Added Mass Force Formulation

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Added Mass Tensor

A good way to think of the added mass components, m_{ij}, is to think of each term as mass associated with a force on the body in the i^{th} direction due to a *unit* acceleration in the j^{th} direction.

$$\underline{F} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} & m_{15} & m_{16} \\ m_{21} & m_{22} & m_{23} & m_{24} & m_{25} & m_{26} \\ m_{31} & m_{32} & m_{33} & m_{34} & m_{35} & m_{36} \\ m_{41} & m_{42} & m_{43} & m_{44} & m_{45} & m_{46} \\ m_{51} & m_{52} & m_{53} & m_{54} & m_{55} & m_{56} \\ m_{61} & m_{62} & m_{63} & m_{64} & m_{65} & m_{66} \end{bmatrix} \begin{pmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \dot{u}_3 \\ \dot{u}_4 \\ \dot{u}_5 \\ \dot{u}_6 \end{pmatrix}$$

$$\underline{F} = F_i \text{, where } i = \underbrace{1,2,3}_{\text{Linear Forces}}, \underbrace{4,5,6}_{\text{Moments}}$$

$$\dot{u}_i = [\dot{u}_1, \dot{u}_2, \dot{u}_3, \dot{u}_4, \dot{u}_5, \dot{u}_6]$$

added mass matrix $[m_a]$

m_{ij} where $i, j = 1,2,3,4,5,6$

Vector Velocity

Velocities:

$$Translation\ Velocity : \vec{U}(t) = (U_1, U_2, U_3)$$

$$Rotational\ Velocity : \vec{\Omega}(t) = (\Omega_1, \Omega_2, \Omega_3) \equiv (U_4, U_5, U_6)$$

All rotation is taken with respect to Origin of the coordinate system (often placed at the center of gravity of the object for simplicity!).

Accelerations:

$$\dot{u}_i = [\dot{u}_1, \dot{u}_2, \dot{u}_3, \dot{u}_4, \dot{u}_5, \dot{u}_6]$$

Added Mass Forces and Moments

Forces: (force in the j^{th} direction). ($i = 1, 2, 3, 4, 5, 6$ and $j, k, l = 1, 2, 3$)

$$F_j = -\dot{U}_i m_{ij} - \varepsilon_{jkl} U_i \Omega_k m_{li}$$

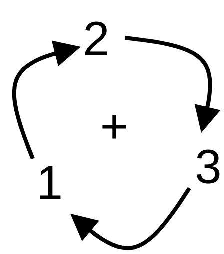
Moments: ($i = 1, 2, 3, 4, 5, 6$ and $j, k, l = 1, 2, 3$)

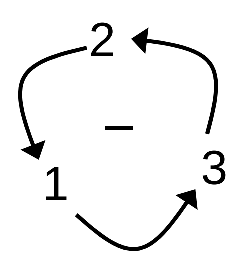
$$M_j = -\dot{U}_i m_{j+3,i} - \varepsilon_{jkl} U_i \Omega_k m_{l+3,i} - \varepsilon_{jkl} U_k U_i m_{li}$$

Tensor Notation

The alternating tensor ε_{jkl}

$$\varepsilon_{jkl} = \begin{cases} 0; & \text{if any } j,k,l \text{ are equal} \\ 1; & \text{if } j,k,l \text{ are in cyclic order} \\ -1; & \text{if } j,k,l \text{ are in anti-cyclic order} \end{cases}$$





Einstein Summation

$$F_j = -\dot{U}_i m_{ij} - \varepsilon_{jkl} U_i \Omega_k m_{li}$$

Sum up the terms for all i,j,k,l options: $(\, i = 1,2,3,4,5,6 \text{ and } j,k,l = 1,2,3)$

For example take: *j* =1 for the Force in the 1-direction (x-component)
Sum over all *i* = 1:6:

$$\underbrace{F_1}_{j=1} = -\underbrace{\dot{U}_1 m_{11}}_{i=1} - \underbrace{\dot{U}_2 m_{21}}_{i=2} - \underbrace{\dot{U}_3 m_{31}}_{i=3} - \underbrace{\dot{U}_4 m_{41}}_{i=4} - \underbrace{\dot{U}_5 m_{51}}_{i=5} - \underbrace{\dot{U}_6 m_{61}}_{i=6}$$

$$- \underbrace{\varepsilon_{1kl} U_1 \Omega_k m_{l1}}_{i=1} - \underbrace{\varepsilon_{1kl} U_2 \Omega_k m_{l2}}_{i=2} - \underbrace{\varepsilon_{1kl} U_3 \Omega_k m_{l3}}_{i=3} - \underbrace{\varepsilon_{1kl} U_4 \Omega_k m_{l4}}_{i=4}$$

$$- \underbrace{\varepsilon_{1kl} U_5 \Omega_k m_{l5}}_{i=5} - \underbrace{\varepsilon_{1kl} U_6 \Omega_k m_{l6}}_{i=6}$$

