta\epsilon^\circ + M\delta\kappa) \, \mathrm{d}x = \int_0^l q(x)\delta w \, \mathrm{d}x + \overline{N}\delta u^\circ - \overline{M}\delta\theta + \overline{V}\delta w \tag{3.46}$$

The above principle will be used to derive approximate solutions of the beam problems and also to obtain the equations of equilibrium and boundary conditions.

END OF ADVANCED TOPIC

3.5 Derivation of Equation of Equilibrium for Beams from the Principle of Virtual Work

The needed mathematical apparatus is the integration by parts. The starting point in Eq.(3.27) which is put in an alternative form

$$\frac{\mathrm{d}a}{\mathrm{d}x}b = \frac{\mathrm{d}}{\mathrm{d}x}(ab) - a\frac{\mathrm{d}b}{\mathrm{d}x} \tag{3.47}$$

Integrating both sides of the above equation on gets

$$\int \frac{\mathrm{d}a}{\mathrm{d}x}b\,\mathrm{d}x = ab|_{\text{ends}} - \int a\frac{\mathrm{d}b}{\mathrm{d}x}\,\mathrm{d}x \tag{3.48}$$

To simplify the notation the "prime" convention will be used throughout

$$\frac{d[]}{dx} = []'; \quad \frac{d^2[]}{dx^2} = []'' \tag{3.49}$$

We turn now the left hand side of the principle of virtual work, Eq.(3.14) and recall the definition of beam curvature and axial strain

$$\kappa = -w'' \tag{3.50a}$$

$$\epsilon^\circ = u' \tag{3.50b}$$

The virtual increments are

$$\delta\kappa = -\delta w'' = (\delta w')' \tag{3.51a}$$

$$\delta\epsilon^\circ = \delta u' \tag{3.51b}$$

Substituting Eq.(3.51b) into the LH side of Eq.(3.14) and integrating twice by parts we get

$$\begin{aligned}
\text{LH} &= -\int_0^l M(\delta w')'\,\mathrm{d}x + \int_0^l N\delta u'\,\mathrm{d}x \\
&= -\left(M\delta w'\big|_0^l - \int_0^l M'\delta w'\,\mathrm{d}x\right) + \left(N\delta u\big|_0^l - \int_0^l N'\delta u\,\mathrm{d}x\right) \\
&= -M\delta w'\big|_0^l + M'\delta w\big|_0^l - \int_0^l M''\delta w\,\mathrm{d}x + N\delta u\big|_0^l - \int_0^l N'\delta u\,\mathrm{d}x
\end{aligned} \tag{3.52}$$

The second term represents the work increment at the beam end on downward virtual displacement. Therefore the corresponding generalized force must be the shear force V

$$V = M' \tag{3.53}$$

Introducing Eq.(3.52) into Eq.(3.46) and grouping the terms yields

$$\int_0^l (M''+q)\delta w\,\mathrm{d}x + \int_0^l N'\delta u\,\mathrm{d}x + (M-\overline{M})\delta w'\big|_0^l - (N-\overline{N})\delta u\big|_0^l - (V-\overline{V})\delta w\big|_0^l = 0 \tag{3.54}$$

The above equation should hold not for one specific incremental displacement but for arbitrary variations $(\delta w, \delta w', \delta u)$, independent inside $0 < x < l$ and on the boundaries. Therefore by the first lemma of the calculus of variation, the local (strong) form of the beam equilibrium follows

$$M'' + q = 0 \tag{3.55a}$$
$$N' = 0 \tag{3.55b}$$

along with the boundary conditions

$$(M - \overline{M})\delta w' = 0 \tag{3.56a}$$
$$(N - \overline{N})\delta u = 0 \tag{3.56b}$$
$$(V - \overline{V})\delta w = 0 \tag{3.56c}$$

In order to satisfy the boundary condition

$$\text{either } M = \overline{M} \text{ or } \delta w' = 0 \tag{3.57a}$$
$$\text{either } V = \overline{V} \text{ or } \delta w = 0 \tag{3.57b}$$
$$\text{either } N = \overline{N} \text{ or } \delta u = 0 \tag{3.57c}$$

The quantities with a bar denotes the quantities prescribed at the ends of a beam. In particular $\overline{M}$, $\overline{V}$, and $\overline{N}$ could be equal to zero. The first column in Eq.(3.57) represents the *static boundary conditions* while the second column the *kinematic boundary conditions*. There are also *mixed boundary conditions*. The following combinations satisfy all boundary conditions, Fig. (3.19).