$\text{for } k,l = 1,2,3$

Next consider k = 1,2,3 then l = 1,2,3 →

For
$k = 1,2,3$

$$\varepsilon_{jkl} = \begin{cases} 0; & \text{if any } j,k,l \text{ are equal} \\ 1; & \text{if } j,k,l \text{ are in cyclic order} \\ -1; & \text{if } j,k,l \text{ are in anti-cyclic order} \end{cases}$$

Since we are considering the F_1 component where j = 1, then all terms with ε in them where j = k = 1 will be zero. So there is no reason to consider k = 1 here. So we just sum up the terms where k = 2 and k = 3:

$$\underbrace{F_1}_{j=1} = \underbrace{-\dot{U}_1 m_{11}}_{i=1} \underbrace{-\dot{U}_2 m_{21}}_{i=2} \underbrace{-\dot{U}_3 m_{31}}_{i=3} \underbrace{-\dot{U}_4 m_{41}}_{i=4} \underbrace{-\dot{U}_5 m_{51}}_{i=5} \underbrace{-\dot{U}_6 m_{61}}_{i=6}$$

same as before)

Let: k = 2

$$\underbrace{-\underbrace{\varepsilon_{12l} U_1 \Omega_2 m_{l1}}_{i=1} - \underbrace{\varepsilon_{12l} U_2 \Omega_2 m_{l2}}_{i=2} - \underbrace{\varepsilon_{12l} U_3 \Omega_2 m_{l3}}_{i=3} - \underbrace{\varepsilon_{12l} U_4 \Omega_2 m_{l4}}_{i=4} - \underbrace{\varepsilon_{12l} U_5 \Omega_2 m_{l5}}_{i=5} - \underbrace{\varepsilon_{12l} U_6 \Omega_2 m_{l6}}_{i=6}}_{k=2}$$

Next Let: k = 3

$$\underbrace{-\underbrace{\varepsilon_{13l} U_1 \Omega_3 m_{l1}}_{i=1} - \underbrace{\varepsilon_{13l} U_2 \Omega_3 m_{l2}}_{i=2} - \underbrace{\varepsilon_{13l} U_3 \Omega_3 m_{l3}}_{i=3} - \underbrace{\varepsilon_{13l} U_4 \Omega_3 m_{l4}}_{i=4} - \underbrace{\varepsilon_{13l} U_5 \Omega_3 m_{l5}}_{i=5} - \underbrace{\varepsilon_{13l} U_6 \Omega 3 m_{l6}}_{i=6}}_{k=3}$$

Next look at $l = 1,2,3$

$$\varepsilon_{jkl} = \begin{cases} 0; & \text{if any } j,k,l \text{ are equal} \\ 1; & \text{if } j,k,l \text{ are in cyclic order} \\ -1; & \text{if } j,k,l \text{ are in anti-cyclic order} \end{cases}$$

Since we are considering the F_1 component where j = 1, then all terms with ***e*** in them where *j* = *l* = 1 will be zero. So there is no reason to consider *l* = 1 here. So we just sum up the terms where *l* = 2 and *l* = 3:

$$\underbrace{F_1}_{j=1} = -\underbrace{\dot{U}_1 m_{11}}_{i=1} - \underbrace{\dot{U}_2 m_{21}}_{i=2} - \underbrace{\dot{U}_3 m_{31}}_{i=3} - \underbrace{\dot{U}_4 m_{41}}_{i=4} - \underbrace{\dot{U}_5 m_{51}}_{i=5} - \underbrace{\dot{U}_6 m_{61}}_{i=6}$$

(same as before)

Let: l = 3 *Note that any term where k = l then ε is zero*

$$\underbrace{-\underbrace{\varepsilon_{123} U_1 \Omega_2 m_{31}}_{i=1} - \underbrace{\varepsilon_{123} U_2 \Omega_2 m_{32}}_{i=2} - \underbrace{\varepsilon_{123} U_3 \Omega_2 m_{33}}_{i=3} - \underbrace{\varepsilon_{123} U_4 \Omega_2 m_{34}}_{i=4} - \underbrace{\varepsilon_{123} U_5 \Omega_2 m_{35}}_{i=5} - \underbrace{\varepsilon_{123} U_6 \Omega_2 m_{36}}_{i=6}}_{k=2;\, l=3}$$