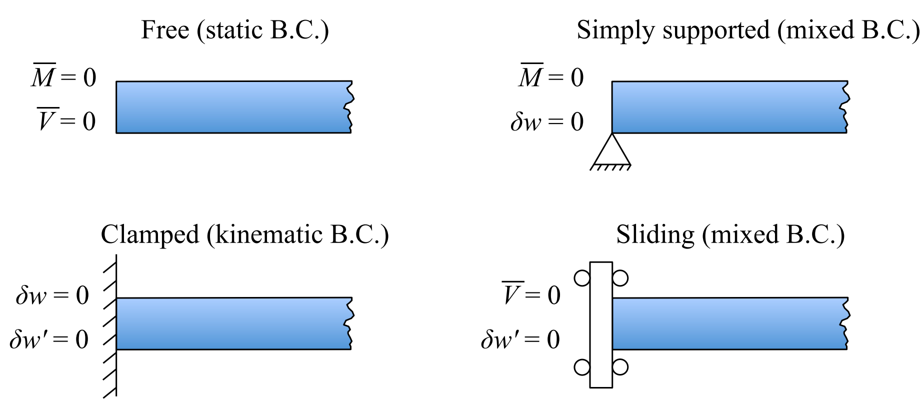

Figure 3.19: Boundary conditions in the axial direction.

In addition the beam could freely slide at the end along x-axis or can be restrained from sliding, Fig. (3.20).

In the case of symmetric loading of the beam, it suffices to consider only one half of the beam with the *symmetry boundary condition*. The symmetry B.C. is identical to the sliding boundary conditions, as explained in Fig.(3.21).

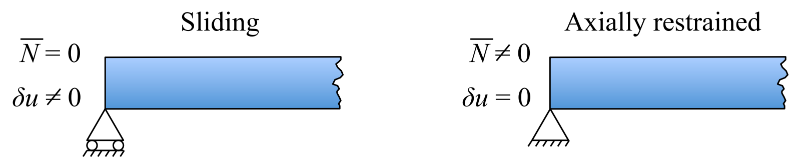

Figure 3.20: Shear force V and rotation angle $\delta w'$ vanishes at the symmetry plane.

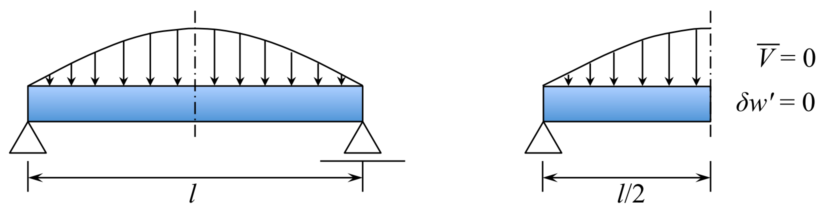

Figure 3.21: Shear force V and rotation angle $\delta w'$ vanishes at the symmetry plane.

ADVANCED TOPIC

3.6 Mathematical Theory of Beams

The equations of equilibrium of a beam with a rectangular cross-section can be derived in an elegant way from the 3-D equilibrium equation. With *zero body forces*, the equation of equilibrium in the compact index notation is

$$\sigma_{ij,j} = 0 \tag{3.58}$$

or the expanded notation

$$i = 1,\ \sigma_{1j,j} = 0 \ \rightarrow\ \sigma_{11,1} + \sigma_{12,2} + \sigma_{13,3} = 0 \tag{3.59a}$$

$$i = 2,\ \sigma_{2j,j} = 0 \ \rightarrow\ \sigma_{21,1} + \sigma_{22,2} + \sigma_{23,3} = 0 \tag{3.59b}$$

$$i = 3,\ \sigma_{3j,j} = 0 \ \rightarrow\ \sigma_{31,1} + \sigma_{32,2} + \sigma_{33,3} = 0 \tag{3.59c}$$

In the engineering notation, the full set of equilibrium equation, already given by Eq.(3.18), is

$$\begin{pmatrix} \dfrac{\partial\sigma_{xx}}{\partial x} & \boxed{\dfrac{\partial\sigma_{xy}}{\partial y}} & \dfrac{\partial\sigma_{xz}}{\partial z} \\ \boxed{\dfrac{\partial\sigma_{yx}}{\partial x}} & \dfrac{\partial\sigma_{yy}}{\partial y} & \boxed{\dfrac{\partial\sigma_{yz}}{\partial z}} \\ \dfrac{\partial\sigma_{zx}}{\partial x} & \boxed{\dfrac{\partial\sigma_{zy}}{\partial y}} & \dfrac{\partial\sigma_{zz}}{\partial z} \end{pmatrix} \tag{3.60}$$

which of the components of the stress tensor will survive beam assumption. Consider a rectangular cross-section beam ($h \times b$) undergoing planar bending, Fig.(3.22).

The beam is subjected to pressure loading p at the plane $z = -\dfrac{h}{2}$. In the case of planar bending there must be no gradient of stresses in the y-direction. Therefore the term $\dfrac{\partial\sigma_{xy}}{\partial y} = 0$. The surviving components lie outside the shaded box in Eq.(3.60), and are:

$$\frac{\partial\sigma_{\mathrm{xx}}}{\partial x} + \frac{\partial\sigma_{\mathrm{xz}}}{\partial z} = 0 \tag{3.61a}$$

Equation in the y-direction is satisfied identically (3.61b)

$$\frac{\partial\sigma_{\mathrm{zx}}}{\partial x} + \frac{\partial\sigma_{\mathrm{zz}}}{\partial z} = 0 \tag{3.61c}$$

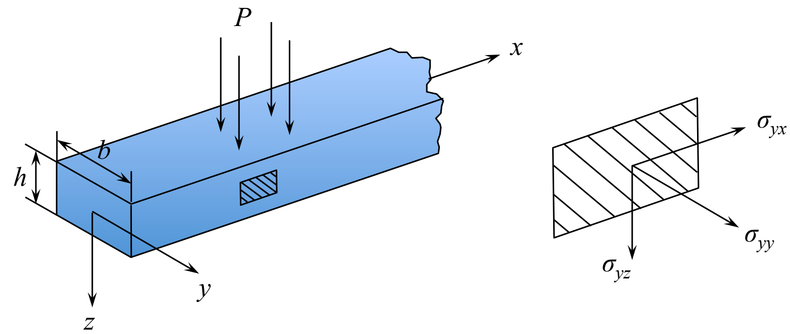

Figure 3.22: Vanishing components of the stress vector.

Boundary Conditions

Boundary conditions are specified by the Cauchy formula. The lateral surfaces $y = \pm\frac{b}{2}$ as well as the lower shelf $z = \frac{h}{2}$ are stress free. Consider only the upper shelf $z = -\frac{h}{2}$, defined by the unit normal vector $\boldsymbol{n}[0, 0, -1]$. Assume that no shear loading is applied, so that the distributed load is directed along z-axis.

The components of surface traction on the upper shelf are $\boldsymbol{T}[0, 0, p]$. From the Cauchy formula, calculate the z-component of the surface traction $T_3 = \sigma_{31}n_1 + \sigma_{32}n_2 + \sigma_{33}n_3$. Only the last term, for which $n_3 = -1$ remains and so

$$T_3 = p = -\sigma_{33} = -\sigma_{zz} \tag{3.62}$$

It was straightforward to see that the pressure p must be equilibrated by the σ_{zz} component. However, for a precise determination of the sign, the Cauchy formula happened to be useful. All other components of the stress tensor on the lateral surface of the beam are zero.

The derivation consists of three steps. First Eq.(3.61a) is integrated with respect to z and multiplied by the beam width b

$$b\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{xx}}{\partial x}\,\mathrm{d}z + b\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{xz}}{\partial z}\,\mathrm{d}z = 0 \tag{3.63}$$

Next, for a definite integral, the differentiation of the integrant in the first is equivalent to the differentiation of the integral. The second term can be integrated to give

$$\frac{\mathrm{d}}{\mathrm{d}x}\int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} b\,\mathrm{d}z + \sigma_{xz}\Big|_{-\frac{h}{2}}^{\frac{h}{2}} = 0 \tag{3.64}$$