Next Let: l = 2

$$\underbrace{-\underbrace{\varepsilon_{132} U_1 \Omega_3 m_{21}}_{i=1} - \underbrace{\varepsilon_{132} U_2 \Omega_3 m_{22}}_{i=2} - \underbrace{\varepsilon_{132} U_3 \Omega_3 m_{23}}_{i=3} - \underbrace{\varepsilon_{132} U_4 \Omega_3 m_{24}}_{i=4} - \underbrace{\varepsilon_{132} U_5 \Omega_3 m_{25}}_{i=5} - \underbrace{\varepsilon_{132} U_6 \Omega 3 m_{26}}_{i=6}}_{k=3;\, l=2}$$

Total Force:

$$\underbrace{F_1}_{j=1} = -\underbrace{\dot{U}_1 m_{11}}_{i=1} - \underbrace{\dot{U}_2 m_{21}}_{i=2} - \underbrace{\dot{U}_3 m_{31}}_{i=3} - \underbrace{\dot{U}_4 m_{41}}_{i=4} - \underbrace{\dot{U}_5 m_{51}}_{i=5} - \underbrace{\dot{U}_6 m_{61}}_{i=6}$$

$$\underbrace{-\underbrace{\varepsilon_{123} U_1 \Omega_2 m_{31}}_{i=1} - \underbrace{\varepsilon_{123} U_2 \Omega_2 m_{32}}_{i=2} - \underbrace{\varepsilon_{123} U_3 \Omega_2 m_{33}}_{i=3} - \underbrace{\varepsilon_{123} U_4 \Omega_2 m_{34}}_{i=4} - \underbrace{\varepsilon_{123} U_5 \Omega_2 m_{35}}_{i=5} - \underbrace{\varepsilon_{123} U_6 \Omega_2 m_{36}}_{i=6}}_{k=2;\, l=3}$$

$$\underbrace{-\underbrace{\varepsilon_{132} U_1 \Omega_3 m_{21}}_{i=1} - \underbrace{\varepsilon_{132} U_2 \Omega_3 m_{22}}_{i=2} - \underbrace{\varepsilon_{132} U_3 \Omega_3 m_{23}}_{i=3} - \underbrace{\varepsilon_{132} U_4 \Omega_3 m_{24}}_{i=4} - \underbrace{\varepsilon_{132} U_5 \Omega_3 m_{25}}_{i=5} - \underbrace{\varepsilon_{132} U_6 \Omega 3 m_{26}}_{i=6}}_{k=3;\, l=2}$$

On the second row of the equation above, the indices of the alternating tensor, ε_{jkl}, are in cyclic order *jkl = 123* ($\varepsilon_{123} = +1$). In the third row, the indices are in anti (or reverse) cyclic order: $\varepsilon_{132} = -1$ where *jkl = 132.*

Example

Example: For a body moving in the fluid with velocity

$$\vec{V} = (1,0,1,0,0,1) = (U_1, 0, U_3, 0, 0, U_6) = (U_1, 0, U_3, 0, 0, \Omega_3)$$

$$\vec{a} = (1,0,0,0,0,1) = (\dot{U}_1, 0, 0, 0, 0, \dot{U}_6)$$

The force in the x-direction is F_1

First substitute "1" for every instance of j

$$F_{j=1} = F_1 = -\dot{U}_i\, m_{i1} - \varepsilon_{1kl}\, U_i \Omega_k m_{li}$$

Next we need to "cycle" through the possible values for i (i = *1,2,3,4,5,6*)

Only need to look at values of i = 1,3,6

Force becomes:

$$F_1 = -\underbrace{\dot{U}_1 m_{11}}_{i=1} - \underbrace{\dot{U}_6 m_{61}}_{i=6} - \underbrace{\varepsilon_{1kl} U_1 \Omega_k m_{l1}}_{i=1} - \underbrace{\varepsilon_{1kl} U_3 \Omega_k m_{l3}}_{i=3} - \underbrace{\varepsilon_{1kl} U_6 \Omega_k m_{l6}}_{i=6}$$