Noting that $b\,\mathrm{d}z = \mathrm{d}A$, the first integral represents the axial force N, according to the definition, Eq.(3.36b). The shear stress σ_{xz} is non-zero inside the beam height but vanishes at the lower and upper shelves, $\sigma_{xz} = 0$, at $z = \frac{h}{2}$ and $z = -\frac{h}{2}$. Only the first term survives, which is the force equilibrium in the x-direction

$$\frac{\mathrm{d}N}{\mathrm{d}x} = 0 \text{ or } N' = 0 \tag{3.65}$$

In the second step both sides of Eq.(3.61a) are multiplied by bz and integrated again with respect to z

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{xx}}{\partial x} z(b\mathrm{d}z) + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{xz}}{\partial z} z(b\mathrm{d}z) = 0 \tag{3.66}$$

The second term is now integrated by parts

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xx} z \,\mathrm{d}A + \{\sigma_{xz} bz\big|_{-\frac{h}{2}}^{\frac{h}{2}} - \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xz} \,\mathrm{d}A\} = 0 \tag{3.67}$$

The second term vanishes, as before. The first integral is the bending moment M while the second one-the shear force V (see the definitions, Eqs. (3.36a-3.36c). So, the above equation represent the moment equilibrium

$$\frac{\mathrm{d}M}{\mathrm{d}x} - V = 0 \tag{3.68}$$

In the third, final step, Eq.(3.61c) is integrated with respect to z after being multiplied by b.

$$\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{xz}}{\partial x}(b\mathrm{d}z) + \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\partial \sigma_{zz}}{\partial z}(b\mathrm{d}z) = 0 \tag{3.69}$$

After integration one gets

$$\frac{\mathrm{d}}{\mathrm{d}x} \int_{-\frac{h}{2}}^{\frac{h}{2}} \sigma_{xz} \,\mathrm{d}A + b\left[\sigma_{zz}\big|_{\frac{h}{2}} - \sigma_{zz}\big|_{-\frac{h}{2}}\right] = 0 \tag{3.70}$$

Recalling the boundary condition $\sigma_{zz}\big|_{\frac{h}{2}} = 0$ and $\sigma_{zz}\big|_{-\frac{h}{2}} = -p$, Eq.(3.70) yields

$$\frac{\mathrm{d}V}{\mathrm{d}x} + b(-1)(-p) = 0 \tag{3.71}$$

or using $bp = q$

$$\frac{\mathrm{d}V}{\mathrm{d}x} + q = 0 \tag{3.72}$$

Eliminating the shear force between Eq.(3.68) and Eq.(3.72), the beam equilibrium equation is obtained.

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} + q(x) = 0 \tag{3.73}$$

This equation is identical to the one derived from the principle of virtual work.

END OF ADVANCED TOPIC

3.7 Equilibrium in the Theory of Moderately Large Deflections of Beams

In Lecture 2, it was shown that finite rotation of the beam element introduced the additional term $\frac{1}{2}\theta^2$ in the expression for the axial strain. Let's see if consideration of finite slope would require modification of the equation of the equilibrium.

In Fig.(3.21) the beam element is shown in the theory of small deflections (infinitesimal rotation) and moderately large deflections (finite rotation).

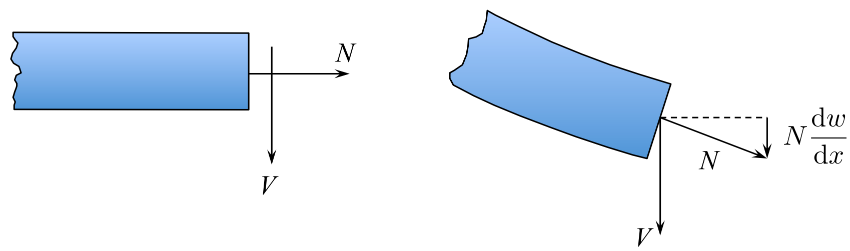

Figure 3.23: In finite rotation the axial force contributes to the total shear force.

The so-called effective shear force V^* is a sum of the cross-sectional shear V and projection of the axial force into the vertical direction. Thus,

$$V^* = V + N\frac{\mathrm{d}w}{\mathrm{d}x} \tag{3.74}$$

Note that this result is valid as long as $\cos\theta \approx 1$ and $\sin\theta \approx \tan\theta \approx \theta$. Will the derivation of the force equilibrium change? The answer is no.

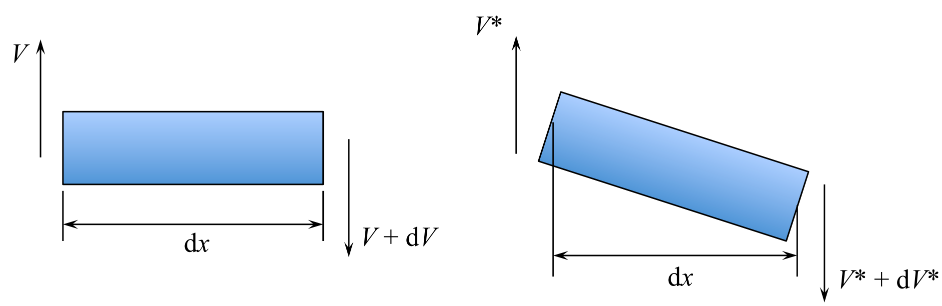

Figure 3.24: Direction of forces to equilibrate the infinitesimal beam element.

To ensure vertical equilibrium

$$(V^* + \mathrm{d}V^*) - V^* + q\mathrm{d}x = 0 \tag{3.75}$$

or

$$\frac{\mathrm{d}V^*}{\mathrm{d}x} + q = 0 \tag{3.76}$$

where V^* is defined by Eq.(3.74). Equilibrium of the horizontal (axial) forces stays the same as before since $\cos\theta \approx 1$.

$$\frac{\mathrm{d}N}{\mathrm{d}x} = 0 \tag{3.77}$$

Eliminating V^* between Eq.(3.74) and Eq.(3.76) gives

$$\frac{\mathrm{d}V}{\mathrm{d}x} + \frac{\mathrm{d}}{\mathrm{d}x}\left(N\frac{\mathrm{d}w}{\mathrm{d}x}\right) + q = \frac{\mathrm{d}V}{\mathrm{d}x} + \frac{\mathrm{d}N}{\mathrm{d}x}\frac{\mathrm{d}w}{\mathrm{d}x} + N\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + q = 0 \tag{3.78}$$

The second term vanishes on account of Eq.(3.77). The modified force equilibrium equation becomes

$$\frac{\mathrm{d}V}{\mathrm{d}x} + N\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + q = 0 \tag{3.79}$$

The new nonlinear term vanishes if (i) the axial force is zero or (ii) for small deflections and rotation. The moment equilibrium equation, Eq.(3.68) is not affected by moderately large rotations. Together with Eq.(3.79) we arrive at the governing equation of the theory of moderately large deflection of beams

$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} + N\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + q = 0 \tag{3.80}$$

On closing this section, two important remarks should be made. All equations of equilibrium for infinitesimal deformations of 3-D bodies and small deflections of beams involved only static quantities and their gradients (M, V, N). In the theory of moderately large deflections there is coupling between static and kinematic quantities through the second nonlinear terms.

Secondly, Eq.(3.80) includes leading in the in-plane direction (through N) and out-of-plane direction through q. Therefore, it is after referred as the equation describing a beam/columns.

3.8 Equilibrium of Rectangular Plates

A step-by-step derivation of the equation of equilibrium and boundary conditions for rectangular plates is presented in the lecture notes of the course 2.081 Plates and Shells. This equation takes the following form in the tensor notation

$$M_{\alpha\beta,\alpha\beta} + p = 0 \tag{3.81}$$

and in the extended notation

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_{yy}}{\partial y^2} + p = 0 \tag{3.82}$$

Recall that the dimensions of the bending moments in plates are [Nm/m] = [N]. In the case of cylindrical bending the twist M_{xy} and M_{yy} vanish. Multiplying Eq.(3.82) by the width b, one gets

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}[bM_{xx}] + q = 0 \tag{3.83}$$

which is identical to the previously derived equation equilibrium of a beam, Eq.(3.73). Therefore wide beams are a special class of rectangular plates.

The boundary conditions for plates are similar to those for beams in the local coordinate system at the edges, (n, t), Fig.(3.25).