Now look at the *k-index*: $(k \neq j \therefore k = 2,3)$

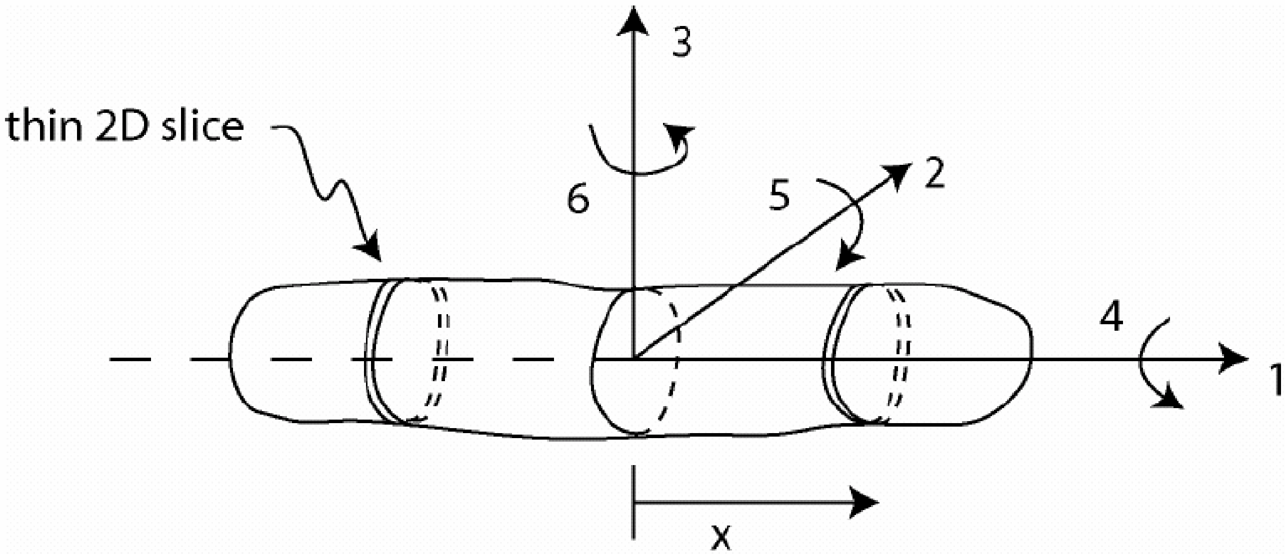
Since velocity is $\vec{V} = (1,0,1,0,0,1)$ then $\Omega_2 = 0 \text{ and } \Omega_3 \neq 0$

so we only have to deal with $k = 3$.

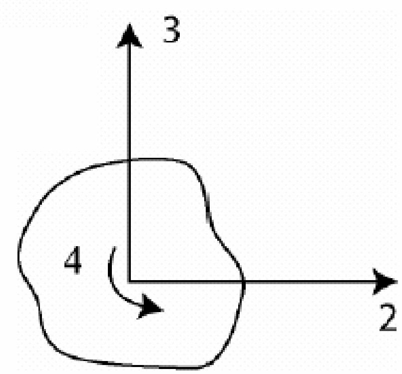
Now the only non-zero terms are for $l = 2$ therefore

$$F_1 = -\underbrace{\dot{U}_1 m_{11}}_{i=1} - \underbrace{\dot{U}_6 m_{61}}_{i=6} \underbrace{- \underbrace{\varepsilon_{132} U_1 \Omega_3 m_{21}}_{i=1} - \underbrace{\varepsilon_{132} U_3 \Omega_3 m_{23}}_{i=3} - \underbrace{\varepsilon_{132} U_6 \Omega_3 m_{26}}_{i=6}}_{k=3;\, l=2}$$

Slender Body



Slender body oriented with the long axis in the 1-direction.



2D cross-sectional slice of slender body.

Added Mass Matrix

	1	2	3	4	5	6
1						
2		$m_{22} = \int_L a_{22}\, dx$	$m_{23} = -\int_L a_{23}\, dx$	$m_{24} = \int_L a_{24}\, dx$		$m_{26} = \int_L x\, a_{22}\, dx$
3			$m_{33} = \int_L a_{33}\, dx$		$m_{35} = -\int_L x\, a_{33}\, dx$	
4				$m_{44} = \int_L a_{44}\, dx$		$m_{46} = \int_L x\, a_{24}\, dx$
5					$m_{55} = \int_L x^2\, a_{33}\, dx$	
6						$m_{66} = \int_L x^2\, a_{22}\, dx$

The 2D coefficients will be written as a_{ij} whereas the 3D coefficients are written as m_{ij} .

Figure removed for copyright reasons. Please see:

Table 4.3 in

Newman, J. "Added-Mass Coefficients for Various Two Dimensional Bodies." In *Marine Hydrodymanics*. Cambridge MA: MIT Press, 1977. ISBN: 0262140268.