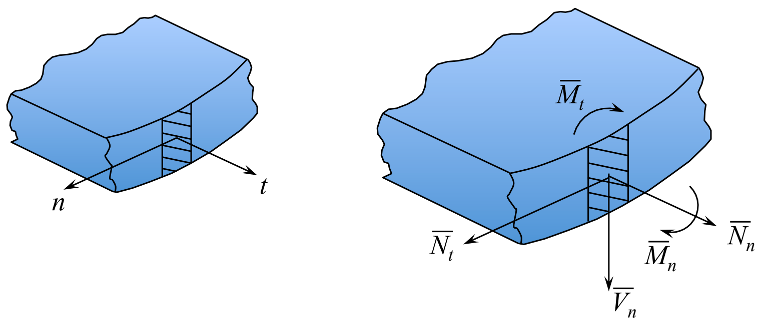

Figure 3.25: Local coordinate system at the plate edge with applied generalized forces.

Therefore Eq.(3.56) for beams should now read

$$(M_n - \overline{M}_n)\delta w' = 0 \tag{3.84a}$$
$$(V_n - \overline{V}_n)\delta w = 0 \tag{3.84b}$$
$$(N_n - \overline{N}_n)\delta u_n = 0 \tag{3.84c}$$
$$(N_t - \overline{N}_t)\delta u_t = 0 \tag{3.84d}$$

3.9 Circular Plates

It is relatively easy to derive the equation of equilibrium of a circular plate from the principle of virtual work. Bending and in-plane responses is considered separately

$$\int_{R_1}^{R_2} (M_r \delta\kappa_r + M_\theta \delta\kappa_\theta) r \, \mathrm{d}r = \int_{R_1}^{R_2} p\delta w r \, \mathrm{d}r + r\overline{M}_r \delta w' \big|_{R_1}^{R_2} + r\overline{V}_r \, \delta w \big|_{R_1}^{R_2} \tag{3.85}$$

where the radial and circumferential curvatures and their variations are defined (without proof) by

$$\kappa_r = \frac{\partial^2 w}{\partial r^2}, \quad \delta\kappa_r = \frac{\partial^2 (\delta w)}{\partial r^2} \tag{3.86a}$$
$$\kappa_\theta = \frac{1}{r}\frac{\partial w}{\partial r}, \quad \delta\kappa_\theta = \frac{1}{r}\frac{\partial (\delta w)}{\partial r} \tag{3.86b}$$

Integrating the left hand side of Eq.(3.85) by parts and using similar arguments as in the case of a beam, one gets equilibrium:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}M_r}{\mathrm{d}r}\right) + \frac{\mathrm{d}M_r}{\mathrm{d}r} - \frac{dM_\theta}{dr} = pr \tag{3.87}$$

and boundary conditions

$$(M_r - \overline{M}_r)\delta w' = 0 \tag{3.88a}$$

$$(V_r - \overline{V}_r)\delta w = 0 \tag{3.88b}$$

where

$$V_r = \frac{\mathrm{d}}{\mathrm{d}r}(rM_r) \tag{3.89}$$

When $R_1 = 0$, we have a circular plate. Otherwise the plate is annular with the inner and outer radius R_1 and R_2, respectively.

When the circular plate is loaded in the in-plane direction only, it remains flat and the components of the mid-surface extensions and their variations are

$$\epsilon_r^\circ = \frac{\mathrm{d}u_r}{\mathrm{d}r}, \quad \delta\epsilon_r^\circ = \frac{\mathrm{d}}{\mathrm{d}r}(\delta u_r) \tag{3.90a}$$

$$\epsilon_\theta^\circ = \frac{u_r}{r}, \quad \delta\epsilon_\theta^\circ = \frac{\delta u_r}{r} \tag{3.90b}$$

The principle of virtual work can be easily established in the form

$$\int_{R_1}^{R_2} (N_r \delta\epsilon_r^\circ + N_\theta \delta\epsilon_\theta^\circ) r\mathrm{d}r = rN_r\delta u_r\Big|_{R_1}^{R_2} \tag{3.91}$$

The equation of equilibrium in the in-plane direction are easily derived by integrating by parts

$$\frac{\mathrm{d}}{\mathrm{d}r}(rN_r) - N_\theta = 0 \tag{3.92}$$

subject to the boundary condition

$$(N_r - \overline{N}_r)\delta u_r\Big|_{R_1}^{R_2} = 0 \tag{3.93}$$

Note that $\overline{N}_\theta$ is zero at the boundaries, ensuring that there will be no in-plane shearing force $N_{r\theta}$ and the radial and hoop membrane forces are principal forces.

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